Imprecise probabilities to propagate uncertainties: a tour

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Lecture goal/content

What you will find in this talk

- Information representation seen by a non-statistician (mostly IA/engineer) guy
- Imprecise probabilities: when to (not) use it?
- Imprecise probabilities: definition and practical representation
- Merging imprecise probabilistic representations
- (In)dependence modelling and uncertainty propagation
- How (not) to decide with imprecise probabilities

What you will not find in this talk

- A deep and exhaustive study of a particular topic
Outline

1. Introductory elements
2. Imprecise probabilities: use and misuse
3. Representing partial (probabilistic) knowledge
4. Merging partial (probabilistic) knowledge
5. Independence and propagation
6. Decision in presence of imprecision
A quantity of interest $X$ can be

- **Generic**, when it refers to a population, or a set of situations.

**Generic quantity example**
The distribution of mother tongue within French inhabitants

- **Singular**, when it refers to an individual or a peculiar situation

**Singular quantity example**
My own mother tongue
Ontic and epistemic information [9]

An item of information $I$ possessed by an agent about $X$ can be

- **Ontic**, if it is a faithful, perfect representation of $X$

**Ontic information example**

A set $X$ representing the exact set of languages spoken by me

e.g.: $X = \{ \text{French, English, Spanish} \}$

- **Epistemic**, if it is an imperfect representation of $X$

**Epistemic information example**

A set $E$ containing my mother tongue

e.g., $E = \{ \text{French, Dutch, English} \}$
Everything is possible

We can have

- **Ontic** information about a **singular** quantity: the hair colour of a suspect; the mother tongue of someone

- **Epistemic** information about a **singular** quantity: the result of the next dice toss; the set of possible mother tongues of someone

- **Ontic** information about a **generic** quantity: the exact distribution of pixel colours in an image

- **Epistemic** information about a **generic** quantity: an interval about the frequency of French persons higher than 1.80 m
Uncertainty definition

Uncertainty: when our information $I$ about the quantity of interest $X$ is insufficient to answer with certainty to assertions

→ In this view, uncertainty is necessarily epistemic, as it reflects an imperfect knowledge of the agent.

Can concern both:

- **Singular quantity**
  - items in a data-base, values of some logical variables, time before failure of a component

- **Generic quantity**
  - parameter values of classifiers/regression models/probability distributions, time before failure of components, truth of a logical sentence ("birds fly")
The room example

Heights of people in a room: generic quantity

- Generic question: are 90% of people in room less than 1m80?
  ⇒ No, with **full certainty**

- Specific question: is the last person who entered less than 1m80?
  ⇒ Probably, with 60% chance (**uncertain answer**)
Uncertainty main origins [6, Ch. 3]

- **Variability** of a population applied to a peculiar, singular situation

Variability example
The result of one dice throw when knowing the probability of each face

- **Imprecision and incompleteness** due to partial information about the quantity $S$

Imprecision example
Observing limited sample of the population, describing suspect as "young", limited sensor precision

- **Conflict and unreliability** of different sources of information

Conflict example
Two redundant data base entries with different information for an attribute, two sensors giving different measurements of a quantity
Uncertainty propagation revisited

$\text{Model } f(x_1, \ldots, x_n) = y$

Propagation: usual steps

1. **Represent**: provide an uncertainty model for $x_1, \ldots, x_n$
2. **Merge**: if multiple models given for $x_i$, merge into a single one
3. **Combine**: specify (in)dependencies between $x_i$'s to get global model
4. **Propagate**: propagate to get uncertainty over $y$
5. **Decide**: once uncertainty on $y$ estimated, decide on an action

Output genericity: same as most generic input variable/parameter
Many notions making sense for generic quantities, make no or poor sense at all for singular ones:

- frequencies and "objective" true probability
- any statistic requiring population (variance, mean, median, ...)
- learning from samples
- stochastic independence

Mathematically equivalent notions may model something about your knowledge of the singular quantity, not about the quantity itself.
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1. Introductory elements

2. Imprecise probabilities: use and misuse
   - Motivation without probabilities
   - A short word on interpretation
   - Some further reasons

3. Representing partial (probabilistic) knowledge

4. Merging partial (probabilistic) knowledge

5. Independence and propagation

6. Decision in presence of imprecision
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A non-probabilistic example

Assume the following:
- A function linking $y$ and $x$ with $f(y) = x^2$
- We want to estimate $f(y)$ but only know $x \in [-1, 6]$

We acknowledge our imprecise knowledge
Our final answer is that $f(x) \in [0, 36]$
Full imprecise knowledge + "uniform" selection

Assume the following:
- A function linking $y$ and $x$ with $f(y) = x^2$
- We want to estimate $f(y)$ but only know $x \in [-2, 5]$

We choose an "equiprobable" guess given $x$ interval: $x^* = 2$
Our final answer is that $f(x^*) = 4 \rightarrow$ is it what we want?
Full imprecise knowledge + "uniform" selection

Assume the following:
- A function linking $y$ and $x$ with $f(y) = x^2$
- We want to estimate $f(y)$ but only know $x \in [-2, 5]$

\[
f(x) = x^2
\]

- We choose the worst case given $x$ interval: $x^w = 0$
- Our final answer is $f(x^w) = 0 \rightarrow$ not easy to find? what we want?
Lesson from example

Two strategies:

1. take account of our knowledge as faithfully as possible
2. reduce it to something more manageable:
   - +: may make computations easier (not always)
   - -: selection will introduce a (possibly wanted) bias, whatever it is
   - -: "reference" point (uniform) may induce an unwanted bias

If you are fine with option 2, you can go for it. Another strategy:

3. Outer-approximate initial information for computational convenience

⇒

same remarks apply when a probability is ill-known
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Imprecise probabilities for generic information

Probabilities as frequencies

\[ P(A) = \text{frequency with which } A \text{ has been observed/is observed} \]

Imprecise probabilities as robust/sensitivity analysis models:

- "true" \( P \) only known to belong to some set \( \mathcal{P} \)
- \( P(A) \) only known to lie in \([\underline{P}(A), \overline{P}(A)]\)
- imprecise observations, limited sample, expert bounds

Eventually, with enough information, get to \( P \) or a small \( \mathcal{P} \)
Imprecise probabilities: use and misuse

A short word on interpretation

Imprecise probabilities for singular information

Probabilities as subjective degrees

\[ P(A) = \text{degree of belief that the true value will be in } A \]

Imprecise probabilities as models of beliefs:

- validity of probability to model partial belief or ignorance questionable
- separate notions of certainty and plausibility to encode ignorance
- asking for a precise \( P \) very demanding
- no notion of "true" \( P \) within \( \mathcal{P} \)

Eventually, with enough information, get the true value
Two views of imprecise probabilities

\( f \): true or "ideal" uncertainty model
\( \hat{f} \): estimated model/representation

**The robust/sensitivity view**
- Probabilities
- Probability sufficient in theory
- Hard to precisely obtain in practice

**The richer model view**
- Imprecise probabilities
- Probabilities not universal
- Accurate modelling may require richer theory
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Probability as a model of (partial) ignorance

The assumption ignorance=uniform probability has some issues

- Assume we know nothing about $S \in [1, 2]$, then ignorance is $p(s) \sim \mathcal{U}[1, 2]$
- Yet, if we consider the variable $1/s$, change of variable induce non-uniform probability over $[1/2, 1]$

$\rightarrow "mathematically right", but model of ignorance should be insensitive to variable changes$
The possibility of incomparability

Given two events $A, B$, whatever this event:
- a probabilistic model $P$ will always output
  - $P(A) > P(B)$
  - $P(A) < P(B)$
  - $P(A) = P(B)$ (not possible for every pair $A, B$, though)
- in the case of $P$, you can end up with

$$A \succ \prec B \text{ if } [P(A), \overline{P}(A)] \cap [P(B), \overline{P}(B)]$$

As a direct consequence of lack of knowledge (rather than derive it through a detour $\rightarrow$ variance/sensitivity)
Imprecision in input $\neq$ in outputs

Assume $x$ best guess is 3, $\pm 2 \Rightarrow$ get $[f(x)]$

$$f(x) = x^2$$

Propagating then adding imprecision $\neq$ propagating imprecision
Imprecision in input ≠ in outputs

Assume $x$ best guess is 3 $\Rightarrow$ get $f(x) \pm 2$

$$f(x) = x^2$$

Propagating then adding imprecision ≠ propagating imprecision
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Uncertainty propagation revisited

**Model**

\[ f(x_1, \ldots, x_n) = y \]

**Distrib. of data/parameter**

- **Generic**
- **Singular**

**Propagate**

- Output genericity: same as most generic input variable/parameter

**Prediction**

**Propagation: usual steps**

1. **Represent**: provide an uncertainty model for \( x_1, \ldots, x_n \)
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Basic framework

Quantity $S$ with possible **exclusive** states $S = \{s_1, \ldots, s_n\}$

$\triangleright$ $S$: input variable, component state, model parameter, ... 

Basic tools

A confidence degree $P : 2^{|S|} \rightarrow [0, 1]$ is such that

- $P(A)$: confidence $S \in A$
- $P(\emptyset) = 0, P(S) = 1$
- $A \subseteq B \Rightarrow P(A) \leq P(B)$

Uncertainty modelled by 2 degrees $\underline{P}, \overline{P} : 2^{|S|} \rightarrow [0, 1]$:

- $\underline{P}(A) \leq \overline{P}(A)$ (monotonicity)
- $\underline{P}(A) = 1 - \overline{P}(A^c)$ (duality)
Probability

Basic tool

A probability distribution \( p : S \rightarrow [0, 1] \) from which

\[
\begin{align*}
P(A) &= \bar{P}(A) = \mu(A) = \sum_{s \in A} p(s) \\
P(A) &= 1 - P(A^c): \text{auto-dual}
\end{align*}
\]

Main interpretations

- **Frequentist [37]**: \( P(A) = \) number of times \( A \) observed in a population
  - only applies to generic quantities (populations)

- **Subjectivist [24]**: \( P(A) = \) price for gamble giving 1 if \( A \) happens, 0 if not
  - applies to both singular and generic quantities
Sets

Basic tool

A set $E \subseteq S$ with true value $S \in E$ from which

- $E \subseteq A \rightarrow \mathbb{P}(A) = \overline{\mathbb{P}}(A) = 1$ (certainty truth in $A$)
- $E \cap A \neq \emptyset, E \cap A^c \neq \emptyset \rightarrow \mathbb{P}(A) = 0, \overline{\mathbb{P}}(A) = 1$ (ignorance)
- $E \cap A = \emptyset \rightarrow \mathbb{P}(A) = \overline{\mathbb{P}}(A) = 0$ (truth cannot be in $A$)

$\mathbb{P}, \overline{\mathbb{P}}$ are binary $\rightarrow$ limited expressiveness

Classical use of sets:

- Interval analysis [26] ($E$ is a subset of $\mathbb{R}$)
- Propositional logic ($E$ is the set of models of a KB)

Other cases: robust optimisation, decision under risk, ...
Example

Assume that it is known that pH value \( E \in [4.5, 5.5] \), then

- if \( A = [5, 6] \), then \( P(A) = 0, \overline{P}(A) = 1 \)

  ![Diagram 1]

- if \( A = [4, 7] \), then \( P(A) = \overline{P}(A) = 1 \)

  ![Diagram 2]

- if \( A = [6, 9] \), then \( P(A) = \overline{P}(A) = 0 \)

  ![Diagram 3]
In summary

Probabilities . . .
- (+) very informative quantification (do we need it?)
- (-) need lots of information (do we have it?)
- (-) if not enough, requires a choice (do we want to do that?)
- use probabilistic calculus (convolution, stoch. independence, . . . )

Sets . . .
- (+) need very few information
- (-) very rough quantification of uncertainty (Is it sufficient for us?)
- use set calculus (interval analysis, Cartesian product, . . . )

→ Need representations bridging these two
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Possibility distributions

Basic tool

A distribution $\pi : S \rightarrow [0, 1]$, usually with $s_i$ such that $\pi(s_i) = 1$, from which

- $\overline{P}(A) = \max_{s \in A} \pi(s)$
- $\underline{P}(A) = 1 - \overline{P}(A^c) = \min_{s \in A^c} (1 - \pi(s))$

Interval/set as special case

The set $E$ can be modelled by the possibility distribution $\pi_E$ such that

$$\pi_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{else} \end{cases}$$
Representing partial (probabilistic) knowledge
Possibility distributions

**A nice characteristic: Alpha-cut [10]**

**Definition**

\[ A_\alpha = \{ s \in S | \pi(s) \geq \alpha \} \]

- \( P(A_\alpha) = 1 - \alpha \)
- If \( \beta \leq \alpha \), \( A_\alpha \subseteq A_\beta \)

Simulation: draw \( \alpha \in [0, 1] \) and associate \( A_\alpha \)

\( \Rightarrow \) Possibilistic approach ideal to model **nested structures**
A basic distribution: simple support

A set $E$ of most plausible values
A confidence degree $\alpha = P(E)$

Interesting case:
- Expert providing most plausible values $E$

Extend to multiple sets $E_1, \ldots, E_p$:
- confidence degrees over nested sets [32]

pH value $\in [4.5, 5.5]$ with $\alpha = 0.8$ (\(~"quite probable"\))
Normalized likelihood as possibilities [20] [7]

\[ \pi(\theta) = \frac{\mathcal{L}(\theta|x)}{\max_{\theta \in \Theta} \mathcal{L}(\theta|x)} \]

Binomial situation:
- \( \theta \) = success probability
- \( x \) number of observed successes
- \( x = 4 \) succ. out of 11
- \( x = 20 \) succ. out of 55
Partially specified probabilities [2] [18]

Triangular distribution: \([\underline{P}, \overline{P}]\) encompass all probabilities with
- mode/reference value \(M\)
- support domain \([a, b]\).

Getting back to \(pH\)
- \(M = 5\)
- \([a, b] = [3, 7]\)
Other examples

- Statistical inequalities (e.g., Chebyshev inequality) [18]
- Linguistic information (fuzzy sets) [15]
- Approaches based on nested models
Possibility: limitations

For a given event \( A \), we can only have

\[
\underline{P}(A) > 0 \Rightarrow \overline{P}(A) = 1
\]

\[
\overline{P}(A) < 1 \Rightarrow \underline{P}(A) = 0
\]

\[
\Rightarrow \text{ interval } [\underline{P}(A), \overline{P}(A)] \text{ either}
\]

- \([\alpha, 1]\) or
- \([0, \beta]\),

Hence cannot model any \([\underline{P}(A), \overline{P}(A)]\) with \(\underline{P}(A) = \overline{P}(A)\)

Possibility distributions **do not include** probabilities as special case.
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Random sets and belief functions

Basic tool

A positive distribution $m : 2^{|S|} \rightarrow [0, 1]$, with $\sum_E m(E) = 1$ and usually $m(\emptyset) = 0$, from which

- $\overline{P}(A) = \sum_{E \cap A \neq \emptyset} m(E)$
- $P(A) = \sum_{E \subseteq A} m(E) = 1 - \overline{P}(A^c)$

$m(E_1)$
$m(E_2)$
$m(E_3)$
$m(E_4)$
$m(E_5)$

$A$

- Mix set and probabilities by putting probability mass over sets rather than points
- Other approach: consider a (convex) set of probability masses
A characteristic of belief functions

Complete monotonicity

If $P$ is a belief measure if and only if it satisfies the inequality

$$P(\bigcup_{i=1}^{n} A_i) \geq \sum_{A \subseteq \{A_1, \ldots, A_n\}} (-1)^{|A|+1} P(\bigcap_{A_i \in A} A_i)$$

for any collection of events.

Simply the exclusion/inclusion principle with an equality.
special cases

Measures $[\mathbb{P}, \overline{\mathbb{P}}]$ include:

- Probability distributions: mass on atoms/singletons
- Possibility distributions: mass on nested sets

→ "simplest" model including both sets and probabilities as subcases!
Frequencies of imprecise observations

Imprecise poll: "Who will win the next Wimbledon tournament?"

- N(adal)
- F(ederer)
- D(jokovic)
- M(urray)
- O(ther)

60% replied \{N, F, D\} → m(\{N, F, D\}) = 0.6

15% replied "I do not know" \{N, F, D, M, O\} → m(S) = 0.15

10% replied Murray \{M\} → m(\{M\}) = 0.1

5% replied others \{O\} → m(\{O\}) = 0.05

...
P-box [21]

A pair $[F, \bar{F}]$ of cumulative distributions

Bounds over events $[-\infty, x]$:

- Percentiles by experts;
- Kolmogorov-Smirnov bounds;

Can be extended to any pre-ordered space [17], [36] ⇒ multivariate spaces!

Expert providing percentiles:

- $0 \leq P([-\infty, 12]) \leq 0.2$
- $0.2 \leq P([-\infty, 24]) \leq 0.4$
- $0.6 \leq P([-\infty, 36]) \leq 0.8$
Other means to get random sets/belief functions

- Extending modal logic: probability of provability [34]
- Parameter estimation using pivotal quantities [28]
- Statistical confidence regions [16]
- Modify source information by its reliability [30]
- ...
Limits of random sets

- Not yet satisfactory extension of Bayesian/subjective approach
- Still some items of information it cannot model in a simple way, e.g.,
  - probabilistic bounds over atoms $s_i$ (imprecise histograms, . . .) \cite{13};
  - comparative assessments such as $2P(B) \leq P(A)$ \cite{29}
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Imprecise probabilities

Basic tool
A set $\mathcal{P}$ of probabilities on $S$ or an equivalent representation

- $\overline{P}(A) = \sup_{P \in \mathcal{P}} P(A)$ (Upper probability)
- $\underline{P}(A) = \inf_{P \in \mathcal{P}} P(A) = 1 - \overline{P}(A^c)$ (Lower probability)

**Note:** lower/upper bounds on events alone cannot model any convex $\mathcal{P}$

$[\underline{P}, \overline{P}]$ as

- subjective lower and upper betting rates [38]
- bounds of an **ill-known probability measure**

$P \Rightarrow \underline{P} \leq P \leq \overline{P}$ [5] [39]
Illustrative example

\[ p(x_1) = 0.2, \ p(x_2) = 0.5, \ p(x_3) = 0.3 \]
Illustrative example

\[ p(x_1) \in [0.2, 0.3], \ p(x_2) \in [0.4, 0.5], \ p(x_3) = [0.2, 0.3] \]

\[
\begin{array}{cccccc}
\{x_1\} & \{x_2\} & \{x_3\} & \{x_1, x_2\} & \{x_1, x_3\} & \{x_2, x_3\} \\
0.2 & 0.4 & 0.2 & 0.7 & 0.5 & 0.7 \\
\end{array}
\]

\[ \Rightarrow \text{not a belief function! By computing the corresponding } m, \text{ we have} \]

\[ m(\{x_1, x_2, x_3\}) = -0.1 \]
Means to get Imprecise probabilistic models

- Include all representations mentioned so far . . .
- . . . and a couple of others
  - probabilistic comparisons
  - density ratio-class
  - expectation bounds
  - . . .
- fully coherent extension of Bayesian approach

\[ P(\theta|x) = L(\theta|x)P(\theta) \]

→ often easy for "conjugate prior" [31]
- make probabilistic logic approaches imprecise [25, 14]
Example of Bayesian extension: the IDM

IDM: Imprecise Dirichlet Model

- Set of possibilities $\mathcal{X} = \{x_1, \ldots, x_n\}$
- "Parameters" $\Theta = (\theta_1, \ldots, \theta_n) \in [0, 1]^n$ with $\theta_i = p(x_i)$
- Observation vector $x = (a_1, \ldots, a_n)$ with $a_i = \#x_i$ and $\sum_i a_i = N$
- Likelihood
  \[
  L(\theta|x) = P(x|\theta) = \binom{N}{x} \theta_1^{a_1} \ldots \theta_n^{a_n}
  \]
- Prior $P(\theta) \sim Diri(\nu \phi)$ with
  - $\nu \in \mathbb{R}^+$: prior strength, $\sim \#$unobserved samples ($\nu = 0 \rightarrow$ no strength)
  - $\phi = (\phi_1, \ldots, \phi_n) \in [0, 1]^n$ with $\sum_i \phi_i = 1$: prior frequencies
- IDM: fix $\nu$, let $\phi \in \Phi$ with $\Phi$ subset of $n - 1$ unit simplex
Possible prior sets

Unknown prior

\( \phi_1 \)

\( \phi_2 \)

\( \phi_3 \)

\( x_2 \) more likely (=modal value)

\( \phi_1 \)

\( \phi_2 \)

\( \phi_3 \)

All results equally likely

\( \phi_1 \)

\( \phi_2 \)

\( \phi_3 \)

Observation vector \( x = (3, 6, 1) \) and \( v = 3 \)

\[ P(\theta_2 | x) = \frac{9}{13} \]

\[ P(\theta_2 | x) = \frac{6}{13} \]

\[ P(\theta_2 | x) = \frac{9}{13} \]

\[ P(\theta_2 | x) = \frac{7}{13} \]

\[ P(\theta_2 | x) = \frac{7}{13} \]
Other "imprecised" classical models

- Exponential family [31, 4]
- Bayesian Model Averaging [8]
- Gaussian process [27]
- Dirichlet process [35, 3]
A crude summary

Possibility distributions

-+: very simple, natural in many situations (nestedness), extend set-based approach

-: at odds with probability theory, limited expressiveness

Random sets

-+: include probabilities and possibilities, include many models used in practice

-: general models can be intractable, limited expressiveness

Imprecise probabilities

-+: most consistent extension of subjective probabilistic approach, very flexible

-: general models can be intractable
A not completely accurate but useful picture

Able to model variability
Incompleteness tolerant

Imprecise probability
Random sets

Probability
Possibility
Sets

Expressivity/flexibility
General tractability/scalability
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Uncertainty propagation revisited

- **Distrib. of data/parameter**
- **Propagate**
- **Model**
  \[ f(x_1, \ldots, x_n) = y \]
- **Prediction**
- **Single-valued data/parameter**
- **Uncertainty over value of interest**

Output genericity: same as most generic input variable/parameter

Propagation: usual steps

1. **Represent**: provide an uncertainty model for \( x_1, \ldots, x_n \)
2. **Merge**: if multiple models given for \( x_i \), merge into a single one
3. **Combine**: specify (in)dependencies between \( x_i \)’s to get global model
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5. **Decide**: once uncertainty on \( y \) estimated, decide on an action
Merging: Definition and goals [19]

Combine items of information $I_1, \ldots, I_S$ on quantity $X \in \mathcal{X}$ given by $S$ sources:

$$f(I_1, \ldots, I_S) = I^*$$

- Usually, $X$ assumed to have a true, yet unknown value in $\mathcal{X}$
- In principle $S$ can be the (multi-dim) real space, finite space of elements/classes, space of functions, ...
- $I_i$ and $I^*$ are generally uncertainty models of the same theory (framework closeness)

**Goal** of information merging: how to pick $f$ to
- Gain information from $I_1, \ldots, I_S$
- Increase the reliability (trust) in my final result
The three basic fusion schemes

- **Conjunction:**
  \[ f = \cap, \quad I^* = \cap_{i=1}^{S} I_i \]
  Assumes that all sources provide reliable information (no important conflict allowed)

- **Disjunction:**
  \[ f = \cup, \quad I^* = \cup_{i=1}^{S} I_i \]
  Assumes that at least one source is reliable (very conservative assumption)

- **(Weighted) average:**
  \[ f = \sum w_i, \quad I^* = \sum_{i=1}^{S} w_i I_i \]
  Assumes that most sources are ok (equivalent to counting)
Probabilities and merging

Assume that we have $P_1$, $P_2$ as opinions:

- Conjunction is impossible, as $P_1 \cap P_2$ exists only if $P_1 = P_2$.
  $\rightarrow$ Product $P_1 \cdot P_2$ may be considered as a surrogate to "intersection"

- Disjunction (or its convex hull) provides $P_1 \cup P_2$, not a single probability!

- Average is ok, $\alpha P_1 + (1 - \alpha) P_2$ still a probability
Sets and merging

Assume that we have $E_1, E_2$ as opinions:

- Conjunction is possible, provided by $E_1 \cap E_2 \neq \emptyset$ (no conflict)
- Disjunction gives $E_1 \cup E_2$, again a set, possibly quite big
- Average $\frac{1}{2}E_1 + \frac{1}{2}E_2$ gives a random set, not a set!
Using non-parametric Kolmogorov-Smirnov bounds.

Useful when small samples and no idea about the possible shape of the distribution (if it exists)

Example: variable $X \in [0, 16]$, observations (1; 1.5; 3; 3.5; 4; 6; 10; 11; 14; 15)
Expert providing a finite set of possible percentiles.

Exemple:

\[ 0 \leq P([\infty, 4]) \leq 0.2 \]
\[ 0.1 \leq P([\infty, 8]) \leq 0.3 \]
\[ 0.5 \leq P([\infty, 12]) \leq 0.7 \]
Combining those two sources

Different ways to combine this information

\[ [F, \overline{F}]_2 \quad [F, \overline{F}]_1 \quad [F, \overline{F}]_{\Sigma} \quad [F, \overline{F}]_{\cap} \quad [F, \overline{F}]_{\cup} \]
Outline

1. Introductory elements
2. Imprecise probabilities: use and misuse
3. Representing partial (probabilistic) knowledge
4. Merging partial (probabilistic) knowledge
5. Independence and propagation
   - Independence as a strong information
   - Independence with imprecise probabilities
   - Propagating imprecise probabilities
6. Decision in presence of imprecision
Independence and propagation

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Uncertainty propagation revisited

Output genericity: same as most generic input variable/parameter

Propagation: usual steps

1. **Represent**: provide an uncertainty model for $x_1, \ldots, x_n$
2. **Merge**: if multiple models given for $x_i$, merge into a single one
3. **Combine**: specify (in)dependencies between $x_i$’s to get global model
4. **Propagate**: propagate to get uncertainty over $y$
5. **Decide**: once uncertainty on $y$ estimated, decide on an action
Independence statement = strong information

- For one $X$, uniformity $\neq$ lack of knowledge
  $\rightarrow$ Symmetry of knowledge $\neq$ knowledge of symmetry
- For two $X, Y$, independence $\neq$ lack of knowledge about interaction
  $\rightarrow$ No knowledge of interaction $\neq$ knowledge of no interaction
- Statistically speaking, stating independence requires just as much data as stating dependence

IP tools instrumental to consider sets of dependence assumptions, even when marginal distributions are well-known.
A small reliability example

- Two pumps $X$ and $Y$, either functioning $(x, y)$ or not $(\neg x, \neg y)$
- Overall system $\phi(X, Y)$ works if and only if one of the pump works (XOR):
  - no pump functioning means no pumping
  - two pumps functioning means overload

\[
\phi(X, Y) = \begin{cases} 
1 & \text{if } x \neg y \lor \neg xy \\
0 & \text{else } (xy \lor \neg x \neg y)
\end{cases}
\]

- Probability of the system functioning is

\[
P(\phi(X, Y) = 1) = P(x \neg y \lor \neg xy) \\
= P(x \neg y) + P(\neg xy)
\]
Independent case

Assume \( p_X(x) = 0.7, p_Y(y) = 0.6 \) and the resulting joint

\[
\begin{array}{ccc}
\text{x} & \neg \text{x} & \sum \\
\text{y} & 0.7 \cdot 0.6 & 0.3 \cdot 0.6 & 0.6 \\
\neg \text{y} & 0.7 \cdot 0.4 & 0.3 \cdot 0.4 & 0.4 \\
\sum & 0.7 & 0.3 & \\
\end{array}
\]

\[
P(\phi(X, Y) = 1) = P(x\neg y) + P(\neg xy) \\
= p_X(x)p_Y(\neg y) + p_X(\neg x)p_Y(y) = 0.46
\]

\( \rightarrow \) less chance of working than not working
Unknown dependence case: upper bound

Assume $p_X(x) = 0.7$, $p_Y(y) = 0.6$ and the table

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$\neg x$</th>
<th>$\sum$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.3</td>
<td>$\min(p_X(\neg x), p_Y(y)) = 0.3$</td>
<td>0.6</td>
</tr>
<tr>
<td>$\neg y$</td>
<td>$\min(p_X(x), p_Y(\neg y)) = 0.4$</td>
<td>0.3</td>
<td>0.4</td>
</tr>
<tr>
<td>$\sum$</td>
<td>0.7</td>
<td>0.3</td>
<td></td>
</tr>
</tbody>
</table>

$$P(\phi(X, Y) = 1) = \max P(x \neg y) + P(\neg xy)$$

$$= 0.3 + 0.4 = 0.7$$
Independence and propagation

Unknown dependence case: lower bound

Assume $p_X(x) = 0.7, p_Y(y) = 0.6$ and the table

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$\neg x$</th>
<th>$\sum$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$\min(p_X(x), p_Y(y)) = 0.6$</td>
<td>$0$</td>
<td>$0.6$</td>
</tr>
<tr>
<td>$\neg y$</td>
<td>$0.1$</td>
<td>$\min(p_X(\neg x), p_Y(\neg y)) = 0.3$</td>
<td>$0.4$</td>
</tr>
<tr>
<td>$\sum$</td>
<td>$0.7$</td>
<td>$0.3$</td>
<td></td>
</tr>
</tbody>
</table>

$P(\phi(X, Y) = 1) = \min P(x \neg y) + P(\neg xy)$

$= 0.1 + 0 = 0.1$

$\rightarrow [P, \overline{P}] = [0.1, 0.7]$, incomparability of working vs not working
Independence and propagation

Partially assumed dependence: common case failure

Assume \( p_X(x) = 0.7, p_Y(y) = 0.6 \) and the following bounds

\[
\begin{array}{ccc}
\text{x} & \text{\neg x} & \sum \\
\hline 
y & p_X(x) \cdot p_Y(y) + \epsilon & p_X(\neg x) \cdot p_Y(y) - \epsilon & 0.6 \\
\neg y & p_X(x) \cdot p_Y(\neg y) - \epsilon & p_X(\neg x) \cdot p_Y(\neg y) + \epsilon & 0.4 \\
\sum & 0.7 & 0.3 & \\
\end{array}
\]

with \( 0 \leq \epsilon \leq 0.08 \), mild assumption of common cause failure

\( \rightarrow [P, \overline{P}] = [0.3, 0.46] \), no change in conclusions despite imprecision
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On independence and interpretation [11, 12]

Use of Independence $\sim$ facilitate computations in multi-variate problems

Meaning for two quantities $X, Y$ to be independent, when they are:

- **generic**: in this case, $X, Y$ associated to distributions $P_X, P_Y$ and stochastic independence may apply $\rightarrow$ imprecise probabilities $=$ sensitivity analysis of an "objective" concept.

- **singular**: $X, Y$ are supposed to have one true value.
  Independence here is "subjective", and purely concerns beliefs, not how the values of $X$ and $Y$ can affect each others.

In singular case, much less clear how it should be modelled and even measured?
Two views of independence

In general, two ways to express independence of $X$, $Y$:

- Compositional (stochastic) independence:

  \[ P_{X,Y}(X \in A, Y \in B) = P_X(X \in A)P_Y(Y \in B) \]

  Clear if $P \simeq$ frequencies, less if $P = $ degrees of belief

- Conditional ("epistemic") independence of $Y$ w.r.t. $X$:

  \[ P_X(X \in A | Y \in B) = P_X(X \in A) \]

  Express that learning $B$ about $Y$ do not change belief about $X$
  - A non-symmetric notion, but with precise probabilities become symmetric
  - With precise probabilities, reduces to the first definition

When $P$ becomes imprecise, the two notions extends in different ways.
Three different definitions

Assume I have $\mathcal{P}_X, \mathcal{P}_Y$ on finite spaces

- **Strong independence** ($SI$)
  \[
  \mathcal{P}^{SI}_{XY} = \{ p | p(x, y) = p(x)p(y), p(x) \in \mathcal{P}_X, p(y) \in \mathcal{P}_Y \}
  \]

- **Epistemic irrelevance** ($IR$) of $X$ w.r.t. $Y$
  \[
  \mathcal{P}^{IR}_{X \rightarrow Y} = \{ p | p(x, y) = p(y|x)p(x), p(x) \in \mathcal{P}_X, p(y|x) \in \mathcal{P}_Y \}
  \]
  We can have $p(y|x) \neq p(y|x')$ for $x \neq x'$, and $\mathcal{P}^{IR}_{X \rightarrow Y} \neq \mathcal{P}^{IR}_{Y \rightarrow X}$

- **Random set independence**, if $\mathcal{P}_X, \mathcal{P}_Y$ representable by $m_X, m_Y$
  \[
  \mathcal{P}^{RI}_{XY} = \{ p | P(C) \geq \sum_{AxB \subseteq C} m_X(A)m_Y(B) \}
  \]
  equivalent to consider joint mass $m_X Y(A \times B) = m_X(A)m_Y(B)$
Inclusion relationship

In general, we have

\[ \mathcal{P}^{SI}_{XY} \subseteq \left\{ \begin{array}{l} \mathcal{P}^{IR}_{X\rightarrow Y} \\ \mathcal{P}^{IR}_{Y\rightarrow X} \end{array} \right\} \subseteq \mathcal{P}^{RI}_{XY} \]

Allowing to use one principle to approximate another, for example for computational convenience.

In the precise case, they all collapse to the same formal definition.
Assume now $p_X(x) \in [0.6, 0.8]$ and $p_Y(y) \in [0.5, 0.7]$

$$P(\phi(X, Y) = 1) = p_X(x)p_Y(\neg y) + p_X(\neg x)p_Y(y) = 0.46$$

$$\overline{P}(\phi(X, Y) = 1) = \overline{p}_X(x)\overline{p}_Y(\neg y) + \overline{p}_X(\neg x)\overline{p}_Y(y)$$

$$0.8 \cdot 0.3 + 0.2 \cdot 0.7 = 0.38$$

$$\overline{P}(\phi(X, Y) = 1) = \overline{p}_X(x)\overline{p}_Y(\neg y) + \overline{p}_X(\neg x)\overline{p}_Y(y)$$

$$0.6 \cdot 0.4 + 0.2 \cdot 0.5 = 0.5$$
IP and random set independence: upper case

$p_X(x) \in [0.6, 0.8]$ and $p_Y(y) \in [0.5, 0.7]$ give masses $m_X, m_Y$. Apply product rule to get $m_{X,Y}$

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$\neg x$</th>
<th>$x, \neg x$</th>
<th>$\sum$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.3</td>
<td>0.1</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>$\neg y$</td>
<td>0.18</td>
<td>0.06</td>
<td>0.06</td>
<td>0.3</td>
</tr>
<tr>
<td>$y, \neg y$</td>
<td>0.12</td>
<td>0.04</td>
<td>0.04</td>
<td>0.2</td>
</tr>
<tr>
<td>$\sum$</td>
<td>0.6</td>
<td>0.2</td>
<td>0.2</td>
<td></td>
</tr>
</tbody>
</table>

$P(\phi(X, Y) = 1) = \sum \text{[green]} = 0.28$

$\overline{P}(\phi(X, Y) = 1) = \sum \text{[green]} + \sum \text{[pink]} = 0.74$
A summary so far

Imprecise probability uses:
- Model a random variable whose distribution is ill-known
- Model beliefs about a deterministic but ill-known variable
- Relax the need to specify a single (in)dependence assumption
- Rather than making one computation per hypothesis, directly compute bounds over set of hypothesis

In practice, most operations can be achieved/approximated through linear optimisation, once propagation through $f$ is done.

Some warnings
- Just as for probabilities, making exact computations for complex models difficult
- Some different notions reducing to the same mathematical tools in probability (independence) may have various extensions
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Uncertainty propagation revisited

Model
\[ f(x_1, \ldots, x_n) = y \]

Output genericity: same as most generic input variable/parameter

Propagation: usual steps

1. Represent: provide an uncertainty model for \( x_1, \ldots, x_n \)
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Random set propagation: principle

Assume:
- each $X_i$ is associated to mass $m_i$
- random set independence holds
- only intervals receive positive mass

Propagation is equivalent to
- Take selections of intervals $E_1, \ldots, E_n$ from $m_1, \ldots, m_n$
- Compute $f(E_1, \ldots, E_n)$ (main bottleneck, as with probabilities)
- Associate product of masses $m_1(E_1) \ldots m_n(E_n)$ to $f(E_1, \ldots, E_n)$
- Make inferences from that mass
A minimal example

- two polluting elements $X_1, X_2$, expressed in average percentage per $m^3$
- experts tell that
  - $P(X_1 \in [1, 4]) = 0.8$, and in any case $X_1 \in [0, 10]$
  - $P(X_2 \in [2, 5]) = 0.6$, and in any case $X_2 \in [1, 8]$
- We are interested in the value $Y = X_1 + X_2$
### A minimal example continued

<table>
<thead>
<tr>
<th>Interval</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1, 4]</td>
<td>0.8</td>
</tr>
<tr>
<td>[0, 10]</td>
<td>0.2</td>
</tr>
<tr>
<td>[2, 5]</td>
<td>0.6</td>
</tr>
<tr>
<td>[1, 8]</td>
<td>0.4</td>
</tr>
</tbody>
</table>

- Take selections of intervals $E_1, \ldots, E_n$ from $m_1, \ldots, m_n$
A minimal example continued

<table>
<thead>
<tr>
<th>Interval</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1, 4]</td>
<td>0.8</td>
</tr>
<tr>
<td>[0, 10]</td>
<td>0.2</td>
</tr>
<tr>
<td>[2, 5]</td>
<td>0.6</td>
</tr>
<tr>
<td>[3, 9]</td>
<td></td>
</tr>
<tr>
<td>[2, 15]</td>
<td></td>
</tr>
<tr>
<td>[1, 8]</td>
<td>0.4</td>
</tr>
<tr>
<td>[2, 12]</td>
<td></td>
</tr>
<tr>
<td>[1, 18]</td>
<td></td>
</tr>
</tbody>
</table>

- Take selections of intervals $E_1, \ldots, E_n$ from $m_1, \ldots, m_n$
- Compute $f(E_1, \ldots, E_n)$ (main bottleneck, as with probabilities)
#### A minimal example continued

<table>
<thead>
<tr>
<th>Interval</th>
<th>Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1, 4]</td>
<td>0.8</td>
</tr>
<tr>
<td>[0, 10]</td>
<td>0.2</td>
</tr>
<tr>
<td>[2, 5]</td>
<td>0.6</td>
</tr>
<tr>
<td>[3, 9]</td>
<td>0.54</td>
</tr>
<tr>
<td>[2, 15]</td>
<td>0.12</td>
</tr>
<tr>
<td>[1, 8]</td>
<td>0.4</td>
</tr>
<tr>
<td>[2, 12]</td>
<td>0.32</td>
</tr>
<tr>
<td>[1, 18]</td>
<td>0.08</td>
</tr>
</tbody>
</table>

- Take selections of intervals $E_1, \ldots, E_n$ from $m_1, \ldots, m_n$
- Compute $f(E_1, \ldots, E_n)$ (main bottleneck, as with probabilities)
- Associate product of masses $m_1(E_1) \ldots m_n(E_n)$ to $f(E_1, \ldots, E_n)$
A minimal example continued

\[
m([1, 4]) = 0.8 \quad m([0, 10]) = 0.2
\]
\[
m([2, 5]) = 0.6 \quad m([3, 9]) = 0.54 \quad m([2, 15]) = 0.12
\]
\[
m([1, 8]) = 0.4 \quad m([2, 12]) = 0.32 \quad m([1, 18]) = 0.08
\]

\[P(Y \leq 10) = P(Y \in [0, 10]) \in [0.54, 1]\]

- Take selections of intervals \(E_1, \ldots, E_n\) from \(m_1, \ldots, m_n\)
- Compute \(f(E_1, \ldots, E_n)\) (main bottleneck, as with probabilities)
- Associate product of masses \(m_1(E_1) \ldots m_n(E_n)\) to \(f(E_1, \ldots, E_n)\)
- Make inferences from that mass
Random set propagation: practice [1, 23]

- Still assuming random set independence
- Pick $\alpha_1, \ldots, \alpha_N \in [0, 1]$ randomly from uniform

![Graphs showing random set propagation](image)

- Estimate $f([F_1^{-1}(\alpha_1), F_1^{-1}(\alpha_1)], \ldots, [F_N^{-1}(\alpha_N), F_N^{-1}(\alpha_N)])$
- Repeat $R$ times: $R$ imprecise samples of $Y$
A more involved example

- Enbankment construction
- Simplified model of flood levels $h$

\[
\begin{align*}
\text{if } q \geq 0 & : \quad h(q, k, u, d) = \left( \frac{q}{k \sqrt{\frac{u-d}{\ell}}} \right)^{1/3} b^{	ext{cm}} \\
\text{otherwise} & : \quad h(q, k, u, d) = 0
\end{align*}
\]
## Parameter uncertainties

<table>
<thead>
<tr>
<th>significance</th>
<th>units</th>
<th>uncertainty</th>
<th>Repres.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$ flood level</td>
<td>$m$</td>
<td>(model)</td>
<td>p-box</td>
</tr>
<tr>
<td>$q$ maximal flow</td>
<td>$m^3s^{-1}$</td>
<td>Gumbel($\mu \in [1300, 1400]$, $\beta = 715$)</td>
<td></td>
</tr>
<tr>
<td>$b$ river width</td>
<td>$m$</td>
<td>300</td>
<td>value</td>
</tr>
<tr>
<td>$k$ Strickler coefficient</td>
<td>$m^{1/3}s^{-1}$</td>
<td>mode=30, support=[15,35]</td>
<td>possibility</td>
</tr>
<tr>
<td>$u$ upriver level</td>
<td>$m$</td>
<td>mode=55, support=[54,56]</td>
<td>possibility</td>
</tr>
<tr>
<td>$d$ downriver level</td>
<td>$m$</td>
<td>mode=50, support=[49,51]</td>
<td>possibility</td>
</tr>
<tr>
<td>$l$ section length</td>
<td>$m$</td>
<td>6400</td>
<td>value</td>
</tr>
</tbody>
</table>
Exemple: illustration

Result on thresholding events ($P([0, x])$) for independence assumption
Random set propagation: limits and warnings

- Limited expressiveness: not all convex sets of probabilities are random sets
- Independence conservativeness: random set independence gives wider bounds than robust stochastic independence
- Dependence modelling: using copulas on distributions $m$ is not equivalent to a robust applications of copulas, unless all marginal models are $p$-boxes (imprecise cumulative distributions)

Practical application not much more difficult than for precise probabilities (similar computational bottlenecks)
Propagation and imprecise probabilities: recent trends

- Using importance sampling techniques to consider multiple (set of) probabilities with one sample [22, 40]
- Combining imprecise probability tools with surrogate models, i.e., polynomial chaos expansions [33]
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   - General decision rules
Uncertainty propagation revisited

**Plot**
- **Model** $f(x_1, \ldots, x_n) = y$
- **Prediction**
- **Distrib. of data/parameter**
- **single-valued data/parameter**
- **Uncertainty over value of interest**

**Output genericity:** same as most generic input variable/parameter

**Propagation: usual steps**

1. **Represent:** provide an uncertainty model for $x_1, \ldots, x_n$
2. **Merge:** if multiple models given for $x_i$, merge into a single one
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Precise case

Fix a threshold $\tau$, decide whether $P(X \leq \tau) \geq \alpha$ with $\alpha$ critical level
Precise case: example

For instance, take $\alpha = 0.9$

We have $P(X \leq \tau) > \alpha$, acceptable risk
Precise case: example

For instance, take $\alpha = 0.9$

We have $P(X \leq \tau) < \alpha$, unacceptable risk
Imprecise case

Fix a threshold $\tau$, decide whether $P(X \leq \tau) \geq \alpha$ with $\alpha$ critical level

![Graph showing cumulative distribution functions](image-url)
Imprecise case: example

For instance, take $\alpha = 0.9$

We have $\overline{P}(X \leq \tau) > \underline{P}(X \leq \tau) > \alpha$, acceptable risk
Imprecise case: example

For instance, take $\alpha = 0.9$

We have $\underline{P}(X \leq \tau) < \overline{P}(X \leq \tau) < \alpha$, unacceptable risk
Imprecise case: example

For instance, take $\alpha = 0.9$

We have $P(X \leq \tau) < \alpha < \overline{P}(X \leq \tau)$, no clear answer
Imprecise case: three DM different attitudes

- No preference of behaviour: undecided
- Pessimistic: pick worst case $\overline{P}(X \leq \tau)$
- Optimistic: pick best case $\underline{P}(X \leq \tau)$

Key idea: decision maker attitude should be separated from the available information
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Decision-making: classical frames

Given set of uncertain quantities/utilities $X_1, \ldots, X_n$, compare pair-wisely

- By expectation: $X_i \succ_{E} X_j$ if $E(X_i) \geq E(X_j)$ with
  
  $$E(X) = \sum x \cdot p(x)$$

---

**Pros**: most recognized criteria, strong theoretical foundations

---

**Cons**: necessitates utility values to be elicited and well-defined

---

**Risk decision**: specific case with utility function $I_{X \leq \tau}$

- By statistical preference: $X_i \succ_{P} X_j$ if $P(X > Y) > 0.5$

---

**Pros**: only need an ordinal scale, close to the notion of median

---

**Cons**: possible cycles ($X \succ_{P} Y \succ_{P} Z \succ_{P} X$), need dependencies
Decision-making: classical frames

Given set of uncertain quantities/utilities \( X_1, \ldots, X_n \), compare pair-wisely

- By stochastic dominance: \( X_i \succ_F X_j \) if \( P(X_i \leq x) \leq P(X_j \leq x) \)

**Pros:** if \( X_i \succ_F X_j \), \( g(X_i) \succeq_E g(X_j) \) for any increasing \( g \)

**Cons (?):** may lead to incomparability (\( X_i \not\succ_F X_j \) and \( X_j \not\succ_F X_i \))

**Risk decision:** fixing a threshold rather than all of them
Imprecise probability: rough ideas for expectations

When going imprecise, points become intervals → how to compare

\[ [\mathbb{E}(X_i), \overline{\mathbb{E}}(X_i)] \text{ and } [\mathbb{E}(X_j), \overline{\mathbb{E}}(X_j)]? \]

Back to precise comparison (need DM attitude)

- Maximin (pessimist): \( X_i \succ_{Mm} X_j \text{ if } \mathbb{E}(X_i) > \mathbb{E}(X_j) \)
- Maximax (optimist): \( X_i \succ_{MM} X_j \text{ if } \overline{\mathbb{E}}(X_i) > \overline{\mathbb{E}}(X_j) \)
- Hurwicz, in-between: \( X_i \succ_{\gamma} X_j \text{ if } \gamma \mathbb{E}(X_i) + (1 - \gamma)\overline{\mathbb{E}}(X_i) > \gamma \mathbb{E}(X_j) + (1 - \gamma)\overline{\mathbb{E}}(X_j) \)

Acknowledging imprecision and indecision

- Interval dominance: \( X_i \succ_{ID} X_j \text{ if } \mathbb{E}(X_i) > \overline{\mathbb{E}}(X_j) \)
A small example

Imprecise probability model

Three possible states \{a, b, c\}

- \( p(a) \in [0.1, 0.3] \)
- \( p(b) \in [0.3, 0.6] \)
- \( p(c) \in [0.3, 0.6] \)
A small example

Imprecise probability model

Three possible decisions/alternatives \( \{X_1, X_2, X_3\} \)

\[
\begin{pmatrix}
  a & b & c \\
  X_1 & 0 & 1 & 2 \\
  X_2 & 1 & 0.5 & 1 \\
  X_3 & 3 & 1 & 0 \\
\end{pmatrix}
\]

- \( \mathbb{E}(X_1) \in [1, 1.5] \)
- \( \mathbb{E}(X_2) \in [0.7, 0.85] \)
- \( \mathbb{E}(X_3) \in [0.6, 1.3] \)
Summary and main messages

- Importance to know what kind of uncertainty you want to model.
- Imprecise probability models useful to model lack of knowledge, or perform robustness analysis.
- Not useful if you want a precise numbers, comparability of any pair of events or of any decisions, no matter what your knowledge is.
- Computational burden usually higher, but not necessarily much higher than precise models.
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