A new estimation algorithm for more reliable prediction in Gaussian Process Regression

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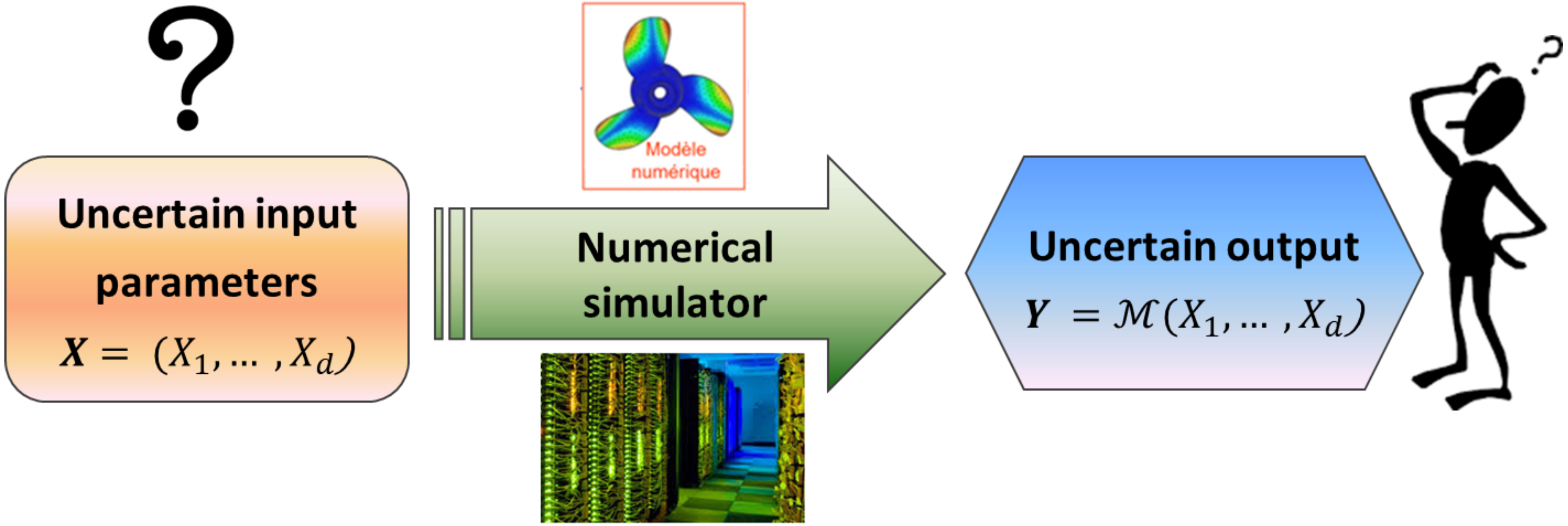
Workshop of the SAMOURAI ANR project - December 2024, Paris.

Risk assessment in nuclear accident analysis

- **Safety studies:** compute a failure risk (margins, rare events) with validated computer/numerical models
- **Numerical simulators:** fundamental tools to understand, model & predict physical phenomena
- **Large number of input parameters,** related to physical and numerical modelling
- **Uncertainty on some inputs → uncertainty on output & safety margins**
- **BEPU (Best-Estimate-Plus-Uncertainties)**: realistic models + uncertain inputs → **Better assessment of the real margins**



Risk assessment in nuclear accident analysis

- **How to deal with uncertainties in numerical simulation?**

→ Probabilistic framework and Monte Carlo-based methods

→ **CPU-expensive simulator** ⇒ Use of **machine learning to mimic the simulator and propagate input uncertainties**

→ **Applicative constraints/framework:**

✓ **Given data for training**: a single inputs/output **sample** $D_S = \left(\boldsymbol{x}^{(i)}, y^{(i)}\right)_{1 \leq j \leq n}$ where $y^{(i)} = \mathcal{M}\left(\boldsymbol{x}^{(i)}\right)$

→ random or quasi-random sample

✓ **Small sample size**: $n \approx$ 100 to 1000 simulations

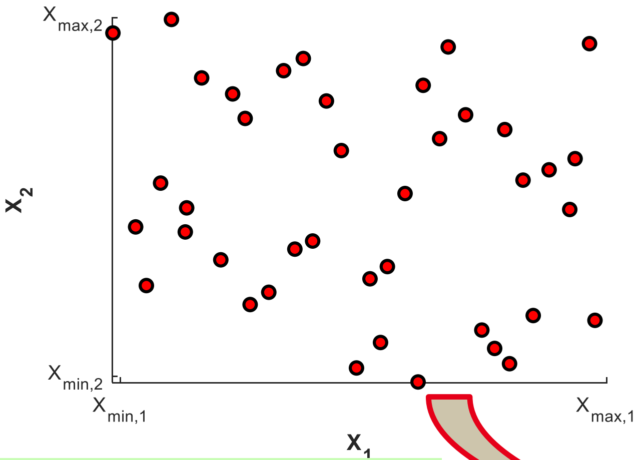
✓ **Large number of uncertain inputs**: d ≈ 10 to 100 inputs

✓ **Required UQ associated to each prediction**

→ Gaussian Process Regression (GPR): particularly well-suited tool ⇒ Very popular

Crucial use of GPR metamodel

Design of numerical experiments → **Numerical simulations** → **Analysis of simulator outputs**



Incertain inputs domain

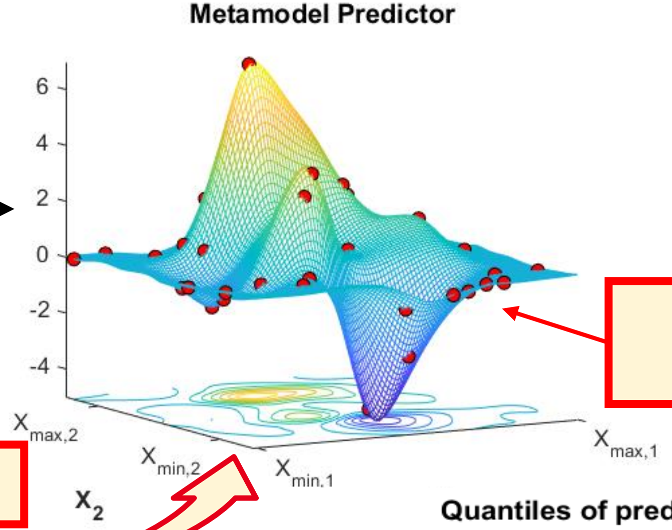
→ Simulator

$Y = \mathcal{M}(X_1, \ldots, X_d)$ →

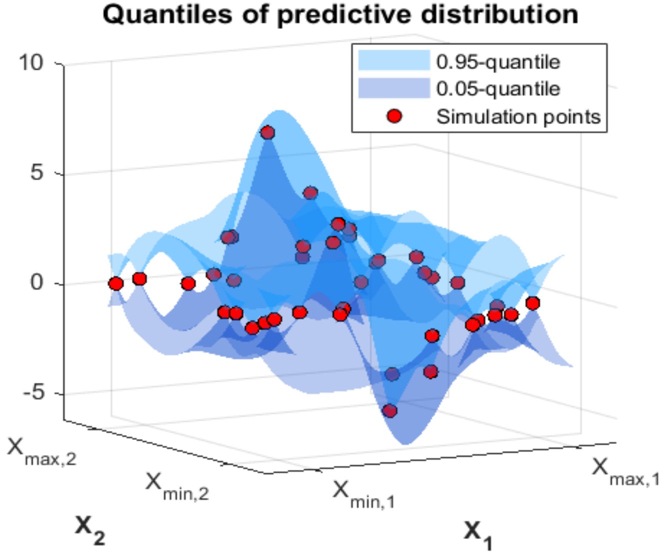
In case of costly $\mathcal{M}$:

Approximation with GPR

Metamodel: $Y_{app} = \widehat{\mathcal{M}}(X) \approx \mathcal{M}(X)$



Probabilistic metamodel



✓ Build from the dataset, GPR mimics the true model $\mathcal{M}$, providing a **GP predictive distribution** for each new evaluation point

$\Rightarrow$ **Intrinsic quantification of prediction error!**

$\Rightarrow$ Very appealing, but in practice calls for **a few good practices**!

Building an efficient GPR in practice

1. Dealing with the large input dimension

Dealing with the large input dimension

How to train the GP in large dimension? (d~10 to 100, e.g.)

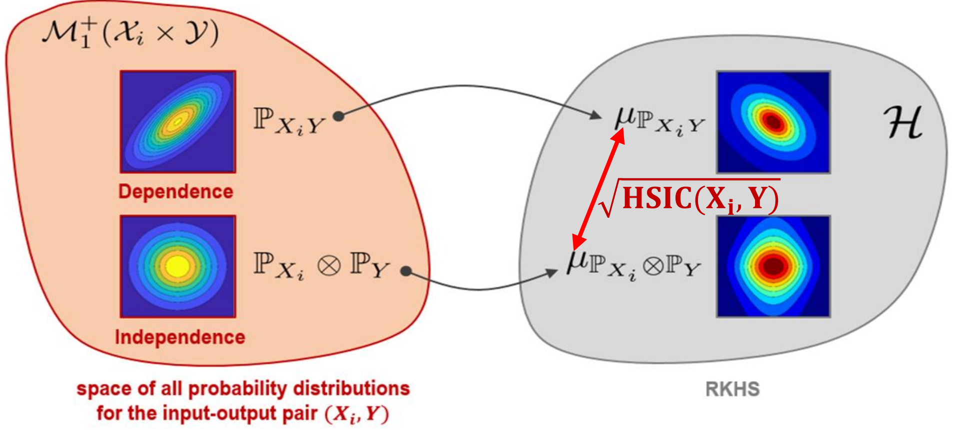
► **Curse of dimensionality $\Rightarrow$ too many GP hyperparameters have to be optimized!**

Preliminary SCREENING for input selection (and thus dimension reduction)

HSIC-based sensitivity measure [GFT+07] $\rightarrow$ dependence measure comparing the RKHS embeddings of joint distribution $\mathbb{P}_{X_i Y}$ and product of marginals $\mathbb{P}_{X_i} \otimes \mathbb{P}_Y$

$$HSIC(X_i, Y) = MMD^2(P_{X_i Y}, P_{X_i} \otimes P_Y) = \left\| \mu_{\mathbb{P}_{X_i Y}} - \mu_{\mathbb{P}_{X_i} \otimes \mathbb{P}_Y} \right\|^2$$

- ✓ HSIC can capture **a large spectrum of** input-output **relationships** (power of RKHS ☺)
- ✓ $\widehat{HSIC}$: Estimation from **a unique random sample**, robust in practice from $n \sim 100$

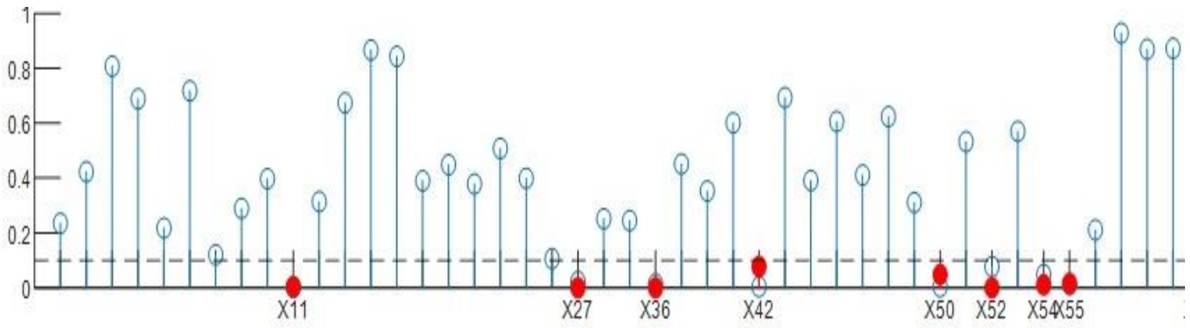


Extract from a presentation by G. Sarazin (CEA)

Dealing with the large input dimension

- **Screening with HSIC-based independence tests** [GFT+07]: $HSIC(X_i, Y) = 0 \Leftrightarrow X_i \perp Y$ (*with characteristic kernels!*)

P-value of global-HSIC tests



● Selected inputs

Selection of **significant inputs** (usually <20)

- ✓ **Explicative inputs** of GPR
- ✓ Non-significant influential inputs captured by an additional variance in GPR (nugget effect)

- **HSIC-based ranking with R²HSIC** [Dav15] : Inputs ordered by degree of influence

Can be used for **more robust sequential GPR estimation**

$\Rightarrow$ "Forward" estimation of GPR hyperparameters: successive inclusion of ordered inputs

See the **"ICSCREAM" methodology** [MIC22]

Building an efficient GPR in practice

1. **Dealing with the large input dimension**

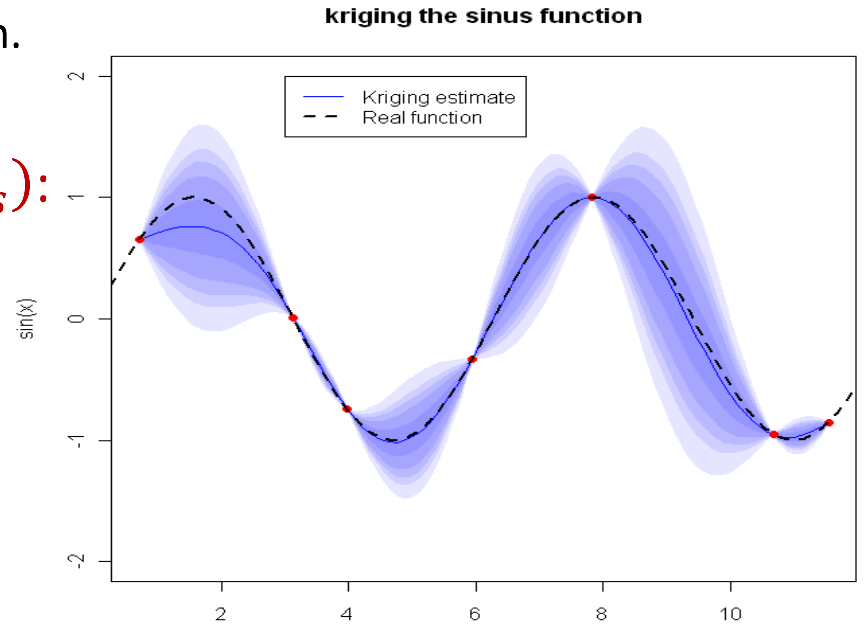
2. **Estimation of hyperparameters and validation**

Reminders on GPR

► **Probabilistic surrogate model**: response is considered as a realization of a random GP field [RW05,Gra21]

$$Y(\boldsymbol{x}) \sim GP(\mu(\boldsymbol{x}), k(\boldsymbol{x}', \boldsymbol{x}))$$

With $\mu(\boldsymbol{x})$ the mean and $k(\boldsymbol{x}', \boldsymbol{x})$ the covariance function.



⇒Predictive GP is the GP conditioned by the observations (X_S, Y_S):

$$Y(\boldsymbol{x}^*)_{|Y(X_S)=Y_S} \sim GP\left(\hat{\mu}(\boldsymbol{x}^*), \hat{s}(\boldsymbol{x}', \boldsymbol{x}^*)\right)$$

With analytical formulations for $\hat{\mu}(\boldsymbol{x}^*)$ *and* $\hat{s}(\boldsymbol{x}', \boldsymbol{x}^*)$

⇒ Conditional mean $\hat{\mu}(\boldsymbol{x}^*)$ serves as the **predictor** at location $\boldsymbol{x}^*$

⇒ Prediction variance (*i.e.* mean squared error) is given by conditional covariance $\hat{s}(\boldsymbol{x}^*, \boldsymbol{x}^*)$

⇒ **Prediction interval** of any level α can be built at any location $\boldsymbol{x}^*$

Reminders on GPR

▶ **In practice:** parametric choices for trend function μ and covariance function k

$$Y(\boldsymbol{x}) \sim GP(\mu(\boldsymbol{x}), k(\boldsymbol{x}', \boldsymbol{x}))$$

⇒ For μ: either **constant** or linear basis

⇒ For k: stationary covariance built-upon tensorized 1-D covariance functions of ν-Matérn

1-Dim ⟶ $k_{\sigma,\nu,\theta}(x,\tilde{x}) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu}h}{\theta}\right)^{\nu} K_\nu\left(\frac{\sqrt{2\nu}h}{\theta}\right)$ ⟶ **3/2 or 5/2 Matérn covariances** offer good properties and « intermediate » regularity

d-Dim ⟶ $k_{\sigma,\nu,\boldsymbol{\theta}}(\mathbf{x},\tilde{\mathbf{x}}) = \sigma^2 \prod_{i=1}^{d} k_{1,\nu,\theta_i}(x_i - \tilde{x}_i)$ with $h = |x - \tilde{x}|$

Hyperparameters
$\boldsymbol{\theta} \in \mathbb{R}^{+,d}$

⇒ Additional variance (nugget effect → nugget hyperparameter $\lambda \in \mathbb{R}^+$)

	$v = \frac{1}{2}$	$v = \frac{3}{2}$	$v = \frac{5}{2}$	$v = +\infty$
Usual name	exponential	3/2-Matérn	5/2-Matérn	Gaussian
$k_{\sigma,\nu,\theta}(x,\tilde{x})$	$\sigma^2 e^{-\frac{h}{\theta}}$	$\sigma^2(1+\sqrt{3}\frac{h}{\theta})e^{-\sqrt{3}\frac{h}{\theta}}$	$\sigma^2\left(1+\sqrt{5}\frac{h}{\theta}+\frac{5}{3}\left(\frac{h}{\theta}\right)^2\right)e^{-\sqrt{5}\frac{h}{\theta}}$	$\sigma^2 e^{-\frac{1}{2}\left(\frac{h}{\theta}\right)^2}$
Differentiability of GP trajectories	$\mathcal{C}^0$	$\mathcal{C}^1$	$\mathcal{C}^2$	$\mathcal{C}^\infty$

Estimation of GPR hyperparameters

$\Rightarrow$ **How to robustly estimate the hyperparameters $\boldsymbol{\theta} \in \mathbb{R}^{+,\boldsymbol{d}}$ from the learning sample?**

How to to ensure that the estimated hyperparameters $\boldsymbol{\theta}$ yield **good predicitivity** but also **reliable GP prediction intervals**?

$\Rightarrow$ Crucial for safety applications

Especially in **« medium » dimension** ($d \in [10, 20]$) and **small dataset** ($n \in [100, 1000]$)

Estimation of GPR hyperparameters

▶ **Usual estimation methods** [KO22,Mur21,Pet22,PBF+23]

→ **Maximum likelihood (MLE)** ⇔ minimization of NLL

→ **Cross-validation and Mean Squared Error** :

minimization of $\mathrm{RMSE} = \left\{\frac{1}{n}\sum_{i=1}^{n}\left(y^{(i)} - \hat{y}_{-i}(\mathbf{x}^{(i)})\right)^2\right\}^{0.5}$

where $\hat{y}_{-i}(\mathbf{x}^{(i)})$ is the metamodel predictor in $\mathbf{x}^{(i)}$ when $(\mathbf{x}^{(i)}, y^{(i)})$ is removed from the learning sample.

Ill-posedness of MLE, problem of **flatness** of functions to be minimized

→ **Bayesian approaches**

CPU ++, delicate choice of priors
Except RobustGAsp method of [GWB18]

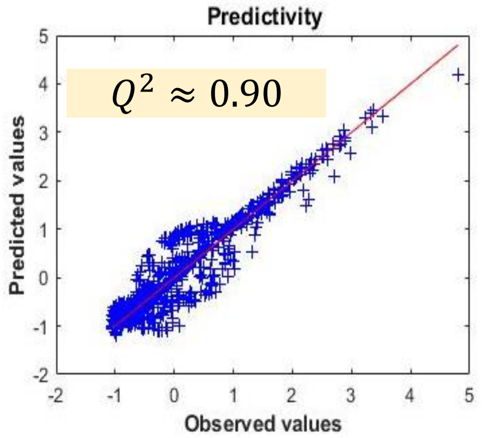
⇒ Could we do better?

⇒ How to check that estimated hyperparameters lead to a "good" GPR metamodel?

Validation of GPR

▶ **Validation criteria computed by cross-validation (LOO or K-fold CV)** [DIG+21, ABG23, MI24a]

→ **Accuracy of the GP predictor (only):** $Q^2 = 1 - \frac{RMSE^2}{\frac{1}{n}\sum_{i=1}^{n}\left(y^{(i)} - \frac{1}{n}\sum_{i=1}^{n} y^{(i)}\right)^2}$



→ **Accuracy of the predictive variance:** $\text{PVA} = \left|\log \frac{1}{n}\sum_{i=1}^{n} \frac{\left(y^{(i)} - \hat{y}_{-i}(\mathbf{x}^{(i)})\right)^2}{\hat{s}_{-i}^2}\right|$

→ **Accuracy of the whole GP conditional distribution**

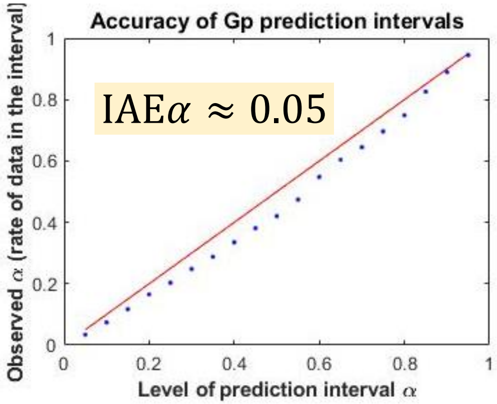
From empirical coverage function for α∈[0,1]: $\hat{\Delta}(\alpha) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{y^{(i)} \in PI_{\alpha,-i}(\mathbf{x}^{(i)})\}$

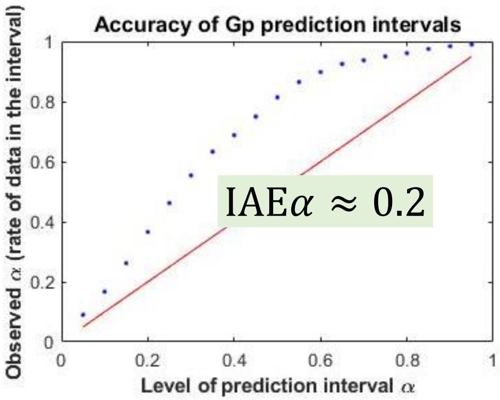
with $PI_{\alpha,-i}(\mathbf{x}^{(i)})$ the α-level GP prediction interval for $\mathbf{x}^{(i)}$ with $(\mathbf{x}^{(i)}, y^{(i)})$ removed from learning sample

⇒ **α-PI Plot**

⇒ **Summarized by Integrated Absolute Error on $\hat{\Delta}(\alpha)$**

$$\text{IAE}\alpha = \int_0^1 \left|\hat{\Delta}(\alpha) - \alpha\right|$$





Building an efficient GPR in practice

1. Dealing with the large input dimension

2. Estimation of hyperparameters and validation

3. New hyperparameter estimation algorithm

From the analysis of estimation & validation criteria...

- **Study of criteria NLL, Q^2, PVA and $IAE\alpha$ on a large benchmark of analytical tests**

  → **Close behavior of NLL and Q^2 $\Rightarrow$ keep NLL as the main estimation objective** to ensure predictivity

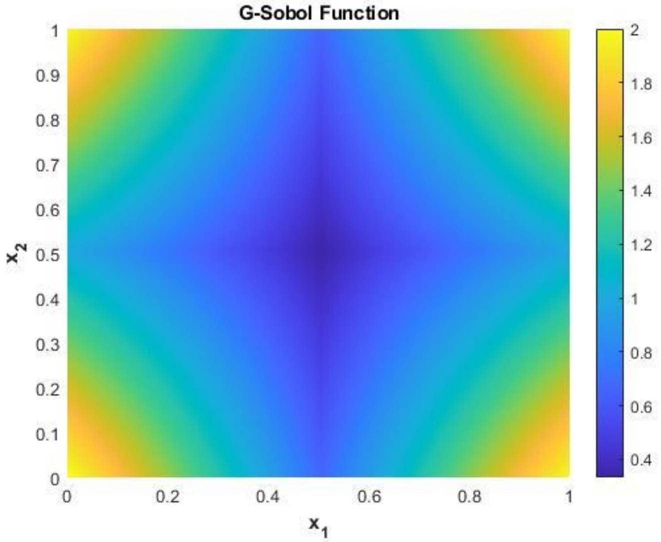
  → *Consistent with [PBF+23,Pet22]*

  → **Similar behavior of PVA and $\mathrm{IAE}\alpha$ but more irregular w.r.t. $\boldsymbol{\theta}$**

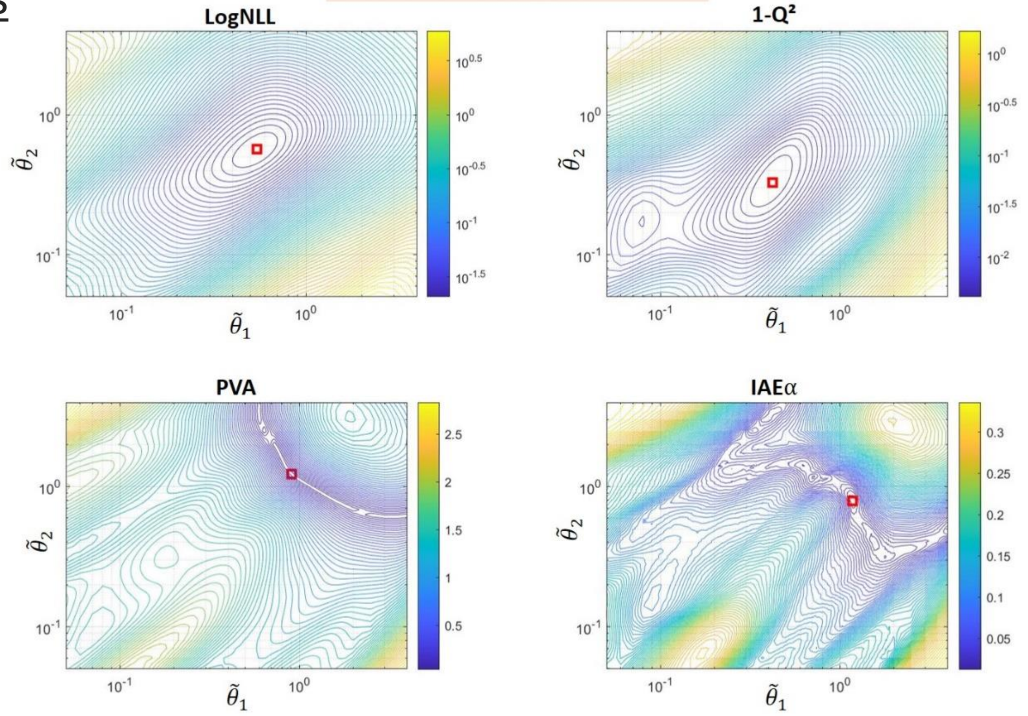
  $\Rightarrow$ Some local minima compatible with optimal values of the other criteria

  $\Rightarrow$ But No to be optimized independently of the others

Illustration with a test on a 2D G-Sobol function



$n = 50$ – Matérn 5/2 Covariance



To a new estimation algorithm!

- **Study of criteria NLL, Q^2, PVA and $IAE\alpha$ on a large benchmark of analytical tests**

→ **Close behavior of NLL and Q^2 $\Rightarrow$ keep NLL as the main estimation objective** to ensure predictivity

→ **$\mathbf{IAE}\alpha$ more directly related to reliable predictive intervals, than PVA**

→ **In the neighborhood of the optimal NLL point, existence of better points θ w.r.t $\mathbf{IAE}\alpha$**, but need to control the possible degradation of Q^2 value, which guarantees the predictivity

$\Rightarrow$ **Optimization based on NLL and $IAE\alpha$ + Control of Q^2**

($IAE\alpha$ and Q^2 estimated by cross validation + use of LOO Dubrule formulas)

$\Rightarrow$ **Proposition of a multi-objective NSGA-II algorithm with constraint on Q^2**

Algorithm flowchart

All details in Marrel and B. Iooss, *Probabilistic surrogate modeling by Gaussian process: A new estimation algorithm for more robust prediction*, Reliability Engineering and System Safety, Volume 247, July 2024, 110120.

Step 1: Initial MLE-based estimation of $\tilde{\theta}$ with standard algorithm

Estimation of hyperparameters by **minimizing the NLL** with a **multistart BFGS algorithm**

↓

$\tilde{\boldsymbol{\theta}}_{MLE}^{init}$

→ Computation of LOO estimator of Q^2 of the GP built-upon $\tilde{\boldsymbol{\theta}}_{MLE}^{init}$:

⇒ $\hat{Q}_{LOO}^{2,init}$

↓

Step 2: Estimation of $\tilde{\theta}$ with constrained multi-objective algorithm

- **2 objectives** to be minimized: **NLL and IAEα** (IAEα computed by LOO)
- **1 constraint** on LOO estimator of Q^2:

  $\hat{Q}_{LOO}^{2,new} \geq \hat{Q}_{LOO}^{2,init} - \gamma$ with $\gamma = 0.05$, e.g.
- Optimization performed by **NSGA-II algorithm, with initial population based on $\tilde{\boldsymbol{\theta}}_{MLE}^{init}$**

↓

Pareto front of n_{Pareto} non-dominated solutions

$\left(\tilde{\boldsymbol{\theta}}_{NGSA,1}^{new}, \dots, \tilde{\boldsymbol{\theta}}_{NGSA,n_{pareto}}^{new}\right)$

→ Computation of LOO estimator of Q^2 for each candidate $\tilde{\boldsymbol{\theta}}_{NGSA}^{new}$:

⇒ $\hat{Q}_{LOO}^{2,new}$

→ ⇒ Each Pareto solution verifies

$\hat{Q}_{LOO,i}^{2,new} \geq \hat{Q}_{LOO}^{2,init} - \gamma$ with Q^2 computed for $\tilde{\boldsymbol{\theta}}_{NGSA,i}^{new}$, for $i = 1, \dots, n_{pareto}$

↓

Step 3: Selection of the "best" solution in Pareto front

3.1 Clustering of Pareto front set of solutions according to the values of both objectives (NLL and IAEα)

→ Number of clusters *k* optimized with **Elbow method**

3.2 Selection of the cluster C1 with optimal NLL (cluster with lower values of NLL)

3.3 Selection of the "best" solution in C1: solution of C1 with optimal (i.e. lowest) value of IAEα

↓

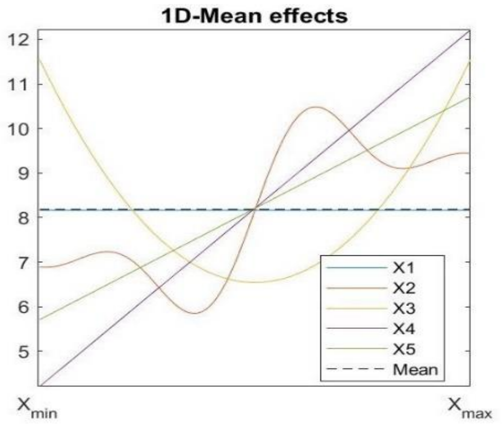
$\tilde{\boldsymbol{\theta}}_{NSGA}^{BestC1}$

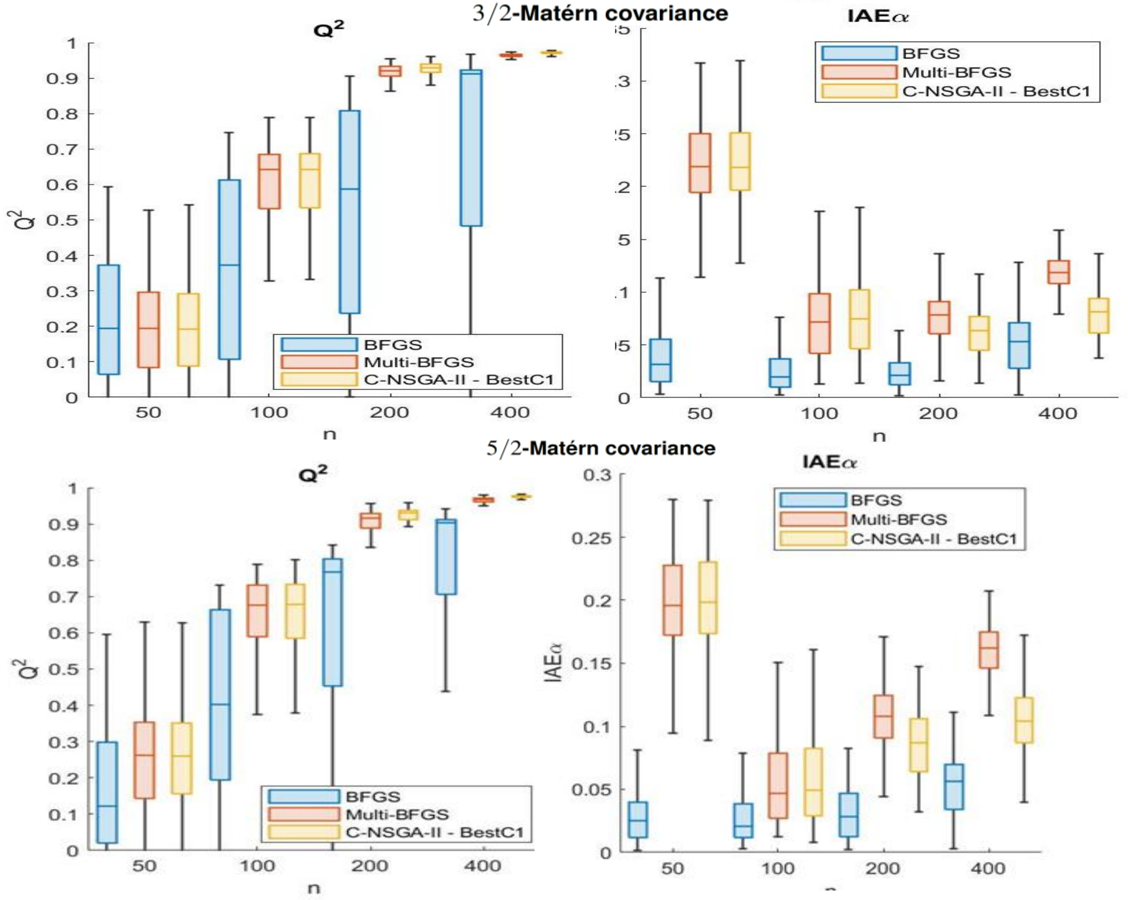
Intensive benchmark on analytical test functions

► Comparison with usual algorithms based on NLL optimization only (BFGS/multistart)

d = 2 to 20, ≠ covariance, ≠ sample sizes, ≠ DoE, with/without nugget effect

Example on Marrel-d20 function : $Y(\boldsymbol{X}) = a_1 \sin[6\pi X_1^{\frac{5}{2}} \left(X_2 - \frac{1}{2}\right) + a_2 \left(X_3 - \frac{1}{2}\right)^2 + a_3 X_4 + a_4 X_5 + r_{X_6,\ldots,X_{15}}$





Results without nugget effect

⇒ **Predictivity with Constr-NSGA-II algorithm at least as good as the simple NLL optimization**

⇒ **Improvement of $IAE\alpha$ especially if** :

- **The model is misspecified, i.e. if the covariance does not match the regularity of the function**
- **When the number of hyperparameters is large** (e.g. large dimension *d* + tensorized anisotropic stationary covariance)

Building an efficient GPR in practice

1. Dealing with the large input dimension

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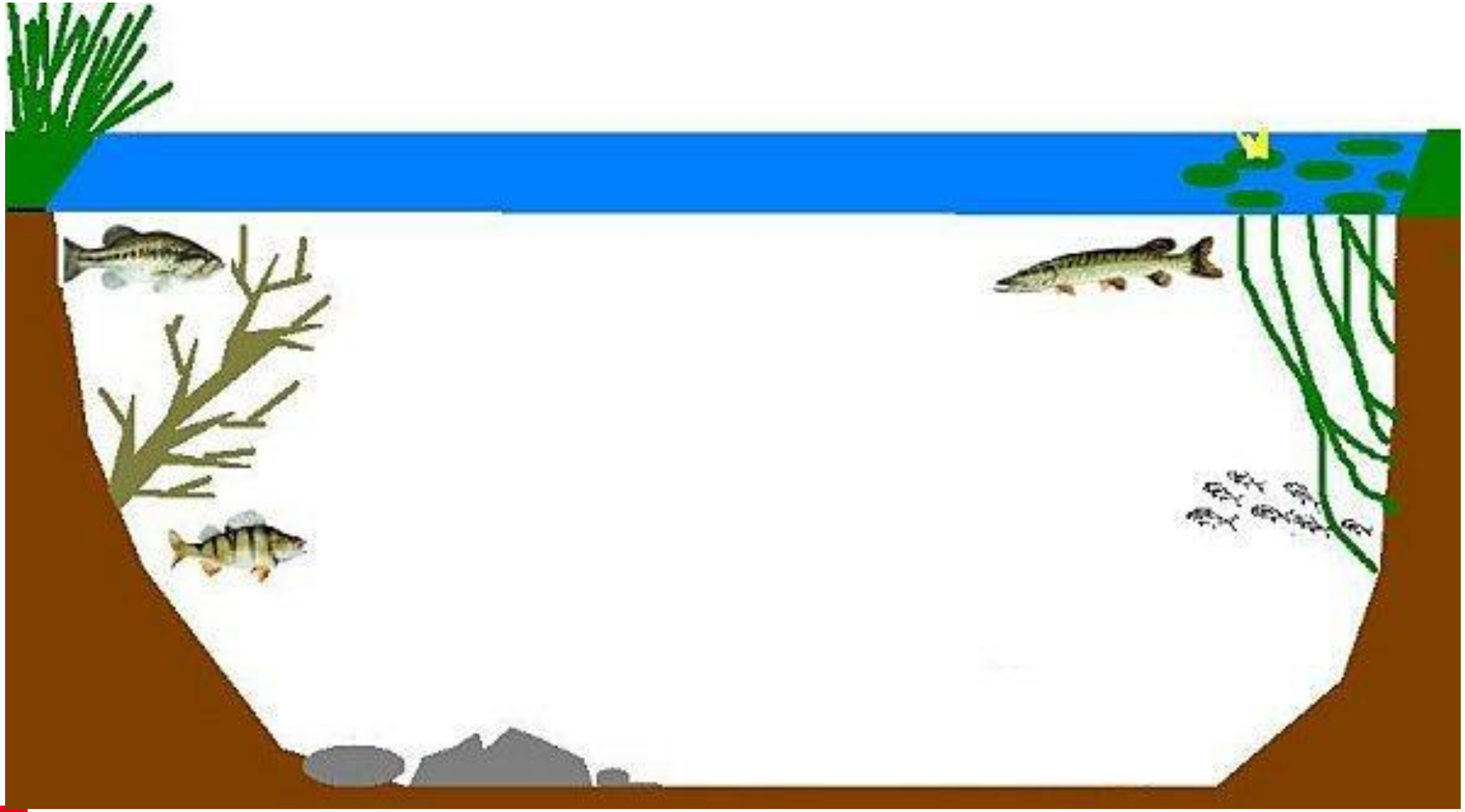
3. New hyperparameter estimation algorithm

4. Illustration on aquatic prey-predator chain model

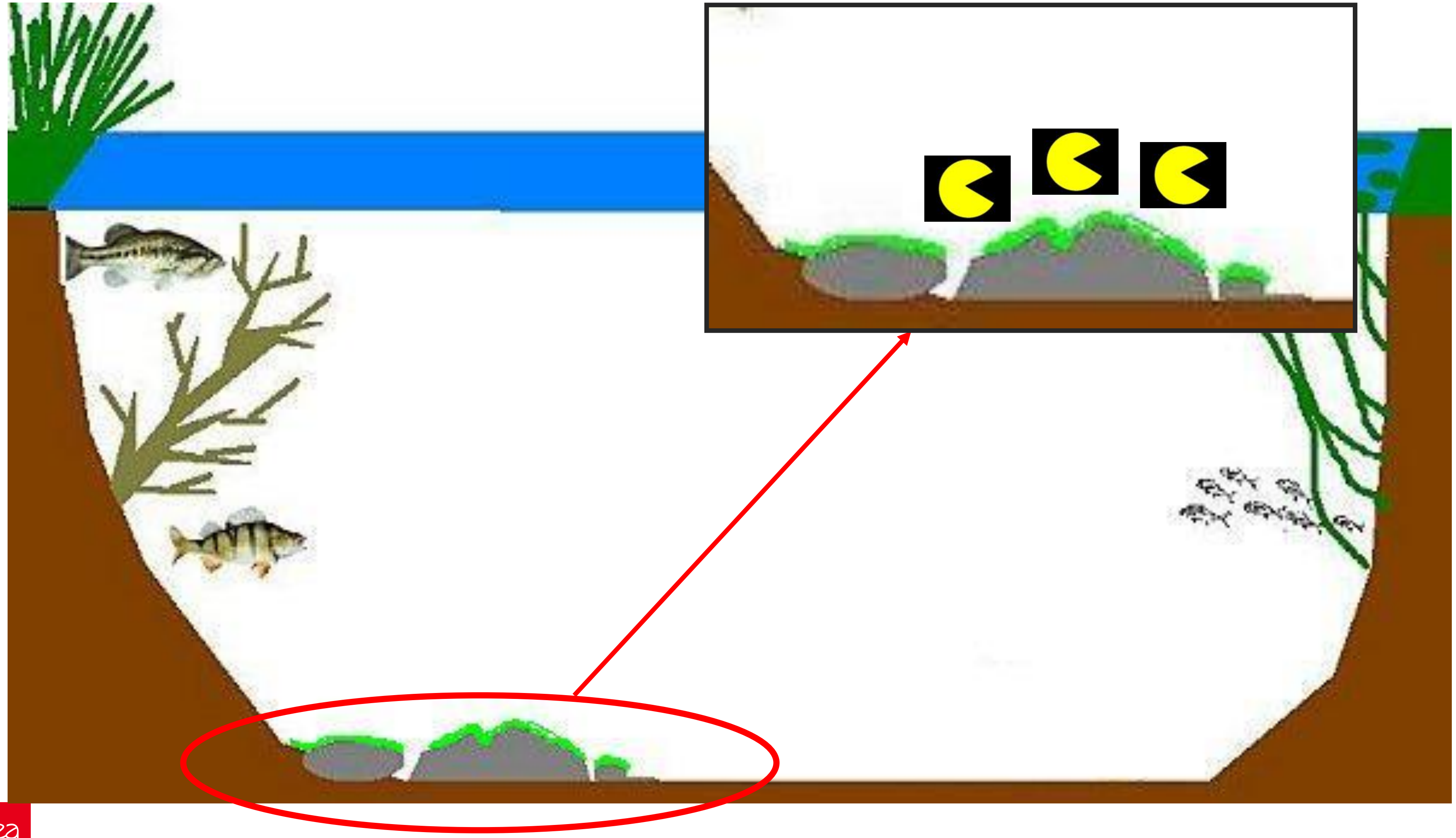
Application: aquatic prey-predator chain model

Studies of biological contamination of rivers



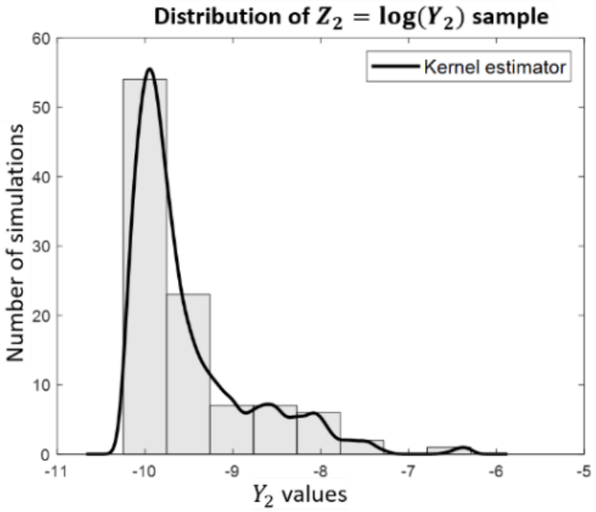
Application: aquatic prey-predator chain model

EDO-type equations describing the growth of microorganisms, grazing and prey-predator interactions



Application: aquatic prey-predator chain model

- **Simulator: MELODY with *d = 20*** uncertain inputs:
  - Periphyton: photosynthesis/mortality/excretion rates, survival temperature, saturation constants, …
  - Grazers: consumption/assimilation/ mortality/excretion rate, survival temperature, …
- **2 outputs of interest**: Periphyton (Y_1) and Grazers (Y_2) biomasses at day 60
- Sample of ***n = 100*** simulations of the model MELODY (from space-filling design)
- Need of **preliminar logarithmic transformation**



⇒ **Lognormal-kriging modeling:**

- Emulation of $Z_i = \log(Y_i)$ with GP regression
- Lognormal-kriging back-transformations to obtain metamodel for Y_i

$$\hat{y}_i(\mathbf{x}) = e^{\left(\hat{z}_i(\mathbf{x}) + 0.5\hat{s}^2_{z_i}(\mathbf{x})\right)}$$

$$\hat{s}^2_Y(\mathbf{x}) = \left(e^{\hat{s}^2_{z_i}(\mathbf{x})} - 1\right) e^{\left(2\hat{z}_i(\mathbf{x}) + \hat{s}^2_{z_i}(\mathbf{x})\right)}$$

- Additional comparison with **Bayesian RobustGaSP approach** *[GWB18]*

Application: aquatic prey-predator chain model

⇒ **With** nugget effect (included in the set of GP hyperparameters to be estimated)

Data	Covariance	Predictivity Coefficient Q^2			IAEα		
		Multi-BFGS	C-NSGA-II-BestC1	RobustGaSP	Multi BFGS	C-NSGA-II-BestC1	RobustGaSP
Y_2	Matern3/2	0,70	0,74	0,25	0,10	0,07	0,04
	Matern5/2	0,77	**0,82**	0,66	0,09	**0,02**	0,07
	Gaussian	0,75	**0,79**	0,66	0,08	**0,02**	0,06

⇒ **Best results with Constr-NSGA-II algorithm**: better Q^2 and IAEα

⇒ **Without** nugget effect

Data	Covariance	Predictivity Coefficient Q^2			IAEα		
		Multi-BFGS	C-NSGA-II-BestC1	RobustGaSP	Multi BFGS	C-NSGA-II-BestC1	RobustGaSP
Y_2	Matern3/2	0,70	**0,75**	0,47	0,10	0,06	0,03
	Matern5/2	0,78	**0,84**	**0,83**	0,08	**0,02**	0,07
	Gaussian	0,70	**0,72**	**0,89**	0,06	**0,03**	0,06

⇒ **Better behavior of RobustGasp without nugget** : best Q^2 but not IAEα

⇒ **Constr-NSGA-II algorithm is more robust to modeling choices** (prior choice of GPR covariance)

Conclusions and remaining challenges

- ✓ GPR benefits greatly from **preliminary HSIC-based screening**
- ✓ GPR calls for **robust estimation of hyperparameters:** considering validation criteria of the whole GP distribution when estimating hyperparameters $\Rightarrow$ enables more robust estimation !
- ✓ Particular attention must be paid to **GP validation**

  $\Rightarrow$ Part of a more general effort to **ensure confidence in machine learning for UQ**

► **Some interesting challenges for UQ applications**

- ✓ Use **more powerful tests** based on SupHSIC [EM24] and HSIC-ANOVA indices [SMD+23]
- ✓ **Screening-free approaches for high dimensional problems** (e.g. beyond 30 to 50 inputs)
- ✓ Learning **outputs with highly irregular**, or even **chaotic behavior** (due to physical threshold phenomena and phenomenological bifurcations, for instance)

References 1/2

Reference of this work

[MI24a] A. Marrel and B. Iooss, Probabilistic surrogate modeling by Gaussian process: A review on recent insights in estimation and validation, Reliability Engineering and System Safety, Volume 247, July 2024, 110120.

[MI24b] A. Marrel and B. Iooss, Probabilistic surrogate modeling by Gaussian process: A new estimation algorithm for more robust prediction, - Reliability Engineering and System Safety, Volume 247, July 2024, 110094.

Work funded by ANR SAMOURAI research project

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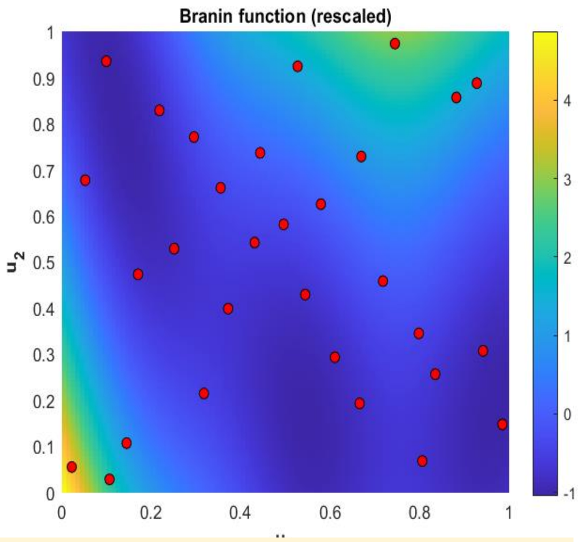
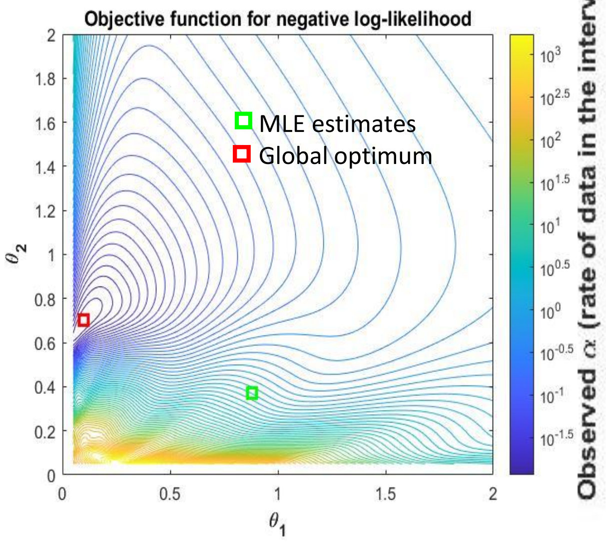
Appendix

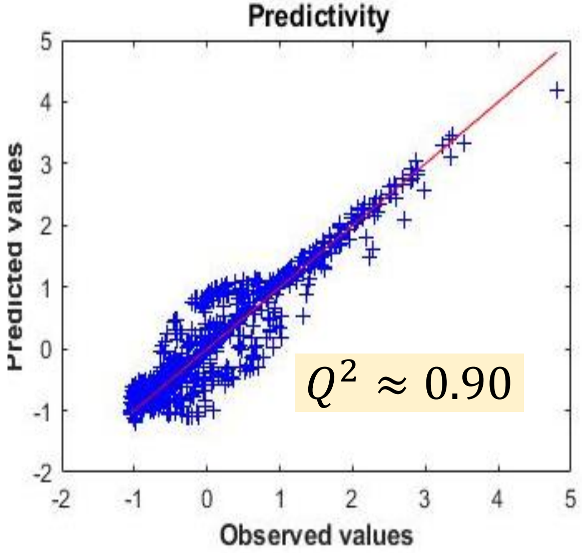
Illustration of criteria for GPR validation [MI24a]

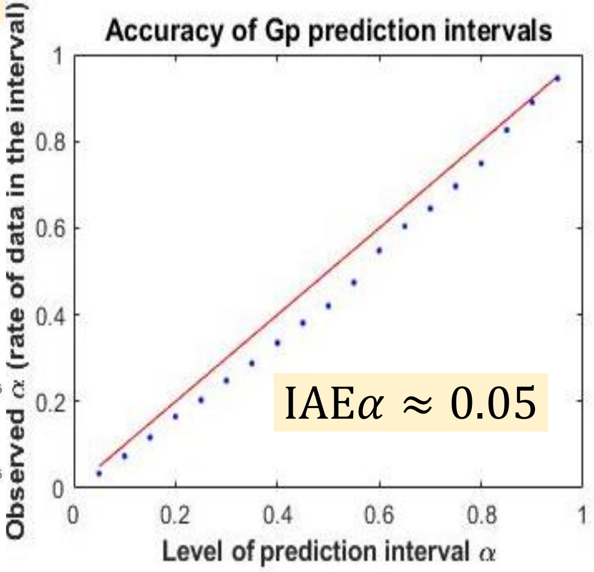


n=30, GPR with constant mean and Gaussian covariance

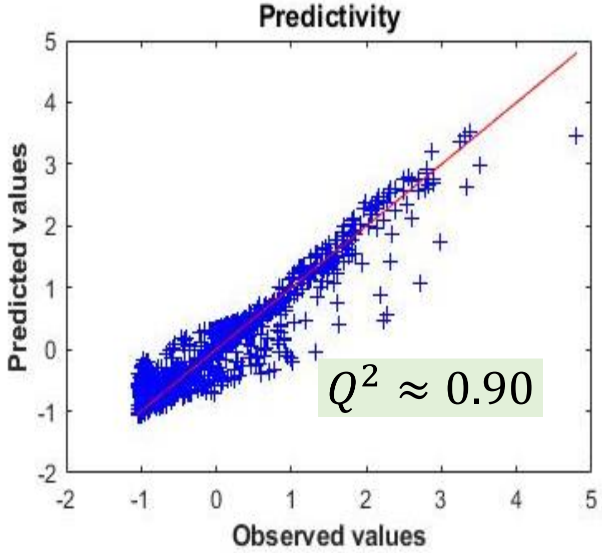


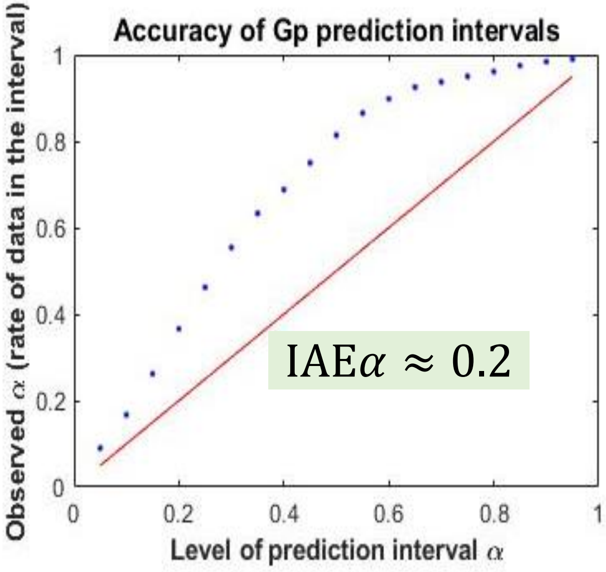
$[\theta_1, \theta_2] = [1.12 \quad 0.8]$



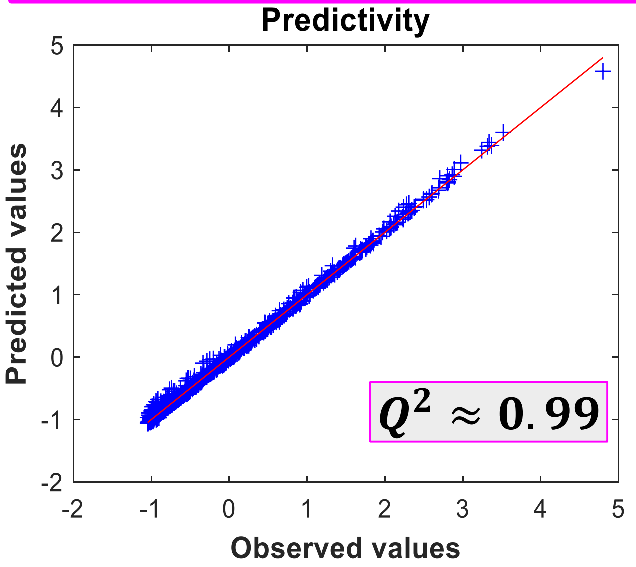


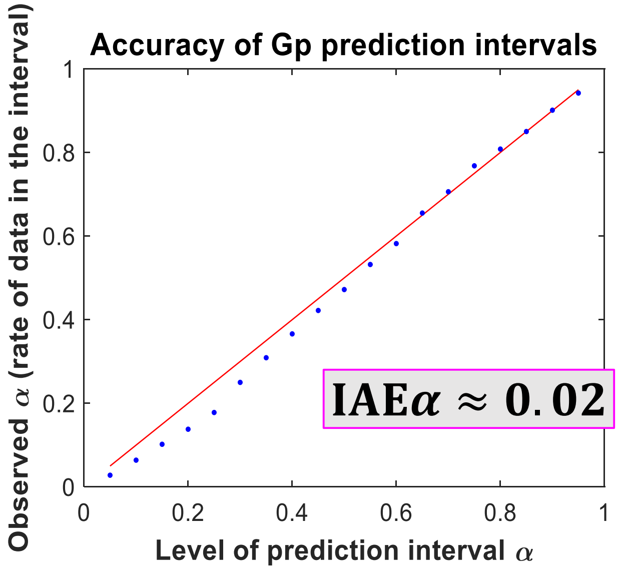
$[\theta_1, \theta_2] = [0.78 \quad 0.52]$





MLE estimates: $[\theta_1, \theta_2] = [0.88 \;\; 0.37]$





Dealing with the large input dimension

- **HSIC-based ranking** [Dav15] **:**

$$R^2_{HSIC} = \frac{HSIC\ (X,Y)}{\sqrt{HSIC(X,X)HSIC\ (Y,Y)}}$$

⇨ $R^2_{HSIC} \in [0,1]$ for easier interpretation

$$\textbf{Influence}(X_{[1]}) > \textbf{Influence}(X_{[2]}) > \cdots > \textbf{Influence}(X_{[d]})$$

Where order $[\cdot]$ is such that $\hat{R}^2_{H,X_{[1]}} > \hat{R}^2_{H,X_{[2]}} > \cdots > \hat{R}^2_{H,X_{[d]}}$

⇒ Use for ranking of inputs

Inputs ordered by degree of influence

Can be used for **more robust sequential GPR estimation**

⇒ "forward" estimation of GPR hyperparameters: successive inclusion of ordered inputs

HSIC review: a kernel-based GSA method

- **MMD² applied between $P_{X_i Y}$ and $P_{X_i} \otimes P_Y \Rightarrow HSIC(X_i, Y)_{\mathcal{H}_{X_i}, \mathcal{H}_Y}$**

$\mathcal{H}_{X_i}$ and $\mathcal{H}_Y$ **RKHS** associated to X_i and Y, resp :

Kernel $k_{X_i}: \mathcal{X}_i \times \mathcal{X}_i \to \mathbb{R}$ with feature space $\mathcal{H}_{X_i}$ and feature map φ_{X_i}

Kernel $k_Y: \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ with feature space $\mathcal{H}_Y$ and feature map φ_Y

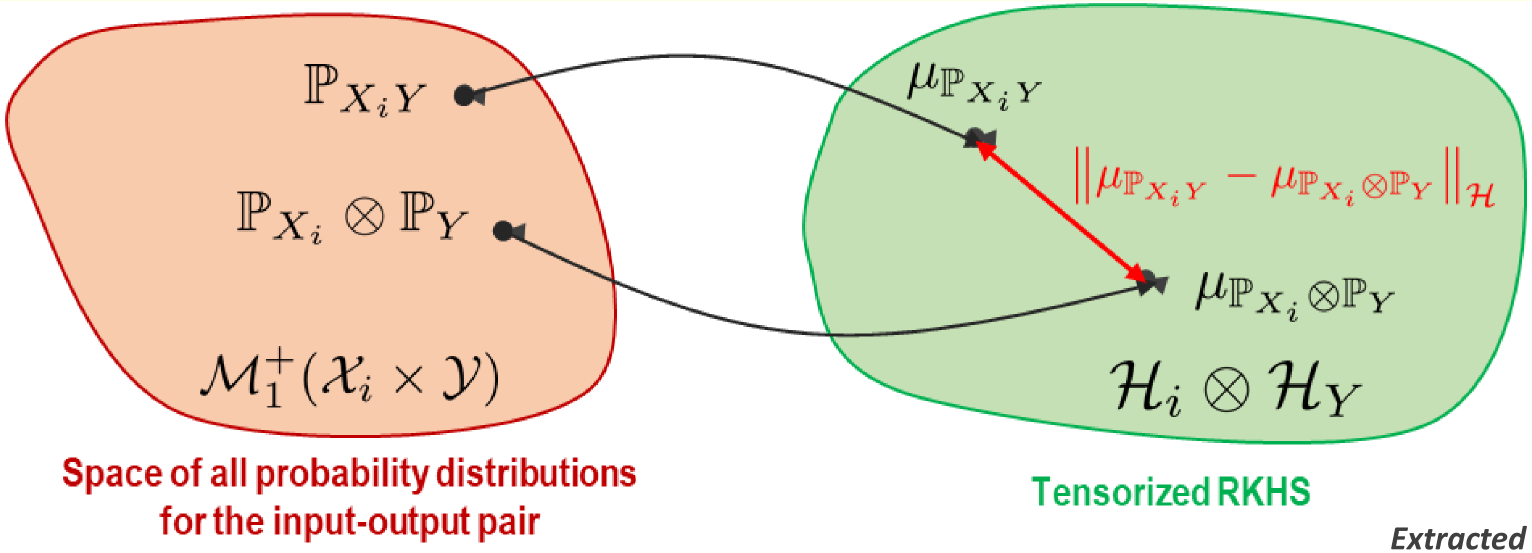
$K_{X_i}(x, x') = \langle \varphi_{X_i}(x), \varphi_{X_i}(x') \rangle_{\mathcal{H}_{X_i}}$ and $K_Y(y, y') = \langle \varphi_Y(y), \varphi_Y(y') \rangle_{\mathcal{H}_Y}$

kernel defines the inner product in the RKHS

HSIC = distance in the RKHS between the images of the two distributions of interest

Gretton et al. [2005]

$$\Rightarrow HSIC(X_i, Y)_{\mathcal{H}_{X_i}, \mathcal{H}_Y} = MMD^2_{\mathcal{H}_{X_i}, \mathcal{H}_Y}(P_{X_i Y}, P_{X_i} \otimes P_Y) = \left\| \mu_{\mathbb{P}_{X_i Y}} - \mu_{\mathbb{P}_{X_i} \otimes \mathbb{P}_Y} \right\|^2_{\mathcal{H}_{X_i}, \mathcal{H}_Y}$$



Extracted from G. Sarazin's (CEA) slides

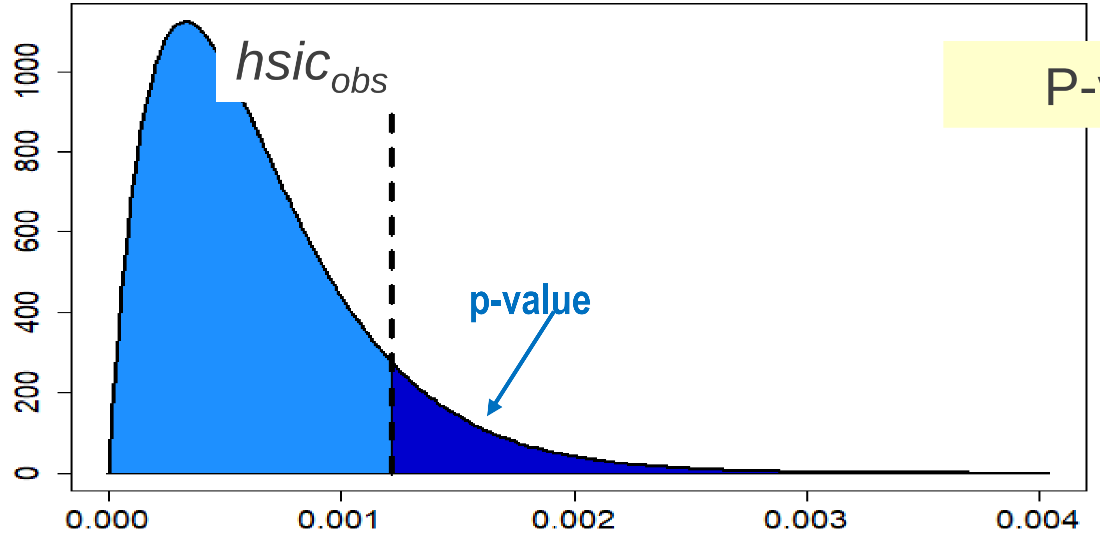
HSIC review: a kernel-based GSA method

HSIC-based independence tests for screening

How to have the distribution $n\widehat{\mathrm{HSIC}}(\mathrm{X_i}, \mathrm{Y})$ under $\mathcal{H}_0$ to compute *p-value*?

- ► **If *n* large: asymptotic test based** on approximation with Gamma law *(Gretton et al. (2008])*
- ► **If *n* small: Permutation-based** approximation *(De Lozzo & Marrel [2016a], Meynaoui [2019], El Amri & Marrel [2021a])*

Gamma distribution



P-value = $Pr\,[\,\widehat{\mathrm{HSIC}}(X_i, Y) > hsic_{obs}\,]$

<u>Interpretation of *p-value* for a level α ($\alpha = 5\%$ or 10%) for screening:</u>

➢ **<u>pval < α</u>** ⇒ H_0 (Independence) rejected ⇒ X_i **is significantly influential**