Sea random models, Rices formula for level sets: crests, specular points and test of isotropy .

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Introduction

When a photography of the sea water is taken one can appreciate different luminous points on the surface. These points are where the reflection of the light happens and have their origin when the light is reflected in agreement to Snell's Law from zones that act as small mirrors (specular points). This a phenomenon that could be described by means of the levels sets of the a random Gaussian field modeling the sea.

Moreover looking at the sea surface, or at the surface of swimming pool, when the light hits it the observer sees luminous curves that evolves for certain time and then collapses. In these curves the light focuses then a flash (twinkle) appears where the curve collapses. Another types of phenomena can be also explain by using the theory of the level sets of random fields. In this talk we will give several examples of the same situation.

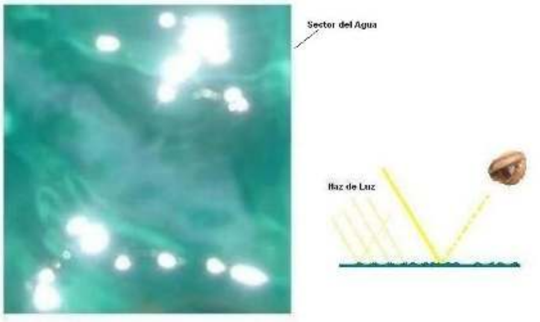


Figure: Specular points

Sea model

The sea level in the location (x, y) and at time t, can be modeled by the following superposition of random waves $X(x, y, t) = \sum_n \mathrm{c}_n \cos(u_n x + v_n y + \omega_n t + \varepsilon_n)$, This model was proposed in the year of 1957 by Longuet-Higgins. The c_n are independent Gaussian random variables and the Airy relation between the spatial and temporal frequencies holds $\omega_n^2 = \sqrt{u_n^2 + v_n^2}$. The above expression could be generalized using the stochastic integral thus $X(x, y, t) = \int_{\mathbb{S}} e^{i(ux+vy+\omega t)} dM(u, v, \omega)$, where $\mathbb{S}$ is the manifold $\{\omega^2 = \sqrt{u^2 + v^2}\}$. This representation is often used for simulations.

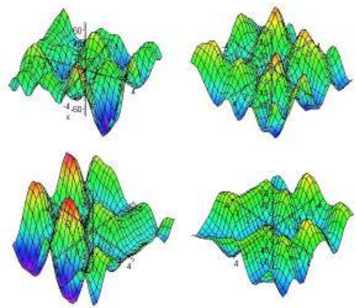


Figure: Some simulations of the random sea

In Figure 1 each of the four images corresponds to simulations of the movement of the sea.

The directional spectrum

There exits another representation often used in engineering literature. It consists in making the polar change of variables $u = \omega^2 \cos\theta$ and $v = \omega^2 \sin\theta$, obtaining

$$X(x, y, t) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} e^{i(\omega^2(x\cos\theta + y\sin\theta) + \omega t)} dS(\omega, \theta).$$

Here dS is a random measure having orthogonal increments. We suppose the existence of the directional spectral measure $f(\omega, \theta) d\omega d\theta = \mathbb{E}[dS(\omega, \theta) dS(\omega_1, \theta_1)]$. In this manner the covariance function results

$$r(x, y, t) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} e^{i(\omega^2(x\cos\theta + y\sin\theta) + \omega t)} f(\omega, \theta) d\omega d\theta,$$

and the spectral moments are

$$m_{klm} = \int_{\mathbb{R}} \int_{-\pi}^{\pi} (\omega^2 \cos\theta)^k (\omega^2 \sin\theta)^l \omega^m f(\omega, \theta) d\omega d\theta.$$

We present an useful example

The directional spectrum of JONSWAP-Cos2s:

The mathematical expression and the graph in polar coordinates of the spectrum of JONSWAP-Cos2s are:

$$\mathrm{S}(\omega,\theta) = \alpha \frac{\mathrm{g}^2}{\omega^5} \exp\left[-\frac{5}{4}\left(\frac{\omega_p}{\omega}\right)^4\right] \gamma^{\delta} \frac{2^{2s-1}}{\pi} \frac{\Gamma^2(s+1)}{\Gamma(2s+1)} \cos^{2s}\left(\frac{\theta-\theta_0}{2}\right)$$

where $-\pi < \theta < \pi$, $s > 0$, $\delta = \exp\left[-\left(\frac{\omega-\omega_p}{\sqrt{2}\sigma_0\omega_p}\right)^2\right]$, ω is the angular frequency, g the gravity acceleration, α is a factor of normalization, $\gamma \simeq 3.3$ and $\sigma_0 = \begin{cases} 0.07 & \text{si } \omega \leq \omega_p \\ 0.09 & \text{si } \omega > \omega_p. \end{cases}$

Below is given the graph of this spectrum for certain parameters.

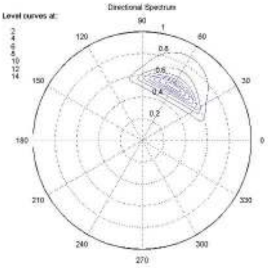


Figure: Directional Spectrum JONSWAP-Cos 2s y $\Theta_0 = \frac{\pi}{3}$

Multidimensional Rice's Formula

The problems that we mentioned in the introduction are in relationship with multidimensional extensions of the well known Rice Formula. We must introduce in this section two theorems that provide the context for the applications that we have in mind.
In what follows σ_d denotes the Lebesgue measure of $\mathbb{R}^d$.
Let us consider $\mathbf{x} \in \mathbb{R}^d$ and $X : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ a vectorial random field and denote $X'(\cdot)$ its Jacobian. We define $N_Q^X(\mathbf{y}) = \#\{\mathbf{x} \in Q \subset \mathbb{R}^d : X(\mathbf{x}) = \mathbf{y}\}$, count the number of roots in Q of the random function X.
The following formula of change of variables for non globally invertible function is known as the Area formula:

$$\int_{\mathbb{R}^d} f(\mathbf{y}) N_Q^X(\mathbf{y}) d\mathbf{y} = \int_Q f(X(\mathbf{x})) |\det X'(\mathbf{x})| d\mathbf{x}$$

An application of the Area formula

Theorem 1 (Azaïs & Wschebor) Let $X : A \subset \mathbb{R}^d \to \mathbb{R}^d$ be a Gaussian random field, A a compact set of $\mathbb{R}^d$, $\mathbf{y}$ a vector of $\mathbb{R}^d$, and the field X satisfying the following conditions:

1. the trajectories of X are continuously differentiable
2. for each $\mathbf{x} \in A$ the distribution of $X(\mathbf{x})$ is no degenerate
3. $\mathbb{P}(\{\exists \mathbf{t} \in \dot{A} : X(\mathbf{t}) = \mathbf{y}\,, \det(X'(\mathbf{t})) = 0\}) = 0$
4. $\lambda_d(\partial(A)) = 0$.

Then

$$\mathbb{E}N_A^X(\mathbf{y}) = \int_A \mathbb{E}[|\det X'(\mathbf{x})|/X(\mathbf{x}) = \mathbf{y}]p_{X(\mathbf{x})}(\mathbf{y})d\mathbf{x}.$$

And both members are finite.

The Azaïs & Wschebor work provides condition under which hypothesis (3) holds.

Let $X : \mathbb{R}^d \to \mathbb{R}^{d'}$ be a continuously differentiable random field, with Jacobian $X'(\cdot)$, and $d' > d$, we define the level set $\mathcal{C}_Q(\mathbf{y}) = \{\mathbf{x} \in Q : g(\mathbf{x}) = \mathbf{y}\} = X^{-1}(\mathbf{y}) \cap Q$, where Q is a compact set of $\mathbb{R}^d$.

If we define as $\sigma_{d-d'}$ the Hausdorf measure for subspaces of dimension $d - d'$. The Coarea formula reads

$$\int_{\mathbb{R}^{d'}} f(\mathbf{y}) \int_{C_Q(\mathbf{y})} Y(\mathbf{z}) d\sigma_{d-d'}(\mathbf{z}) d\mathbf{y} = \int_Q f(X(\mathbf{x}))(\det X'(\mathbf{x}) X'(\mathbf{x})^T)^{1/2} d\mathbf{x}$$

An application of the Coarea formula

Theorem 2 Let $X : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d'}$, with $d > d'$, be a Gaussian random field belonging to $\mathbf{C}^1(\mathbb{R}^d, \mathbb{R}^{d'})$, with $\operatorname{Var} X(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^d$. $X'(\mathbf{x})$ is a.s α-Hölder continuous with $1 - \frac{1}{d+d'} < \alpha \leq 1$.

It holds for all $\mathbf{y} \in \mathbb{R}^{d'}$

$$\mathbb{E}[\sigma_{d-d'}(\mathcal{C}_Q(\mathbf{y}))]$$

$$= \int_Q \mathbb{E}[(\det X'(\mathbf{x})X'(\mathbf{x})^T)^{1/2}/X(\mathbf{x}) = \mathbf{y}]p_{X(\mathbf{x})}(\mathbf{y})d\mathbf{x} \text{ and}$$

$$\mathbb{E}[\int_{\mathcal{C}_Q(\mathbf{y})} Y(\mathbf{x})d\sigma_{d-d'}(\mathbf{x})] =$$

$$\int_Q \mathbb{E}[Y(\mathbf{x})(\det X'(\mathbf{x})X'(\mathbf{x})^T)^{1/2}/X(\mathbf{x}) = \mathbf{y}]p_{X(\mathbf{x})}(\mathbf{y})d\mathbf{x}$$

Specular points, twinkles and other level functionals

We display here two pictures of specular points in the water.

Figure: Specular points in the Guri dam in Venezuela

Figure: Specular points in Malta

- **Number of specular points.**
  We consider now a zero mean stationary Gaussian process $X(t,x)$ depending of two parameters (t,x) say time and location. Defining the process

$$Y(t,x) = X(t,x) + \frac{1}{2}kx^2,$$

we regard it at a fixed time t, for instance $t = 0$. We have

$$Y(x) := X(0,x) + \frac{1}{2}kx^2.$$

The process $X(0,x)$ has zero mean, is stationary and Gaussian. We put $\mathbf{E}X^2 = 1$, to simplify the notation.

Then, at first approximation, we say that there exists a specular point reflected by a beam of light emitted at distance h_1 and seem by an observer at distance h_2 ($k = \frac{1}{h_1} + \frac{1}{h_2}$) if $Y'(x) = 0$ and the quantity $N^S(k) = \#\{0 < x < \infty : Y'(x) = 0\}$, counts the number of these points. Observe that we are looking for the crossings of the derivative of Y. We obtain by using the Rice formula:
$\mathbb{E}[N^S(k)] = \int_0^\infty p_{Y'(x)}(0)E|Y''(x)|dx$

$$= \mathbb{E}|\mathcal{N} + \frac{k}{\sqrt{m_4}}|\frac{1}{2\pi}\sqrt{\frac{m_4}{m_2}}\int_0^\infty e^{-\frac{k^2x^2}{2m_2}}dx,$$

where $\mathcal{N}$ is a standard Gaussian, m_2 and m_4 are the second and forth spectral moments corresponding to $X(0, x)$.

In a joint work with Azïs & Wschebor we have obtained also the following CLT $\frac{1}{\sqrt{k}}(N^S(k) - \mathbb{E}[N^S(k)]) \stackrel{k \to 0}{\to} N(0, \sigma_S^2)$.The limit variance is also explicitly computed.

The heuristic is that if the observer and the emitter are placed at infinite distance from the sea surface, the random variable $N^S(\infty)$ is infinite. The result precises this convergence.

- **Number of twinkles**

  Consider now the same problem but adding time, and use the above approximation, specular points are defined by

$$X_x(x,t) = kx, \qquad \text{again } Y = X_x(x,t) - kx$$

where X_x denotes the derivative with respect to x. This equation defines, in general, a finite number of points that move with time. The implicit function theorem, when it can be applied, shows that a specular point moves at the speed

$$\frac{dx}{dt} = -\frac{X_{xt}}{X_{xx} - k},$$

where, for example, X_{xx} is the second derivative with respect to x. This quantity diverges whenever $X_{xx} - k = 0$. In this conditions, a flash appears and the point is called a "twinkle".

We are interested in computing the expectation of the number of flashes over a certain set of space A and an interval of time $[0, T]$

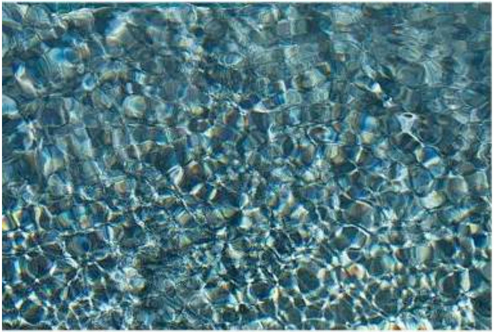


Figure: Twinkles

The number of twinkles is defined as

$$\mathcal{TW}(A,T) := \sharp\{(x,t) \in A \times [0,T] : \mathbf{Y}(x,t) = 0\}.$$

Thus

$$\mathbb{E}\mathcal{TW}(A,T) = \int_A \mathbb{E}[|\det \mathbf{Y}'(0,x)|/\mathbf{Y}(0,x) = \mathbf{0}]p_{0,x}(\mathbf{0})dx.$$

Where we have define $\mathbf{Y} = (Y_x(t,x), Y_{xx}(t,x))^T : \mathbb{R}^2 \to \mathbb{R}^2$, and $p_{0,x}(\cdot)$ the density of the vector $\mathbf{Y}(0,x)$. Again the process $\mathbf{Y}$ must satisfy the hypothesis of Theorem 1.

- **Specular points in two dimensions**
  We consider a two parameter vectorial stationary stochastic field $\mathbf{X}(x_1,x_2)$ describing a random surface that is supposed two times continuously differentiable. By using the same reasoning that in one dimension, defining the field

$\mathbf{Y}(x_1, x_2) = \mathbf{X}'(x_1, x_2) - k(x_1, x_2)^T$, we say that there exists a specular point whenever $\mathbf{Y}(x_1, x_2) = 0$. Denoting $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{y} = (y_1, y_2)^T$ and using independence we get

$$\mathbb{E}N_Q^{\mathbf{Y}}(\mathbf{0}) = \int_Q \mathbb{E}[|\det \mathbf{Y}'(\mathbf{x})|] p_{\mathbf{Y}(\mathbf{x})}(\mathbf{0}) d\mathbf{x}$$

$$= \mathbb{E}[|\det \mathbf{Y}'(\mathbf{0})|] \int_Q p_{\mathbf{Y}(\mathbf{x})}(\mathbf{0}) d\mathbf{x}$$

$$= \mathbb{E}[|\det \mathbf{Y}'(\mathbf{0})|] \int_Q p_{\mathbf{X}'(\mathbf{0})}(-kx_1, -kx_2) dx_1 dx_2. \qquad (1)$$

The difficulty here is to compute $\mathbb{E}[|\det \mathbf{Y}'(\mathbf{x})|]$. Two methods can be used. The first one use a Gaussian regression and Monte Carlo, the other one consists in to use a representation for the absolute value function and then to write the determinant as a quadratic form.

The calculus ends with some elementary but cumbersome computations of complex integrals. Here we show the result obtained by using the first method. Let $f(\lambda_2, \lambda_2)$ the spectral density of process $\mathbf{X}$ and

$m_{ij} = \int_{\mathbb{R}^2} \lambda_1^i \lambda_2^j f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2$, its spectral moments.

Defining the regression model $X_{22} = \alpha X_{11} + \beta X_{12} + \sigma\varepsilon$, where $\varepsilon \perp (X_{11}, X_{12})$, we readily have

$$\alpha = \frac{m_{22}^2 - m_{31}m_{13}}{m_{40}m_{22} - m_{31}^2} \quad \beta = \frac{m_{40}m_{13} - m_{22}m_{31}}{m_{40}m_{22} - m_{31}^2}$$

and $\sigma^2 = m_{04} - \alpha^2 m_{40} - \beta^2 m_{22} - 2\alpha\beta m_{31}$. Thus

$$\mathbb{E}|\det \mathbf{Y}'(\mathbf{0})|] = \sigma \int_{\mathbb{R}^2} |z_1 - k| \mathbb{E}[|\gamma(z_1, z_2) + \varepsilon|] p_{X_{11}, X_{12}}(z_1, z_2) dz_1 dz_2,$$

where $\gamma(z_1, z_2) := \dfrac{\alpha z_1(z_1 - k) + \beta(z_1 - k)z_2 - z_2^2}{(z_1 - k)\sigma}$ and $p_{Y_{11}, Y_{12}}(z_1, z_2)$ is the density of the vector $(X_{11}(x_1, x_2), X_{12}(x_1, x_2))$.

Since

$$\mathbb{E}|\mathcal{N} + m| = m[2\Phi(m) - 1] + 2\,\varphi(m) := G(m),$$

where $\mathcal{N}$ is a standard Gaussian, Φ and φ its distribution and density respectively. It holds

$$\mathbb{E}[|\det \mathbf{Y}'(\mathbf{0})|] = \sigma \int_{\mathbb{R}^2} |z_1 - k| G(\gamma(z_1, z_2)) p_{X_{11}, X_{12}}(z_1, z_2) dz_1 dz_2.$$

This integral is well adapted for numerical computation. Note that that expression (1) only depends of the spectral moments $m_{p,q}$ of the process $\mathbf{X}$ for $0 \leq p + q \leq 4$. The field $\mathbf{X}'$ must satisfy the hypothesis of Theorem 1.

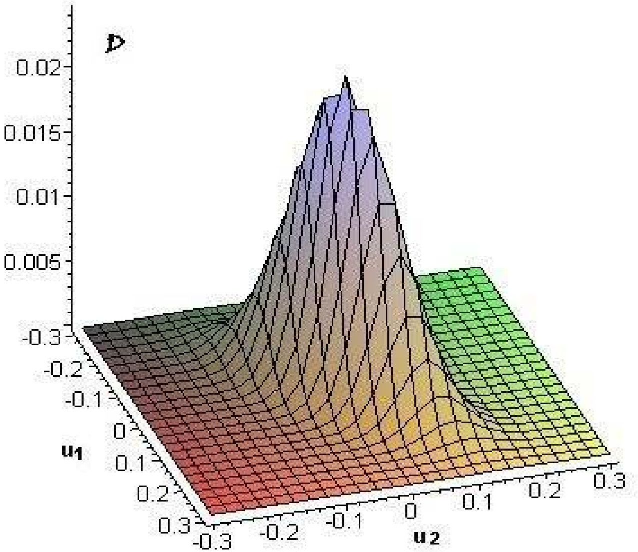
When $(-kx_1, -kx_2) \sim (u_1, u_2)$ the expression (1) can be rewritten as

$$\mathbb{E}N_Q^{\mathbf{Y}}(\mathbf{0}) = \mathbb{E}\left[|\xi_4\xi_6 - \xi_5^2|\right] |Q| p_{X_{11}, X_{12}}(u_1, u_2) \equiv \mathcal{D}. \qquad (2)$$

The following tables give the calculation of $\mathcal{D}$ for different values of $(\mathrm{u}_1, \mathrm{u}_2)$, using the two directional spectrum JONSWAP-Cos with $\theta_0 = \pi/3$ and $|\Omega| = 1$.

u_1	u_2	$\mathcal{D}$
.1800000000	.600000000e-1	.2231681219e-2
.600000000e-1	.3000000000	.1712117707e-11
-.600000000e-1	.3000000000	.2687072643e-16
.2400000000	.1200000000	.5313461740e-3
.2600000000	-.1000000000	.2524271146e-10
.2700000000	.900000000e-1	.1122496705e-3

The graph of $\mathcal{D}$ in terms of u_1 and u_2 is:


D
0.02
0.015
0.01
0.005
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
u1
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
u2

- **A variant of the Isotropy test of Cabaña**
  Suppose that we have a number N of images taken of the sea at N times. We want to make inference with these data. Let $X(t, \mathbf{x}) = X(t, x_1, x_2)$ a stationary zero mean Gaussian random field modeling the sea waves height. Let us suppose that this process satisfies the assumption of Theorem 2. We can write

$$X'(t, \mathbf{x}) = ||X'(t, \mathbf{x})||(\cos\Theta(t, \mathbf{x}), \sin\Theta(t, \mathbf{x}))^T,$$

we first study the level functionals per unit of time. For x_2 fixed let define

$$N^{x_2}_{[0,M_1],t}(u) = \frac{1}{t}\#\{0 < x_1 \leq M_1\, 0 < s < t : X(s, x_1, x_2) = u\}.$$

If X satisfies the conditions of ergodicity, by using the Rice's formula we have

$$\int_0^{M_2} N^{x_2}_{[0,M_1],t}(u)dx_2 \to \mathbb{E}[\int_{\mathcal{C}_Q(u)} |\cos\Theta|\ d\sigma_1]$$

$$= \frac{\sigma_2(Q)}{\pi}\sqrt{\frac{m_{20}}{m_{00}}}\, e^{-\frac{u^2}{2m_{00}}}$$

and also

$$\int_0^{M_1} N^{x_2}_{[0,M_1],t}(u)dx_1 \to \mathbb{E}[\int_{\mathcal{C}_Q(u)} |\sin\Theta|\ d\sigma_1]$$

$$= \frac{\sigma_2(Q)}{\pi}\sqrt{\frac{m_{02}}{m_{00}}}\, e^{-\frac{u^2}{2m_{00}}}.$$

And finally defining $\gamma = \frac{\lambda_-}{\lambda_+}$ where $\lambda_\pm = \frac{(m_{20}+m_{02})\pm\sqrt{(m_{20}-m_{02})^2+4m_{11}^2}}{2}$, we get

$$\mathbb{E}[\sigma_1(\mathcal{C}_Q(u))] = \frac{1}{\pi}(\frac{m_{20}+m_{02}}{m_{00}})^{1/2}(1+\gamma^2)^{-1/2}\mathbf{E}(\sqrt{1-\gamma^2}).$$

Where $\mathbf{E}(k)$ is the elliptic integral of the first kind.

We have that $X := X(t, \mathbf{x})$ stationary in time and in space.

We are now looking the time average distribution of function $\Theta(t, \mathbf{x})$ restricted at the level set: $\mathcal{C}_Q^{M_r}(y)$. Defining $h_{\theta_1,\theta_2}(\cdot) = 1_{[\tan\theta_1\,,\tan\theta_2]}(\cdot)$, for $-\frac{\pi}{2} \leq \theta_1 < \theta_2 \leq \frac{\pi}{2}$ we consider

$$F_t(\theta_2) - F_t(\theta_1)$$

$$:= \frac{1}{t}\int_0^t \sigma_1(\{\mathbf{x} \in Q : X(s, \mathbf{x}) = y\,;\,\theta_1 \leq \Theta((\mathbf{s}, \mathbf{x})) \leq \theta_2\})ds$$

$$= \frac{1}{t}\int_0^t \int_{\mathcal{C}_Q^{M_r}(y)} h_{\theta_1,\theta_2}(\tan\Theta((s, \mathbf{x})))d\sigma_1(s, \mathbf{x})ds.$$

By using the ergodic Theorem we know that this random variable tends when t goes to infinity towards its expectation. Hence to compute the limit let us take expectation in the above expression. By using that X is stationary and the formula provides by Theorem 2 we get:

$F(\theta_2) - F(\theta_1)$
$:= \mathbb{E}[\frac{1}{t}\int_0^t \sigma_1(\{\mathbf{x} \in Q : \ X(s, \mathbf{x}) = y \,;\, \theta_1 \leq \Theta((\mathbf{s}, \mathbf{x})) \leq \theta_2\})ds]$
$= \mathbb{E}\int_{\mathcal{C}_Q^{M_r}(y)} h_{\theta_1,\theta_2}(\tan\Theta((0, \mathbf{x})))d\sigma_1(0, \mathbf{x}) =$
$\sigma_2(Q)\mathbb{E}[h_{\theta_1,\theta_2}(\frac{\partial_y X}{\partial_y X})(\partial_x X^2 + \partial_y X^2)^{1/2}]$.

Recalling that $m_{i,j} = \int_{\mathbb{R}^2} \lambda_1^i \lambda_2^j f(\lambda_1, \lambda_2)d\lambda_1 d\lambda_2$, denoting $\Delta = m_{20}m_{02} - m_{11}$ and putting $\sigma_2(Q) = 1$ for ease of notation, we readily obtain

$F(\theta_2) - F(\theta_1)$

$$= \frac{\lambda_+^{1/2}(1-\gamma^2)}{\sqrt{8\pi}} \int_{\theta_1}^{\theta_2} \frac{1}{(1-\gamma^2\sin^2(\varphi-\kappa))^{3/2}} d\varphi.$$

Where $\lambda_- \leq \lambda_+$ are the eigenvectors of the covariance matrix of the random vector $(\partial_x X(0,0,0), \partial_y X(0,0,0))$, κ is the angle of the rotation that turns diagonal this matrix and $\gamma^2 = (1 - \frac{\lambda_-}{\lambda_+})$.
In the case of an isotropic process we have $m_{20} = m_{02}$ and moreover $m_{11} = 0$ the above expression writes

$$F(\theta_2) - F(\theta_1) = \frac{m_{20}^{1/2}}{\sqrt{8\pi}} (\theta_2 - \theta_1).$$

The above formula says that over the contour the "distribution" of the angle is uniform (cf. Longuet-Higgins (1957), pp. 348).

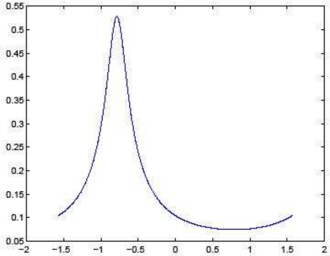


Figure: Density of the Palm distribution of the angle showing important depart from the isotropy

Let us turn to ergodicity. For a given subset Q of $\mathbb{R}^2$ and each t, let us define $\mathcal{A}_t = \sigma\{W(x,y,t) : \tau > t\,;(x,y) \in Q\}$ and consider the σ-algebra of t-invariant events $\mathcal{A} = \bigcap \mathcal{A}_t$. We assume that for each pair (x,y), $\Gamma(x,y,t) \to 0$ as $t \to +\infty$. It is well-known that under this condition, the σ-algebra $\mathcal{A}$ is trivial, that is, it only contains events having probability zero or one. This has the following important consequence in our context. Assume that the set Q has a smooth boundary and for simplicity, unit Lebesgue measure. Let us consider

$$Z(t) = \int_{\mathcal{C}_Q(u,t)} H\big(\mathbf{x},t\big)\, d\sigma_1(\mathbf{x}), \tag{3}$$

with $H\big(\mathbf{x},t\big) = \mathcal{H}\big(W(\mathbf{x},t), \nabla W(\mathbf{x},t)\big)$, where $\nabla W = (W_x, W_y)$ denotes the gradient in the space variables and $\mathcal{H}$ is some measurable function such that the integral is well-defined. This is exactly our case. The process $\{Z(t) : t \in \mathbb{R}\}$ process is strictly stationary, and in our case has a finite mean and is Riemann-integrable. By the Birkhoff-Khintchine ergodic theorem, a.s. as $T \to +\infty$,

$$\frac{1}{T}\int_0^T Z(s)ds \to \mathbb{E}_{\mathcal{B}}[Z(0)],$$

where $\mathcal{B}$ is the σ-algebra of t-invariant events associated to the process $Z(t)$. Since for each t, $Z(t)$ is $\mathcal{A}_t$-measurable, it follows that $\mathcal{B} \subset \mathcal{A}$, so that $\mathbb{E}_{\mathcal{B}}[Z(0)] = \mathbb{E}[Z(0)]$. On the other hand, Rice's formula yields (take into account that stationarity of $\mathcal{W}$ implies that $W(\mathbf{0}, 0)$ and $\nabla W(\mathbf{0}, 0)$ are independent):

$$\mathbb{E}[Z(0)] = \mathbb{E}[\mathcal{H}\big(u, \nabla W(\mathbf{0}, 0)\big)||\nabla W(\mathbf{0}, 0)||]p_{W(\mathbf{0},0)}(u).$$

We consider now the CLT. Let us define

$$\mathcal{Z}(t) = \frac{1}{t}\int_0^t \left[Z(s) - \mathbb{E}(Z(0))\right] ds, \qquad (4)$$

To compute the variance of $\mathcal{Z}(t)$, one can use again the Rice formula for the first moment of integrals over level sets, applied this time to the $\mathbb{R}^2$-valued random field with parameter in $\mathbb{R}^4$,
$\{(W(\mathbf{x}_1, s_1), W(\mathbf{x}_2, s_2))^T : (\mathbf{x}_1, \mathbf{x}_2) \in Q \times Q, s_1, s_2 \in [0, t]\}$ at the level (u, u). We get

$$Var\mathcal{Z}(t) = \frac{2}{t}\int_0^t (1 - \frac{s}{t}) I(u, s) ds,$$

where

$$I(u,s) = \int_{Q^2} \mathbb{E}\Big[H(\mathbf{x}_1,0)H(\mathbf{x}_2,s)\|\nabla W(\mathbf{x}_1,0)\|\|\nabla W(\mathbf{x}_2,s)\| \Big| \, W(\mathbf{x}_1,0)=u\,;W(\mathbf{x}_2,s)=u\Big]$$

$$\times\, p_{W(\mathbf{x}_1,0),W(\mathbf{x}_2,s)}(u,u)d\mathbf{x}_1 d\mathbf{x}_2 \; - \Big(\mathbb{E}[\mathcal{H}(u,\nabla W(\mathbf{0},0))\,||\nabla W(\mathbf{0},0)||]p_{W(\mathbf{0},0)}(u)\Big)^2.$$

Assuming that the given random field is time-δ-dependent, that is,
$\Gamma(x,y,t)=0\ \forall\ (x,y)$ whenever $t>\delta$, we readily obtain

$$t\,Var\mathcal{Z}(t) \to 2\int_0^{\delta} I(u,s)ds := \sigma^2(u) \quad \text{as } t\to\infty. \qquad (5)$$

Using now a variant of the Hoeffding-Robbins Theorem for sums of δ-dependent random variables, we can establish the following theorem.

Theorem

Assume that the random field W and the function H satisfy the conditions of Theorem about the second moment factorial. Assume for simplicity that Q has Lebesgue measure one. Then
- if the covariance $\gamma(x, y, t)$ tends to zero as $t \to +\infty$ for every value of $(x, y) \in Q$, we have

$$\frac{1}{T}\int_0^T Z(s)ds \to \mathbb{E}[\mathcal{H}(u, \nabla W(\mathbf{0}, 0))||\nabla W(\mathbf{0}, 0)||]p_{W(\mathbf{0},0)}(u),$$

where $Z(t)$ is defined by (3).
- if the random fields W is δ- dependent in the sense above, we have

$$\sqrt{t}\mathcal{Z}(t) \Rightarrow N(0, \sigma^2(u)).$$

where $\mathcal{Z}(t)$ is defined by (4) and $\sigma^2(u))$ by (5).

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