Optimal Uncertainty Quantification of Quantiles
Application to Industrial Risk Management.

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ETICS 2018, Roscoff

Optimal Uncertainty Quantification (OUQ) : [Owhadi et al., 2013]

Principle

Find optimal bounds for a quantity of interest $Q(\mu^\dagger)$, functional of an uncertain probability measure $\mu^\dagger$, known only to lie in some subset $\mathcal{A}$ of $\mathcal{M}_1(\mathcal{X})$:

$$\underline{Q}(\mathcal{A}) \leq Q(\mu^\dagger) \leq \overline{Q}(\mathcal{A}),$$

with :

- $\underline{Q}(\mathcal{A}) = \inf_{\mu \in \mathcal{A}} Q(\mu)$
- $\overline{Q}(\mathcal{A}) = \sup_{\mu \in \mathcal{A}} Q(\mu)$
- $\mathcal{A} = \{\mu \in \mathcal{M}_1(\mathcal{X}) \mid \Phi_j(\mu) \leq c_j, j = 1, \dots, N\}$ the *admissible* subset,

Distributionally Robust : Useful in risk-adverse situations when parametric assumptions are hard to justify
$\hookrightarrow$ Many applications in Industrial Risk Management (find two : exercise $\sharp 1$)

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- and also : Probabilistic Seismic Hazard Assessment (PSHA), Extreme weather forecasting, Environmental risk assessment, Ecotoxicology, ...

Robust Bayesian Inference (RBI) : [Rios Insua and Ruggeri, 2000]

RBI as a special case of OUQ

- μ prior distribution on parameter θ in statistical model $Y \sim f_\theta d\lambda$, belonging to a class $\mathcal{A}$ of admissible priors
- $P_Y(\mu)$ posterior distribution of θ given Y, according to Bayes' theorem :

$$dP_Y(\mu)(\theta) = \frac{f_\theta(Y)d\mu(\theta)}{\int_\nu f_\nu(Y)d\mu(\nu)} \quad (1)$$

- Derive optimal bounds on interest quantity of posterior distribution
  - replace $Q(\mu)$ by $Q(P_Y(\mu))$ (or Q by $Q \circ P_Y$)

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OUQ as a special case of RBI

- OUQ corresponds to the **no-data** case ($f_\theta := f$ does not depend on θ)
  - *Proof :* (1) reduces to $P_Y(\mu)$, whence $Q(P_Y(\mu))$ reduces to $Q(\mu)$ $\square$

$\hookrightarrow$ both formulations are equivalent, and special cases of the Dempster Schaeffer Theory (DST) (M. Couplet, private conversation)

Main result

Theorem (Measure affine functionals over generalized moment classes)

If :

- $Q(\mu)$ *is measure affine (e.g.* $Q(\mu) := \mathbb{E}_\mu[q]$, q *bounded above or below)*
- $\mathcal{A} = \{\mu \in \mathcal{M}_1(\mathcal{X}) | \mathbb{E}_\mu[\varphi_j] \leq c_j, j = 1, \ldots, N\}$ *for measurable functions* φ_j
- $\mathcal{A}_\Delta = \{\mu \in \mathcal{A} | \mu = \sum_{0=1}^{N} w_i \delta_{x_i}\}$ *extremal admissible probability measures*

Then :

- $\underline{Q}(\mathcal{A}) = \underline{Q}(\mathcal{A}_\Delta)$; $\overline{Q}(\mathcal{A}) = \overline{Q}(\mathcal{A}_\Delta)$

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Then :

- $\underline{Q}(\mathcal{A}) = \underline{Q}(\mathcal{A}_\Delta)$; $\overline{Q}(\mathcal{A}) = \overline{Q}(\mathcal{A}_\Delta)$

Implementation

To find $\underline{Q}(\mathcal{A})$ (resp.$\overline{Q}(\mathcal{A})$) :

- Minimize (resp. Maximize) $Q(\mu) = \sum_{i=0}^{N} w_i q(x_i)$ wrt : $(w_i, x_i)_{0 \leq i \leq N}$
- subject to : $\sum_{i=0}^{N} w_i \varphi_j(x_i) \leq c_j$, for $j = 1, \ldots, N$

$\hookrightarrow$ **Constrained optimization** problem, solvabe by (almost) off-the shelf methods if q, φ_j analytical functions : **Mystic framework** [McKerns et al., 2012]

A Simple Result

Theorem (Quantile-CDF duality)

Assume $\mathcal{X} = \mathbb{R}^+$

Let :

- $F_\mu(x) = \mathbb{P}_\mu[X \leq x]$ *pointwise cdf evaluation functional*
- $Q_\mu(p) = \inf\{x > 0 | F_\mu(x) \geq p\}$ *order-p quantile*

Then :

$$\overline{Q_{\mathcal{A}}}(p) = \inf\{x > 0 | \underline{F_{\mathcal{A}}}(x) \geq p\}$$

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Then :

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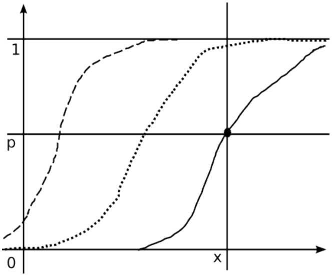
Proof.

- $\forall \mu \in \mathcal{A}$:
  $\{x > 0 | \underline{F_{\mathcal{A}}}(x) \geq p\} \subseteq \{x > 0 | F_\mu(x) \geq p\}$
  $\Rightarrow \inf\{x > 0 | \underline{F_{\mathcal{A}}}(x) \geq p\} \geq Q_\mu(p)$
  $\Rightarrow \inf\{x > 0 | \underline{F_{\mathcal{A}}}(x) \geq p\} \geq \overline{Q_{\mathcal{A}}}(p)$.
- Assuming a strict inequality, $\exists x_0$, s.t. :
  - $x_0 < \inf\{x > 0 | \underline{F_{\mathcal{A}}}(x) \geq p\}$
    $\Rightarrow \underline{F_{\mathcal{A}}}(x_0) < p \Rightarrow \exists \mu, F_\mu(x_0) < p$
  - $\overline{Q_{\mathcal{A}}}(p) < x_0$
    $\Rightarrow Q_\mu(p) < x_0 \Rightarrow F_\mu(x_0) \geq p$,

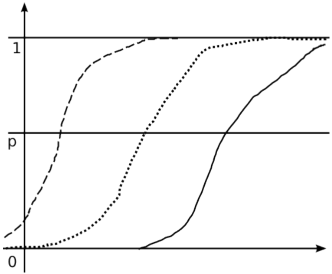


leading to a contradiction $\square$

Application : Inversion of CDF Bounds

Sequential construction of Quantile Upper Bound

- **Initialization ($t = 0$) :**
  $(a_t, b_t) = (0, +\infty)$



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  - Calculate $\underline{F_{\mathcal{A}}}(x_t)$
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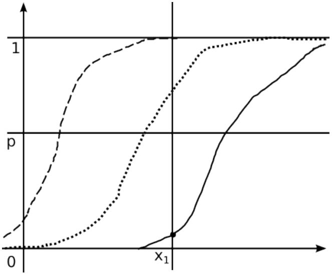

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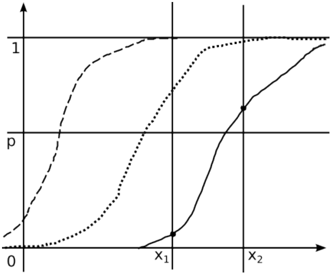

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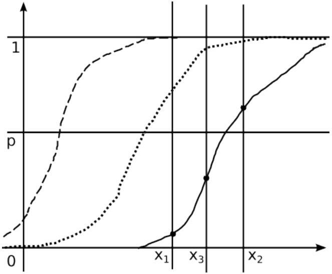

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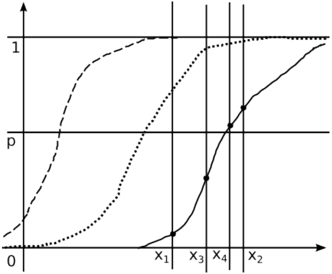

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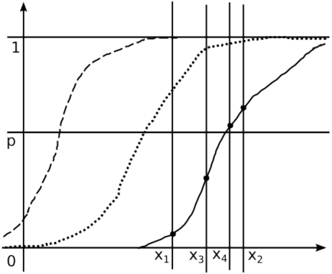

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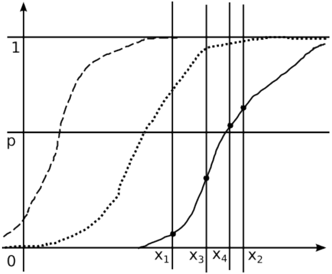

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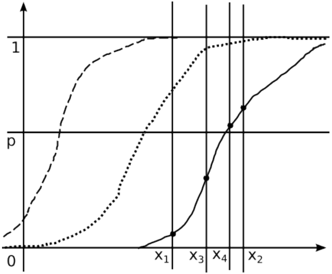

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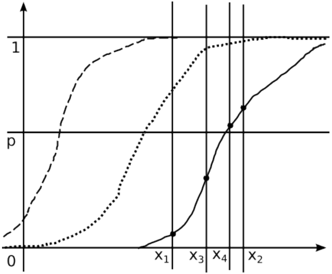

Challenges

- How to 'choose' $x_t \in (a_{t-1}, b_{t-1})$?
- Can we guarantee $\overline{Q_{\mathcal{A}}}(p)$ is nontrivial ($< \infty$) ?
- What if $Q(\mu)$ is an extreme quantile on $Y = G(X)$, with $X \sim \mu$ and G a costly computer model ? Can we develop an 'EGO-like' approach ?

Alternative : Direct quantile optimization

$Q_\mu(p)$ **not a measure affine functional**
However, quantile-CDF duality ensures that main result **still applies :**

OUQ for quantiles

To find $\underline{Q_\mathcal{A}}(p)$ (resp.$\overline{Q_\mathcal{A}}(p)$) :

- Minimize (resp. Maximize) $Q_\mu(p) = x_{(i^*)}$ wrt : $(w_i, x_i)_{0\leq i\leq N}$ where :
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Main difficulty

- Objective function $Q_p(\mu) = Q_p((w_i, x_i)_{0 \le i \le N})$ irregular and non convex

Case study : Quantile of nonlinear transform under product measure

Problem specification (simplified version)

$$Q_\mu(p) = \sup\{y > 0, \mathbb{P}_\mu[G(X) \leq y] \leq p\},$$

where :

- $X = (X_1, \ldots, X_d) \sim \mu$ over $[0,1]^d$
- $G : [0,1]^d \to \mathbb{R}^+$ potentially costly
- $\mathcal{A} = \{\mu = \otimes_{k=1}^d \mu_k \mid \mathbb{E}_{\mu_k}[X_k] = m_k, 1 \leq k \leq d\}$

Extreme set of product measures

[Owhadi et al., 2013] show that $\overline{Q}(\mathcal{A}) = \overline{Q}(\mathcal{A}_\Delta)$ where :

- $Q(\mu)$ measure affine (extendable to quantiles by duality with CDF)
- $\mathcal{A}_\Delta = \{\mu = \otimes_{k=1}^d \left(w_k \delta_{x_{k,0}} + (1 - w_k)\delta_{x_{k,1}}\right) \mid w_k x_{k,0} + (1 - w_k) x_{k,1} = m_k\}$

To be continued. . .

THANKS FOR YOUR ATTENTION!!

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