Statistical characterization for nuclear dismantling applications with small data sets

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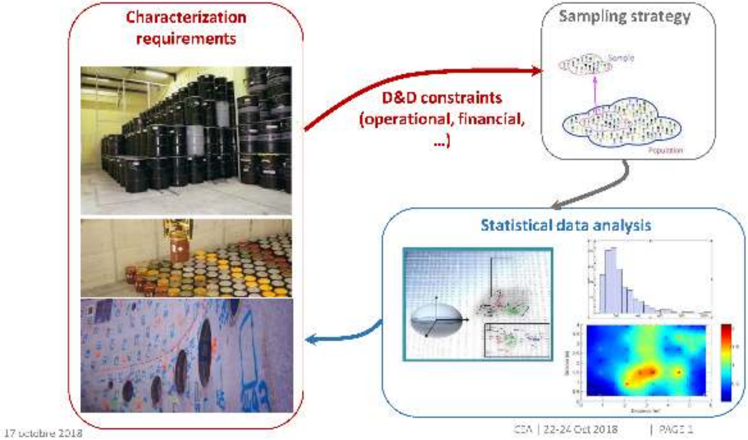
SFdS/MASCOT-NUM meeting - Big ideas for small data

Context


Characterization requirements
D&D constraints (operational, financial, ...)
Sampling strategy
Sample
Population
Statistical data analysis
17 octobre 2018
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| PAGE 1

Problem statement

Context : Radiological characterization of contaminated elements from nuclear facilities

Problem : Small number of available data

$\longrightarrow$ Inappropriate statistical tools (e.g. Gaussian approximation) to determinate risk confidence bounds

Ex : The 2σ rule (95% of values inside $\pm 2\sigma$) works in the Gaussian case

Risks of a wrong estimation of the contamination : Under-estimation (impact on safety) or over-estimation (impact on economic cost)

Strategy : Resort to robust inequalities which only depend on weak assumptions about the statistical distribution of the measured quantity

Outline

1. Prediction, tolerance and confidence intervals
2. Application to real measurements

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Probabilistic framework

- Consider a set of measures $\mathcal{X} = \{X_1, \ldots, X_n\}$ of a given quantity
- They are assumed to be independent copies of a continuous random variable X with unknown distribution (but finite mean and variance)
- In the context of risk analysis, it is relevant to estimate from the data the three following kinds of probabilistic intervals :

Unilateral prediction interval	Unilateral tolerance interval	Bilateral confidence interval on $\mu = \mathbb{E}[X]$
$\mathbb{P}[X \leqslant s] \geqslant \gamma$	$\mathbb{P}[\mathbb{P}[X \leqslant s] \geqslant \gamma] \geqslant \beta$	$\mathbb{P}[s_1 \leqslant \mu \leqslant s_2] \geqslant \gamma$

s, s_1, s_2 : threshold values γ, β : prescribed probabilities (e.g. 95%)

$\alpha = 1 - \gamma$: probabilistic risk bound ; then $\mathbb{P}[X \geqslant s] \leqslant \alpha$

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Intervals based on Gaussian approximation

Notation : $X \sim \mathsf{N}(\mu, \sigma)$, $\mu = \mathbb{E}[X]$, $\sigma^2 = \mathbb{V}\mathrm{ar}[X]$

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$, $z_u = u$-$\mathcal{N}(0,1)$-quantile

Gaussian case with known (μ, σ) : The exact α-prediction interval is

$$s = \mu + \sigma z_{1-\alpha}$$

Gaussian case with unknown (μ, σ) : The exact α/β-tolerance interval is

$$s = \bar{X}_n + t_{n-1,\beta,\sqrt{n}z_{1-\alpha}} \frac{S_n}{\sqrt{n}}$$

However these are only approximations if X is not Gaussian, which may reveal poor if n is small and/or X is highly skewed

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Intervals based on concentration inequalities

We recall that we look for $\mathbb{P}[X \geqslant s] \leqslant \alpha$

Concentration inequalities give

$$s = \bar{X}_n + t \quad \text{and} \quad \alpha = \left(1 + \frac{t^2}{kS_n^2}\right)^{-1}$$

with $t \geq 0$ and k a positive constant

In practice, either s is fixed, either α is fixed (then t is directly recovered)

Inequality name	Value of k	Assumptions
Bienaymé-Chebyshev (BC)	1	None
Camp-Meidell (CM)	$4/9$	Unimodal pdf
Van Dantzig (VD)	$3/8$	Convex pdf tails

Note : Camp-Meidell inequality gives the so-called "3σ rule"

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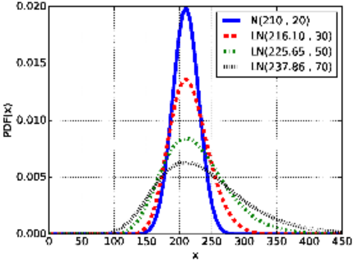
Examples of risk estimates with known distributions of X



$\alpha = 0.05$	$\mathcal{N}(210, 20)$	$\mathcal{LN}(216, 30)$	$\mathcal{LN}(226, 50)$	$\mathcal{LN}(238, 70)$
Gauss	0.05	0.04	0.04	0.03
BC	0.27	0.25	0.23	0.23
CM	0.14	0.13	0.12	0.12
VD	0.12	0.11	0.10	0.10

Extension to tolerance intervals by bootstrapping

A β-confidence level is required due to the empirical estimation of the mean and standard deviation

$$\mathbb{P}\left[\mathbb{P}\left[X \geqslant s\right] \leqslant \alpha\right] \geqslant \beta$$

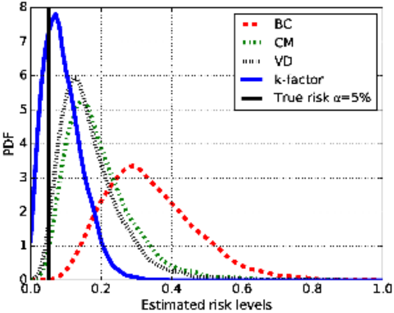
From sample $\mathcal{X} = \{X_1, \ldots, X_n\}$, we repeat B times (e.g. $B = 500$) :

- Create a new n-size sample $\mathcal{X}'$ by sampling with replacement in $\mathcal{X}$,
- Compute $\bar{X}_n$ and S_n,
- If s (resp. α) is fixed, compute t and α (resp. s)

From the B-size sample of α values (resp. s values), take the β-quantile of α (resp. s)

Numerical experiments : $X \sim \mathcal{LN}(238, 70)$ and $n = 30$

Keep $\beta = 95\%$-quantile of bootstrap sample ($B = 500$) of α estimates



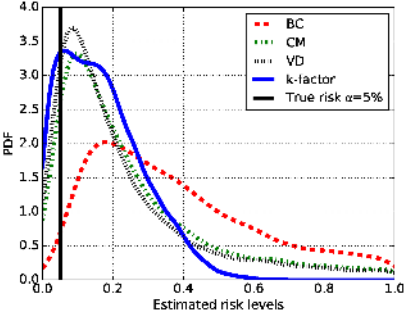
Statistical distributions of the quantiles ($N = 5000$ repetitions)

Proportion of non-conservative estimates of the exact $\alpha = 5\%$:

k-factor	BC	CM	VD
0.25	0.00	0.00	0.01

Numerical experiments : $X \sim \mathcal{LN}(238, 70)$ and $n = 10$

Keep $\beta = 95\%$-quantile of bootstrap sample ($B = 500$) of α estimates



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Outline

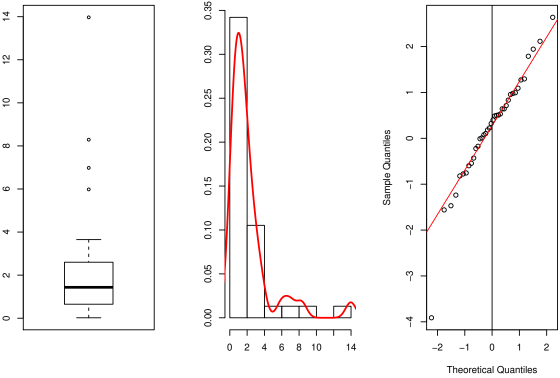
Data : H_2 flow rates of radioactive waste drums

Evaluation of H_2 flow rates (in l / drum / year) required for disposal in final waste repositories

Population of several thousands drums

Data : H_2 flow rates of radioactive waste drums

Measures on a random sample of size $n = 38$, $(\bar{X}_n, S_n) = (2.18, 2.67)$



Adequacy to a parametric distribution (as the log-normal one) is rejected by statistical tests

Some results obtained with the Camp-Meidell inequality

Estimation of the risk α of threshold exceedance ($\beta = 0.95$) :

$$s = 5 \text{ gives } \alpha = 58\%$$
$$s = 10 \text{ gives } \alpha = 11\%$$
$$s = 15 \text{ gives } \alpha = 4\%$$

Estimation of the relative error on the mean flow rate (the empirical mean is equal to 2.18 l/drum/year) :

- $31\% =$ relative error on the estimation of the mean H_2 flow rate with $(\alpha, \beta) = (0.75, 0.95)$
- $93 =$ sample size required to reach a 20%-relative error on the estimation of the mean with $(\alpha, \beta) = (0.75, 0.95)$

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Conclusions and prospects

- Be careful with the Gaussian approximation especially for small data samples
- Concentration inequalities provide robust risk bound and confidence interval for the mean
- Their degrees of conservatism are linked to explicit assumptions on the distribution of the studied variable

- Apply more sophisticated concentration inequalities in order to give tighter bounds

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