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LABORATOIRE NATIONAL DE MÉTROLOGIE ET D'ESSAIS

Université Gustave Eiffel

IMSE

SEQUENTIAL DESIGN FOR MULTIFIDELITY BAYESIAN INVERSION WITH FUNCTIONAL OUTPUTS: A CASE STUDY IN THERMAL CHARACTERIZATION

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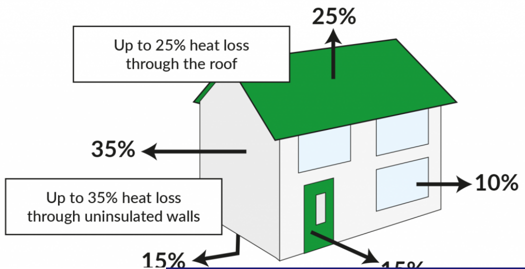
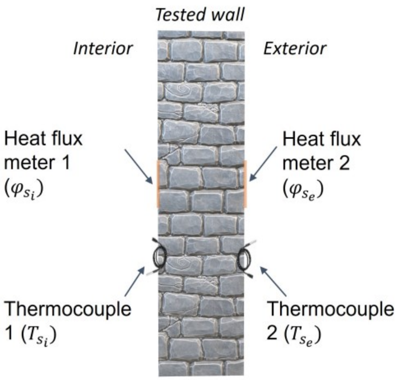
Context

- More than 40% of energy consumption comes from the building sector.
- Good insulation saves 30–50% on heating costs.
- To support renovation and compliance to regulations, a method is needed to determine the **in-situ** thermal resistance of a wall.
- **ANR RESBIOBAT (2022-2026)**: Identification in-situ de la RESistance thermique de parois BIOsourcées et géosourcées de BATiment.

RESBIOBAT



NF ISO 9869-1:2014: Heat flow meter method for the in-situ measurement of thermal resistance



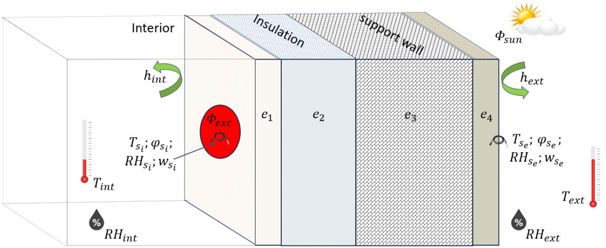
- ✔ thermal resistance $R = \frac{T_{si} - T_{se}}{\phi_{si}}$ from at least 3 days of measurements
- ✔ sensitive to environmental conditions and intrusive for people
- ✔ works well for high interior/exterior temperature difference

Requirements

The new method should be fast (1 day), less intrusive and less sensitive to the environment.

Coupling Measurements with Thermal Models

Principle: When the wall is locally heated, the system undergoes a transient thermal response. Estimating the thermal properties of the wall then requires the use of appropriate thermal models and inverse identification methods.



- ✔ Measured quantities: *Tsi*, ... stored in $\boldsymbol{y}^{\mathrm{mes}}$
- ✔ Unknown quantities: λ_i (thermal conductivity),... stored in $\boldsymbol{z}$ are parameters of the thermal model

Thermal resistance: $R_{Tot} = \sum_{i=1}^{n_\ell} \frac{e_i}{\lambda_i}$

Problematic: Inverse Problems with Costly Simulators

Find $\boldsymbol{z}$ such that the distance between measurements and thermal model outputs be minimal accounting for error terms

$$\boldsymbol{y}^{\text{mes}} = \underbrace{\boldsymbol{y}^{\text{Mod}}(\boldsymbol{z})}_{\text{thermal model output}} + \underbrace{\varepsilon^{\text{mes}}}_{\text{measurement error}} + \underbrace{\varepsilon^{\text{Mod}}}_{\text{thermal model error}}$$

- ✔ Computationally costly thermal models
- ✔ Functional outputs
- ✔ Numerous uncertainty sources

How to combine thermal models with statistical methods to overcome these challenges?

Bayesian framework

$$\pi(\boldsymbol{z}|\boldsymbol{y}^{\text{mes}}) = \frac{\pi(\boldsymbol{z}) \times \pi(\boldsymbol{y}^{\text{mes}}|\boldsymbol{z})}{\int_{\boldsymbol{z}'} \pi(\boldsymbol{z}') \times \pi(\boldsymbol{y}^{\text{mes}}|\boldsymbol{z}')d\boldsymbol{z}'}, \quad \boldsymbol{z} \in \Omega(\mathbb{Z}), \tag{1}$$

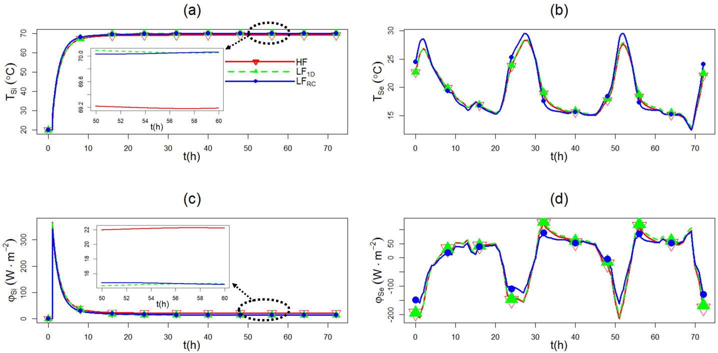
- ✔ $\boldsymbol{y}^{\text{mes}}$: discretized measurements on a grid.
- ✔ $\pi(\boldsymbol{z})$: prior distribution of $\boldsymbol{z}$.
- ✔ $\pi(\boldsymbol{y}^{\text{mes}}|\boldsymbol{z})$: likelihood function, requires calls from a thermal model.
- ✔ $\pi(\boldsymbol{z}|\boldsymbol{y}^{\text{mes}})$: (marginal) posterior distribution of $\boldsymbol{z}$, obtained with Markov Chain Monte Carlo (MCMC) sampling.

Choosing a thermal model... or all?

Thermal model	Advantages	Limitations	Cost
RC	Intuitive analogy with electrical circuits. Very fast, low-order models	Strong simplifications	$C_{LF_{RC}}$ =0.01 s per simulation
1D	Very simple, fast computations; Often analytical solutions exist.	◦ Only valid when heat flow is essentially 1-directional. ◦ Ignores lateral/edge effects.	$C_{LF_{1D}}$ = 2 s per simulation
2D	◦ Captures lateral heat spreading and gradients. ◦ Good for surfaces, fins, layered structures. ◦ More accurate than 1D when variations occur in 2 directions	Higher computational cost than 1D; Usually needs numerical solvers.	not considered
2D axi-symmetric	Efficient reduction of 3D problems with rotational symmetry. Good trade-off between accuracy and cost.	Only valid if geometry and boundary conditions are symmetric	C_{HF} = 24 s

Multifidelity approach is able to combine two or more thermal models for faster estimation

Comparison of the thermal models



- ➔ High correlation between the fidelity levels for the four outputs.
- ➔ Modelling (thermal) models errors as a function of time.
- ➔ Building metamodels of codes that take the correlation structure into account → multi-fidelity meta-modelling

Metamodel-based inversion

$$\boldsymbol{y}^{\text{mes}} = \boldsymbol{y}^{\text{Meta}}(\boldsymbol{z}) + \varepsilon^{\text{mes}} + \varepsilon^{\text{Meta}}(\boldsymbol{z}) + \varepsilon^{\text{Res}}$$

Assumptions :

- $\varepsilon^{\text{mes}} \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_{\text{mes}} = \sigma^2 (\boldsymbol{y}^{\text{mes}})^2\right)$.
- $\varepsilon^{\text{Res}} \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_{\text{Res}} = \theta_{\text{a}}{}^2 \mathbf{I}_{\text{N}_\text{t}} + \theta_{\text{m}}{}^2 (\boldsymbol{y}^{\text{mes}})^2\right)$.
- $\varepsilon^{\text{Meta}} \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_{\text{Meta}}(\boldsymbol{z})\right)$.
- $\varepsilon^{\text{mes}} \perp\!\!\!\perp \varepsilon^{\text{Res}} \perp\!\!\!\perp \varepsilon^{\text{Meta}}$.
- $\boldsymbol{\Sigma}(\boldsymbol{z}) = \boldsymbol{\Sigma}_{\text{mes}} + \boldsymbol{\Sigma}_{\text{Res}} + \boldsymbol{\Sigma}_{\text{Meta}}(\boldsymbol{z})$.

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$$\pi(\boldsymbol{y}^{\mathrm{mes}}|\boldsymbol{z}) = \frac{\exp\left(-\frac{1}{2}\left(\boldsymbol{y}^{\mathrm{mes}} - \boldsymbol{y}^{\mathrm{Meta}}(\boldsymbol{z})\right)^{\top} \boldsymbol{\Sigma}(\boldsymbol{z})^{-1} \left(\boldsymbol{y}^{\mathrm{mes}} - \boldsymbol{y}^{\mathrm{Meta}}(\boldsymbol{z})\right)\right)}{\sqrt{(2\pi)^{N_t} \det(\boldsymbol{\Sigma}(\boldsymbol{z}))}}, \ \boldsymbol{z} \in \Omega(\mathbb{Z}).$$

Most likely value of $\boldsymbol{z}$: $\boldsymbol{z}^{\mathrm{MAP}} \in \arg\max_{\boldsymbol{z}} \pi(\boldsymbol{z}) \times \pi(\boldsymbol{y}^{\mathrm{mes}}|\boldsymbol{z}), \ \boldsymbol{z} \in \Omega(\mathbb{Z})$

Credible intervals of level $\alpha \in (0,1)$: $\int_{CI_\zeta(\alpha)} \pi(z_\zeta|\boldsymbol{y}^{\mathrm{mes}}) dz_\zeta = \alpha, \ \zeta \in d_{\boldsymbol{z}}$.

Multi-fidelity meta-modelling for functional outputs

$$\boldsymbol{y}^{\text{mes}} = \boldsymbol{y}^{\text{Meta}}(\boldsymbol{z}) + \varepsilon^{\text{mes}} + \varepsilon^{\text{Meta}}(\boldsymbol{z}) + \varepsilon^{\text{Res}}$$

➔ **Multi-fidelity meta-model construction with Gaussian Process regression:** builds $\boldsymbol{y}^{\text{Meta}}$, enables to quantify and reduce $\varepsilon^{\text{Meta}}(\boldsymbol{z})$.

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→ **Option: Dimension reduction through weighted PCA:** the outputs of each code are projected onto the m first principal components and multi-fidelity is performed on the projections. Projection error is controlled.

Multi-fidelity meta-modelling for functional outputs

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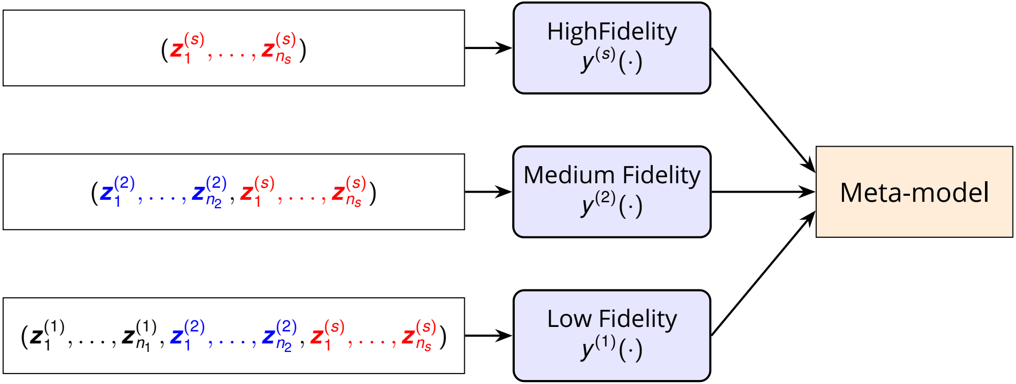
- ➔ **Option: Dimension reduction through weighted PCA:** the outputs of each code are projected onto the *m* first principal components and multi-fidelity is performed on the projections. Projection error is controlled.

- ➔ **Sequential approach for augmenting the DoE:** new points are selected according to a criterion targeting the posterior distribution $\pi(\boldsymbol{z}|\boldsymbol{y}^{mes})$ for a given total cost, while maintaining the nested structure of the multi-level design.

Hierarchical Multi-Fidelity Framework



Recursive Gaussian Process Regression*

- Each code $\boldsymbol{y}_\ell$ is a realization of a Gaussian process $\boldsymbol{Y}_\ell$.
- Recursive and affine structure:

$$\boldsymbol{Y}_\ell(\boldsymbol{z}) = \begin{cases} \boldsymbol{W}_1(\boldsymbol{z}) & \text{if } \ell = 1 \\ \boldsymbol{\rho}_\ell(\boldsymbol{z})\boldsymbol{Y}_{\ell-1}(\boldsymbol{z}) + \boldsymbol{W}_\ell(\boldsymbol{z}) & \text{if } \ell \geq 2 \end{cases}$$

- $\boldsymbol{W}_\ell(\boldsymbol{z}) \sim \mathcal{GP}(\boldsymbol{\mu}_\ell(\boldsymbol{z}), \boldsymbol{C}_\ell(\boldsymbol{z}, \boldsymbol{z}'))$ with values in $\mathbb{R}^{N_t}$.

$$\mathbb{E}\left[\boldsymbol{Y}_s(\boldsymbol{z}) \mid \overline{\boldsymbol{Y}} = \overline{\boldsymbol{y}}\right] = \boldsymbol{m}(\boldsymbol{z}) + \boldsymbol{U}(\boldsymbol{z})\overline{\boldsymbol{V}}^{-1}(\overline{\boldsymbol{y}} - \overline{\boldsymbol{m}})\,.$$

$$\mathrm{Cov}\left(\boldsymbol{Y}_s(\boldsymbol{z}), \boldsymbol{Y}_s(\boldsymbol{z}) \mid \overline{\boldsymbol{Y}} = \overline{\boldsymbol{y}}\right) = \boldsymbol{V}(\boldsymbol{z}) - \boldsymbol{U}(\boldsymbol{z})\overline{\boldsymbol{V}}^{-1}\boldsymbol{U}(\boldsymbol{z})^T.$$

Advantages

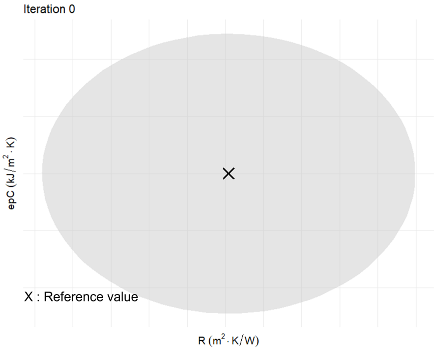
- ✔ Model with uncertainty quantification.
- ✔ Exploits inter-level correlations via $\boldsymbol{\rho}_\ell$.

Limitations

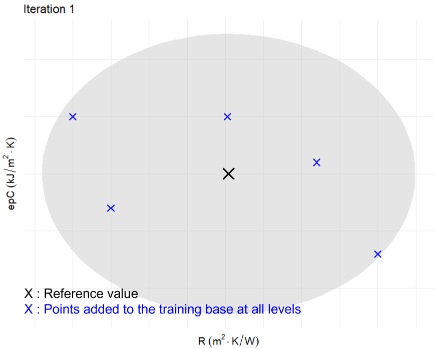
- ✗ Sensitive to hyperparameters.
- ✗ $(N_t \sum_{\ell=1}^{s} n_\ell \times N_t \sum_{\ell=1}^{s} n_\ell)$ matrix to invert.

* Le Gratiet et al. 2014, International Journal for Uncertainty Quantification.

Sequential identification strategy

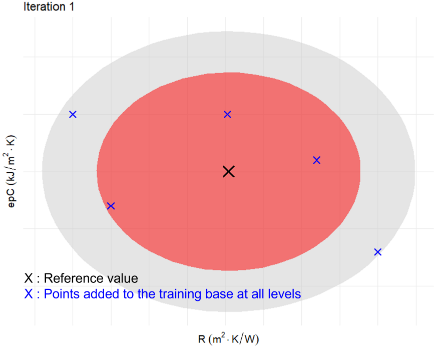


Sequential identification strategy



- $\mathcal{Z}_1^0 = \cdots = \mathcal{Z}_s^0$ selected using Latin Hypercube Sampling (LHS).
- Training data: $\mathcal{D}^0 = \{\boldsymbol{Y}_1^0, \dots, \boldsymbol{Y}_s^0\}$.
- Meta-model: $\boldsymbol{W}^0 \sim \mathcal{GP}(\boldsymbol{\mu}^0, \boldsymbol{C}^0)$.

Sequential identification strategy



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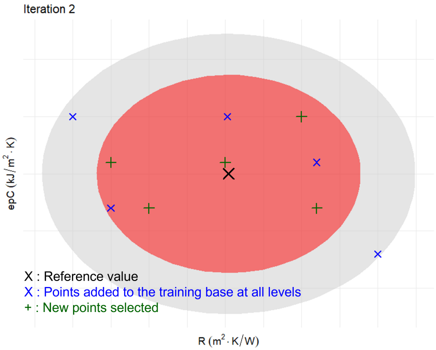
- Meta-model: $\boldsymbol{W}^0 \sim \mathcal{GP}(\boldsymbol{\mu}^0, \boldsymbol{C}^0)$.
- First posterior estimate: $\pi^0(\boldsymbol{z}|\boldsymbol{y}^{\text{mes}})$ using MCMC on set $\mathcal{Z}^0_{\text{MCMC}}$.

Sequential identification strategy



- New points selected via **k-means** on $\mathcal{Z}^{i-1}_{\text{MCMC}}$,
- Fidelity level ℓ chosen :$\ell^i = \arg\max_\ell \left(\mathcal{R}^i_\ell = \frac{\mathcal{G}^i_\ell}{c_\ell}\right)$,
- Meta-model is updated: $\boldsymbol{W}^i \sim \mathcal{GP}(\boldsymbol{\mu}^i, \boldsymbol{C}^i)$,
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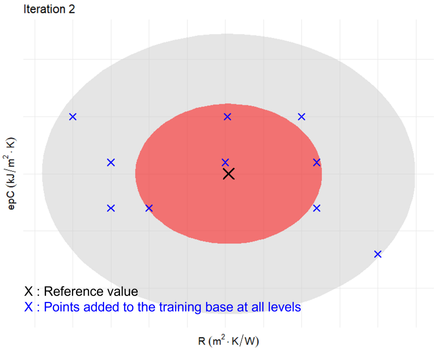

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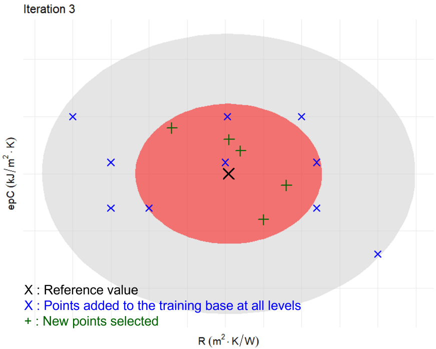

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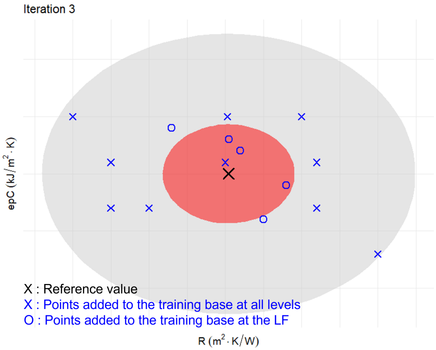

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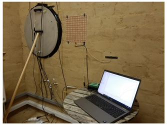
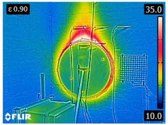
Row earth house in Sense-City equipment in Champs-sur-Marne, France (Paris area)





- ➔ Single raw earth wall with measured thickness $e = 0.3$ m
- ➔ Thermal conductivity evaluated from 8 days measurements in stationary conditions in climatic chamber: $\lambda = 0.86$ W/(K.m)

Need to accelerate the estimation time for less intrusive measurements: using cost-effective multi-fidelity inversion on transient measurements!

Protocol to evaluate thermal conductivity from 1 day meas.

Phase 1: collecting transient measurements

- ➔ Control the environmental conditions using the climatic chamber: $T_{\mathrm{int}} = 20^\circ C$, $T_{\mathrm{ext}} = 10^\circ C$, $\Phi_{\mathrm{sun}} = 0\,\mathrm{W/m^2}$
- ➔ Generate 1 day measurements by heating the surface of the wall

controls the residual error

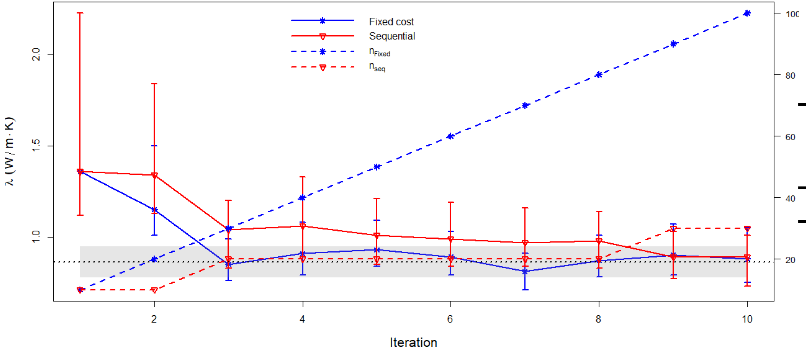
Phase 2: cost-effective statistical analysis

- ➔ Choose 3 fidelity levels: HF=2D-axi, 1D, RC
- ➔ Create the initial nested DoE: evaluate the same $n_0 = 10$ points at each level
- ➔ iterate choice of 10 new points at specific fidelity level, and meta-model updating steps
- ➔ stop at 9 iterations of the algorithm

controls the meta-model error

Comparison of sequential MF three-levels inversion versus HF inversion

- ➔ Fixed cost approach: at each iteration, add 10 points at each level
- ➔ Sequential approach: at each iteration, choose the fidelity level !



- ➔ The sequential approach alternates calls to 1D and 2D-axi models (no lower level RC model).
- ➔ HF 2D-axi points reduce bias.
- ➔ 1D-LF points reduce uncertainty.

Conclusion

- A Bayesian multi-fidelity sequential approach was developed to estimate the thermal parameters of walls from temperature and flux measurements at the surface, at a **minimal cost**.

- New training points are selected iteratively and a fidelity-level selection criterion is used.

- The sequential process progressively refines the meta-model and could be improved e.g. by considering a physics-informed criterion or joint selection of the points and the fidelity level.

Articles (within Hadi Nasser PhD)

[1] H. Nasser, G. Perrin, R. Chakir, S. Demeyer, and J. Waeytens. Sequential Bayesian identification using multi-fidelity Gaussian process surrogates: Application to an inverse transient thermal problem. Journal of computational mathematics - SMAI, 2025. “submitted”

[2] H. Nasser, G. Perrin, R. Chakir, S. Demeyer, and J. Waeytens. On the construction of non-intrusive multifidelity models for computer codes with time-series output: Comparison of three paradigms on a transient thermal problem. Journal of Computational Physics, 2025.

[3] H. Nasser, G. Perrin, R. Chakir, S. Demeyer, and J. Waeytens. Incorporating multi-source uncertainties in fast building wall thermal resistance estimation through physics-based and multi-fidelity statistical learning models. Journal of Building Engineering, 2024.