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Focus on Reliable (and Robust) – Based Design Optimization of structures or systems

ETICS2018

Research School on Uncertainty in Scientific Computing
Roscoff, June 3-8, 2018

UNCERTAINTY QUANTIFICATION IN MECHANICAL ENGINEERING
Using probabilistic methods

APPLIED RESEARCH

Tolerance analysis of products

Fatigue damage analysis of structures

Need numerical tools to reduce the computing effort

Meta-models - based approaches (polynomial chaos, kriging)

Pr. N. Gayton – Past and on going PhD

Past PhD:

- ✓ A. Notin: Polynomial chaos and resampling for the failure probability assessment (2010)
- ✓ B. Echard: The use of kriging for the reliability assessment of structures subjected to fatigue (2012)
- ✓ A. Dumas: Probabilistic methods for the tolerance analysis of over-constraint mechanical systems (2014)
- ✓ S. Bucas: Reliability assessment of tower crane structural members (2014)
- ✓ P. Beaucaire: The use of probabilistic methods for the tolerance analysis and tolerance synthesis of systems

PhD on going:

- ✓ N. Lelievre: The use of AK-methods in high dimension (defense planed at the end of 2018)
- ✓ Q. Huchet: The use of kriging for an efficient wind turbine damage assessment (EDF collaboration, defense planed at the end of 2018)
- ✓ V. Chabridon: Reliability-based sensitivity analysis under distribution parameter uncertainty – Application to aerospace engineering (ONERA collaboration, defense planed at the end of 2018)
- ✓ D. Idriss: Sensitivity analysis for the search of critical dimension in tolerance analysis (Defense planed in 2020)
- ✓ M. Ahmadivala: Optimal maintenance planning of existing structures using monitoring data (Defense planed in 2010)
- ✓ C. Amrane: the use of mixed meta-models for the failure probability assessment (PhD just beginning)

Pr. N. Gayton – Main publications

- ✓ LELIEVRE N., BEAUREPAIRE P., MATTRAND C., GAYTON N., **AK-MCSi: a Kriging-based method to deal with small failure probabilities and time-consuming models**, Structural Safety, Vol. 73, pages 1-11, 2018.
- ✓ CHABRIDON V., BALESDENT M., BOURINET JM., MORIO J., GAYTON N., **Rare event simulation under probability distribution parameter uncertainties: applications to aerospace system reliability assessment**, Aerospace Science and Technology, vol. 69, pp. 526-537, 2017.
- ✓ LELIÈVRE N., BEAUREPAIRE P., MATTRAND C., GAYTON N., OSTMANE A., **On the consideration of uncertainty in design: Optimization - Reliability – Robustness**, Structural and Multidisciplinary Optimization, Educational paper, doi:10.1007/s00158-016-1556-5, 2016.
- ✓ FAURIAT W., MATTRAND C., GAYTON N., BEAKOU A., CEMBRSYNSKI T., **Estimation of road profiles variability from measured vehicle response**, Vehicle system dynamics, Vol. 54(5) pages 585-605, 2016.
- ✓ FAURIAT W., MATTRAND C., GAYTON N.n BEAKOU A., **An application of Stochastic simulation to the study of the variability of road induces fatigue load, Procedia Engineering**, Vol. 133, pages 631 – 645, 2015.
- ✓ DUMAS A., DANTAN J.Y., GAYTON N., **Impact of the behavior model linearization strategy on the tolerance analysis of hyperstatic mechanisms,** Computer Aided Design, vol. 62, DOI: 10.1016/j.cad.2014.11.002, 2015.
- ✓ DUMAS A., GAYTON N., DANTAN J.Y., SUDRET B., **A new system formulation for the tolerance analysis of over-constrained mechanisms,** Probabilistic Engineering Mechanics, Vol. 40, 2015.
- ✓ BUCAS S., RUMELHART P., GAYTON N., CHATEAUNEUF A., **A global procedure for reliability assessment of crane structural elements,** Engineering Failure analysis, Engineering Failure analysis, Vol. 42, pages 143-156, 2014. *Nb.*
- ✓ FAURIAT W., GAYTON N., **AK-SYS: an adaptation of the AK-MCS method for system reliability,** Reliability engineering and System Safety, Vol. 123, pages 137–144, 2014.
- ✓ ECHARD B., GAYTON N., BIGNONNET A., **A reliability analysis method for fatigue design,** International journal of fatigue, Vol. 14, pages 292-300, 2013.
- ✓ BEAUCAIRE P., GAYTON N., DANTAN JY., DUC E., **Statistical tolerance analysis of over-constrained mechanism using system reliability methods,** Computer Aided Design, Vol. 45(2), pages 1547 – 1555, 2013.

Pr. N. Gayton – Main publications

- ✓ ECHARD B., GAYTON N., LEMAIRE M., RELUN N., **A combined Importance Sampling and Kriging reliability method for small failure probabilities with time demanding numerical models,** Reliability Engineering and System Safety, Vol. 111, pages 232-240, 2013.
- ✓ DUMAS A., ECHARD B., GAYTON N., ROCHAT O., DANTAN JY., **AK-ILS: an Active learning method based on Kriging for the Inspection of Large Surfaces,** Precision engineering, Vol. 37, pages 1-9, 2013.
- ✓ BEAUCAIRE P., GAYTON N., DUC E., LEMAIRE M., DANTAN JY., **Statistical tolerance analysis of an hyperstatic mechanism with gaps using system reliability methods,** Computer and Industrial Engineering, Vol. 63, pages 1118-1127, 2012.
- ✓ QURESHI A.J., SABRI V., DANTAN JY., BEAUCAIRE P., GAYTON N., **A statistical tolerance analysis approach for over-constrained mechanism based on optimization and Monte Carlo simulation,** Computer Aided Design, Vol. 44(2), pages 132-142, 2012.
- ✓ GAYTON N., BEAUCAIRE P., DUC E., LEMAIRE M., **The APTA method for the tolerance analysis of products – comparison of capability-based tolerance and inertial tolerance,** Asian International Journal of Science and Technology, Vol. 4(3), pages 24-36, 2011.
- ✓ ECHARD B., GAYTON N., LEMAIRE M., **AK-MCS: an Active learning reliability method combining Kriging and Monte Carlo Simulation,** Structural Safety, Vol. 33, pages 145-154, 2011.
- ✓ GAYTON N., BEAUCAIRE P., BOURINET J.M., DUC E., LEMAIRE M., GAUVRIT L., **APTA: Advanced Probability - based Tolerance Analysis of products**, Mécanique et Industrie, Vol 12, pages 71-85, 2011.
- ✓ NOTIN A., GAYTON N., DULONG J.L., LEMAIRE M., VILLON P., **RPCM: A strategy to perform reliability analysis using polynomial chaos and resampling - application to fatigue design,** European Journal of Computational Mechanics, Vol. 19(8), pages 795-830, 2010.
- ✓ GAYTON N., MOHAMED A., SORENSEN J.D., PENDOLA M., LEMAIRE M., **Calibration methods for reliability-based design codes,** Structural Safety, Vol. 26, pages 91-121, 2004.
- ✓ GAYTON N., BOURINET J.M., LEMAIRE M., **CQ2RS: A new statistical approach to the response surface method for reliability analysis**, Structural Safety, Vol. 25, pages 99-121, 2003.

My use of RBDO

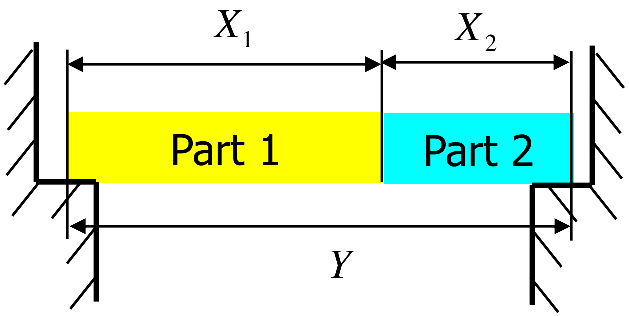
Partial safety factor calibration: an optimization problem with uncertainties but without any constraints

$$\gamma^* = \min_{\gamma} W(\gamma) = \sum_{j=1}^{L} w_j \, \mathrm{M}(\beta_j(\gamma), \beta_c)$$

Optimization of tolerance in a mechanical system: minimize the production cost with respect to a target quality level

$$\mathrm{t}^* = \min_{\mathrm{t}} f(\mathrm{t})$$
$$\text{under} \quad P_f(\mathrm{t}) \leq P_{Target}$$

My use of RBDO



Functional requirement $Y = X_1 + X_2 \in [9,5-10,5]$

$T_1 = 6, T_2 = 4$ $\quad$ $t_1 = t_2 = ?$

Minimize: $f(t_1,t_2) = \frac{c_1}{t_1} + \frac{c_2}{t_2}$

Under: $g(t_1,t_2) = P_f(t_1,t_2) - P_{\text{Target}} \leq 0$

$c_i = 1, P_{Target} = 10^{-6}, \quad t_i = 0,43$
$c_i = 1, P_{Target} = 10^{-5}, \quad t_i = 0,48$
$c_i = 1, P_{Target} = 10^{-4}, \quad t_i = 0,54$
$c_i = 1, P_{Target} = 10^{-3}, \quad t_i = 0,64$
$c_i = 1, P_{Target} = 10^{-2}, \quad t_i = 0,82$
$c_i = 1, P_{Target} = 10^{-1}, \quad t_i = 1,28$

$c_1 = 1, c_2 = 5, P_{Target} = 10^{-6}, t_1 = 0,31, t_2 = 0,52$
$c_1 = 1, c_2 = 5, P_{Target} = 10^{-5}, t_1 = 0,34, t_2 = 0,58$
$c_1 = 1, c_2 = 5, P_{Target} = 10^{-4}, t_1 = 0,39, t_2 = 0,66$
$c_1 = 1, c_2 = 5, P_{Target} = 10^{-3}, t_1 = 0,58, t_2 = 1,00$
$c_1 = 1, c_2 = 5, P_{Target} = 10^{-1}, t_1 = 0,92, t_2 = 1,57$

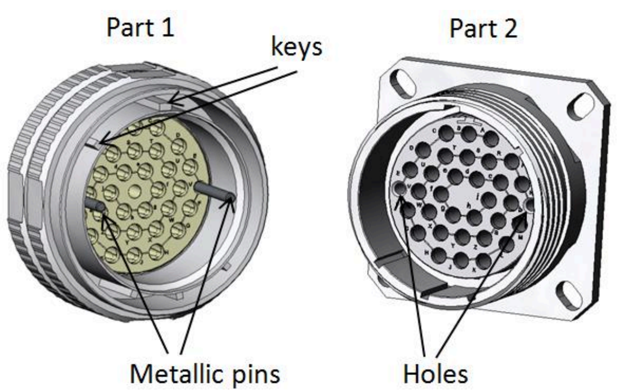
Failure probability estimation of hyper-static mechanisms

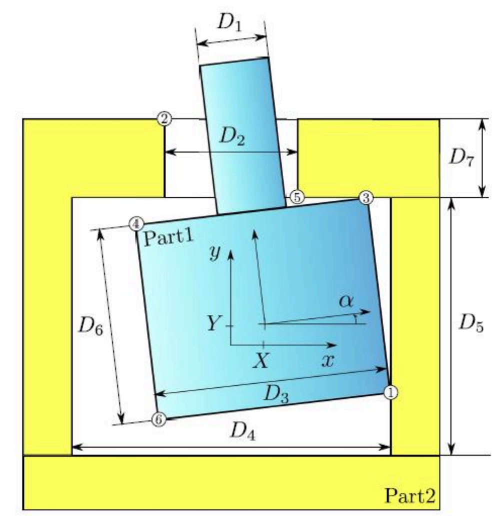
$$P_f = \underset{\gamma}{\text{Prob}}(R(\mathbf{X}) \leq 0)$$

$$\text{where} \quad R(\mathbf{X}) = \min_{g \in \mathbb{R}^m} C_f(\mathbf{X}, \mathbf{g})$$

$$\text{under } C(\mathbf{X}, \mathbf{g}) \leq 0$$

Not really RBDO but nested optimization / uncertainty problem





Who are you ?

- ✓ Students ?
- ✓ PhD ?
- ✓ Researcher ?
- ✓ Teacher ?

What do you expect from this course ?

$$P_f = \text{Prob}\left(g(\mathbf{X}(\omega)) \leq 0 \right) = \int_{g(\mathbf{X}) \leq 0} f_{\mathbf{X}}(\mathbf{X}) \mathrm{d}\mathbf{X}$$

Performance function $g : \mathbb{R}^n \rightarrow \mathbb{R}$

Random variable set $\mathbf{X}$

Joint pdf $f : \mathbb{R}^n \rightarrow \mathbb{R}$

A CLASSIFICATION PROBLEM …

A CLASSIFICATION PROBLEM …

$g(\mathrm{X}) > 0$ Safe domain

$g(\mathrm{X}) \le 0$ Failure domain

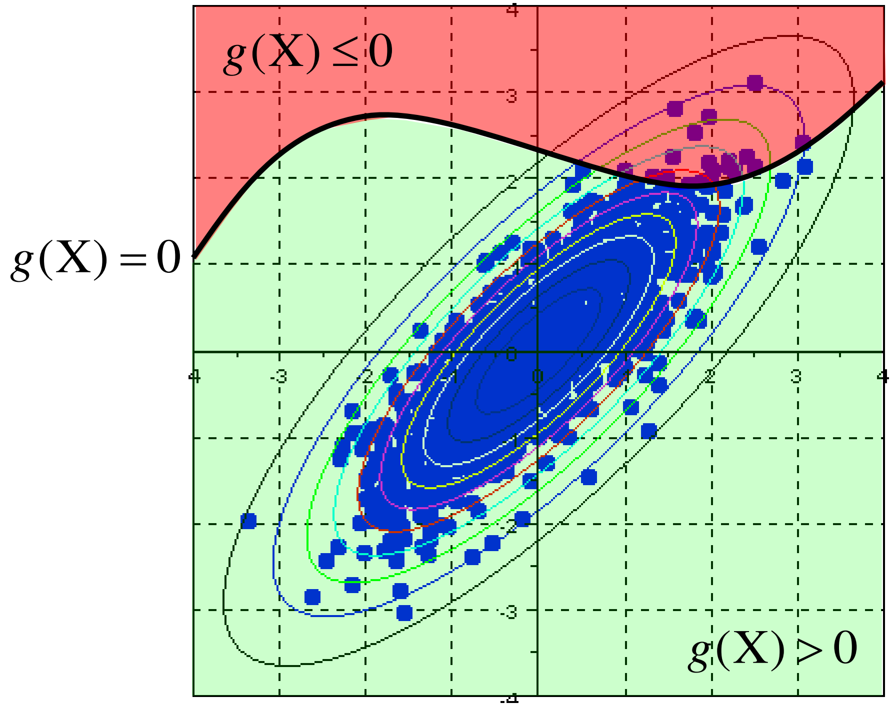
$g(\mathrm{X}) = 0$ Limit state function

$$P_f = \int_{g(\mathrm{X})\le 0} f_{\mathrm{X}}(\mathrm{X})\mathrm{dX}$$

$$= \int_{\mathbb{R}^n} \mathrm{I}_{Df}(\mathrm{X}) f_{\mathrm{X}}(\mathrm{X})\mathrm{dX}$$

$$= \mathrm{E}\left[\mathrm{I}_{Df}(\mathrm{X})\right]$$

$$\mathrm{I}_{Df}(\mathrm{X}) = 1 \quad \text{if} \quad g(\mathrm{X}) \le 0$$
$$= 0 \quad \text{if} \quad g(\mathrm{X}) > 0$$



Monte Carlo estimator

$$P_f = \mathrm{E}\left[\mathrm{I}_{Df}\right] \approx \frac{1}{N}\sum_{r=1}^{N} \mathrm{I}_{Df}^{(r)} = \tilde{P}_f$$

$$\tilde{P}_f\left(1 - 1{,}96\sqrt{\frac{1-\tilde{P}_f}{N\tilde{P}_f}}\right) \le P_f \le \tilde{P}_f\left(1 + 1{,}96\sqrt{\frac{1-\tilde{P}_f}{N\tilde{P}_f}}\right)$$

95% confidence interval

$$P_f = \int_{g(\mathrm{X})\le 0} f_{\mathrm{X}}(\mathrm{X})\mathrm{dX}$$

$$= \int_{\mathbb{R}^n} \mathrm{I}_{Df}(\mathrm{X}) f_{\mathrm{X}}(\mathrm{X})\mathrm{dX}$$

$$= \int_{\mathbb{R}^n} \mathrm{I}_{Df}(\mathrm{X}) \frac{f_{\mathrm{X}}(\mathrm{X})}{h_{\mathrm{X}}(\mathrm{X})} h_{\mathrm{X}}(\mathrm{X})\mathrm{dX}$$

Importance sampling pdf, instrumental pdf

$$= \int_{\mathbb{R}^n} \mathrm{I}_{Df}(\mathrm{X}) W(\mathrm{X}) h_{\mathrm{X}}(\mathrm{X})\mathrm{dX}$$

Importance sampling ratio

$$= \mathrm{E}\left(\mathrm{I}_{Df}(\mathrm{X}) W(\mathrm{X})\right)$$

$$P_f \approx \frac{1}{N}\sum_{r=1}^{N} \mathrm{I}_{Df}(\mathrm{X}^{(r)}) W(\mathrm{X}^{(r)}) = \tilde{P}_f$$

$$\mathrm{Var}(\tilde{P}_f) = \frac{1}{N}\left(\mathrm{E}\left(\left(I(\mathrm{X})\frac{f_X(\mathrm{X})}{h_X(\mathrm{X})}\right)^2\right) - \tilde{P}_f^{\,2}\right)$$

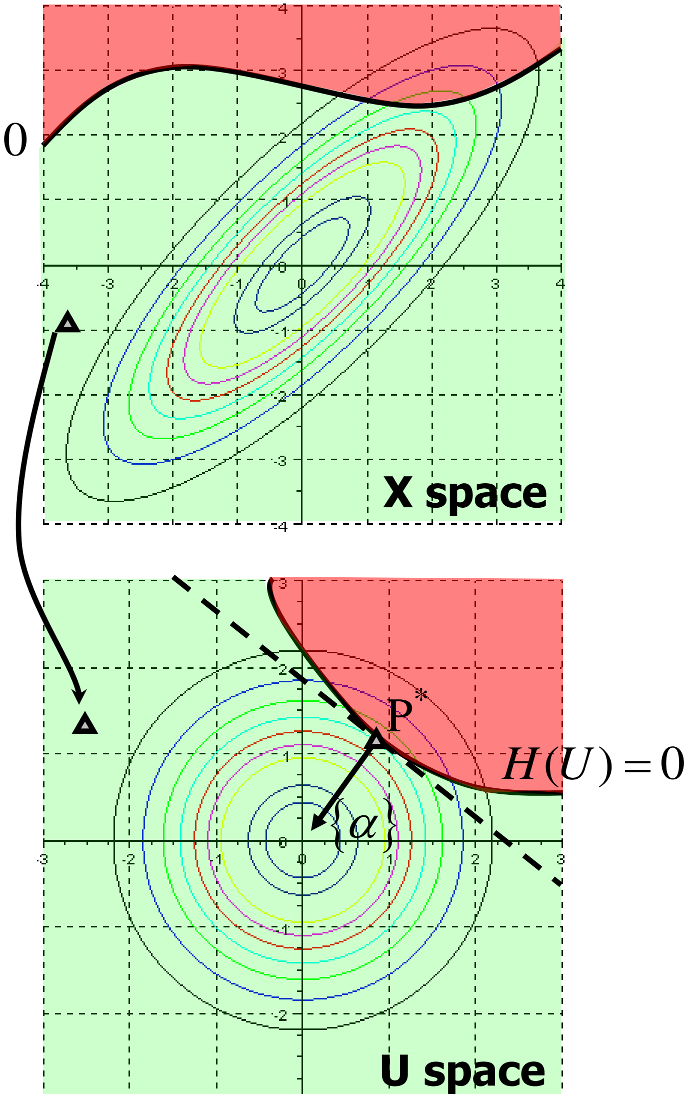
$$U^* = \underset{U \in \mathbb{R}^n}{\operatorname{Argmin}} \left(U^t . U \right)$$

under $H(U) = 0$

with $U = T(X)$

$g(X) = 0$

Iso-probabilistic transformation
$U = T(X)$



FORM approximation: $P_f \approx \Phi(-\beta)$

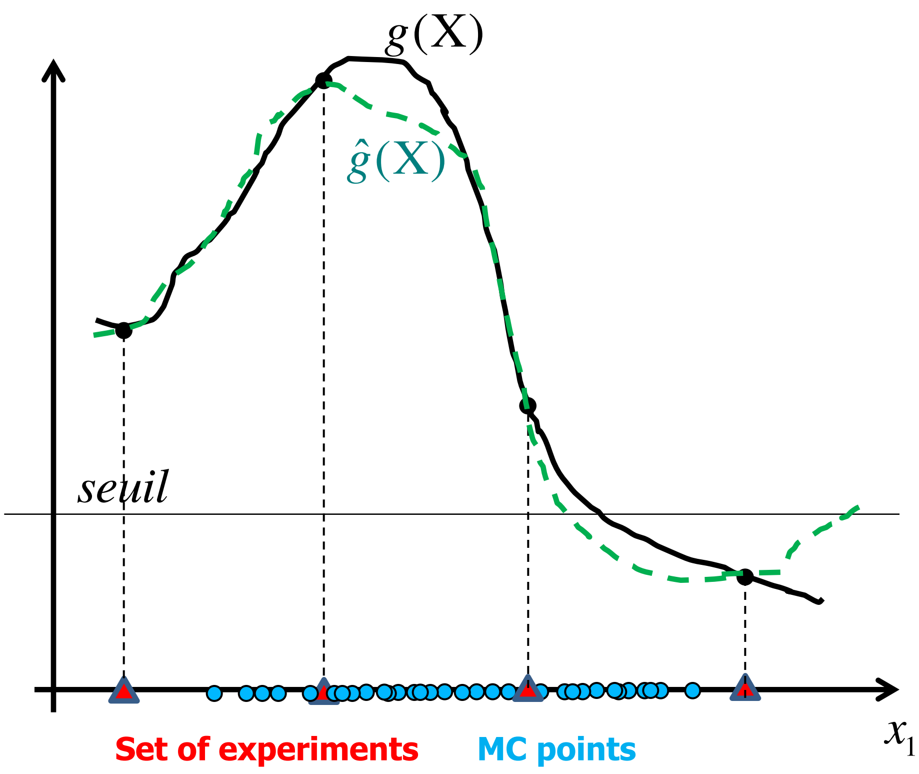
$$\beta = \left\| u^* \right\|$$

Direct cosinus: $$\alpha = \frac{\nabla H\left(U^*\right)}{\left\| \nabla H\left(U^*\right) \right\|}$$

$$U^* = -\beta \alpha$$

Meta-models – based courant strategy for failure probability assessment:

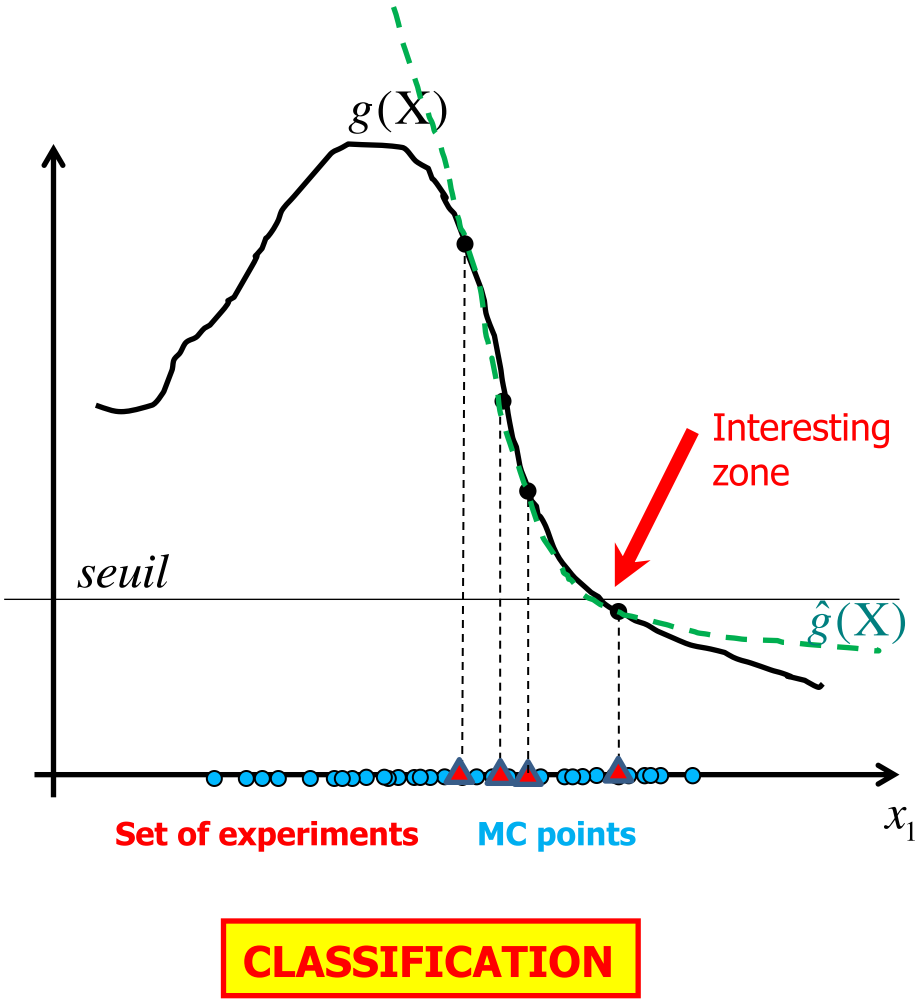
1. Choose a first numerical set of experiments (**FREE**)
2. Call the costly mechanical model (**COSTLY**)
3. Fit a meta-model from the points (**MORE OR LESS FREE**)
4. Quantify the quality of the meta-model approximation. If necessary, go back to 1. to add point to the set of points. If not, go the 5. (**FREE**)
5. Perform a simulation (MC, IS, …) on the meta-model to assess the failure probability (**FREE**)



APPROXIMATION

AK: Active learning and Kriging based strategy:

1. Define the set of candidate points (MC, IS points) (**FREE**)
2. Choose a reduce set of points among the candidate (**FREE**)
3. Call the costly mechanical model (**COSTLY**)
4. Fit a meta-model from the points and go back to 2. (U criteria) while the candidate points are not well classified (**MORE OR LESS FREE**)
5. While all the candidate points are well classified, estimate the failure probability (**FREE**)



AK methods
Active Learning and
Kriging- based methods

Contraints management in optimization

- AK-OPT
- AK-ST
- AK-RBDO

Reliability analysis

- AK-MCS
- AK-IS
- AK-SYS

AK-SS	Huazhong University of Science and Technology, Wuhan, Chine
AK-SSIS	Northeastern University, Shenyang, Chine + Jiangxy University of Science and Technology, Ganzhou, Chine
AK-RBIS	Vanderbilt University, Nashville, US
metaAK-IS	Politecnico di Milano, Italie + Politécnica de Madrid, Espagne + Ecole Centrale Paris et Supelec, France
AK-LS	Northwestern Polytechnical University, Xi'an, Chine
Mixed EGO + AK-MCS	Missouri University of Science and Technology, Rolla, US

Geometrical conformity of parts

- AK-ILS

Optimization under uncertainties

- ✓ **Objective #1** – What are the industrial issues?
- ✓ **Objective #2** – What are the differences between optimization with and without uncertainty?
- ✓ **Objective #3** – What are the different kind of problems and mathematical formulations?
- ✓ **Objective #4** – What are the main difficulties and tools needed to optimize a system under uncertainties?
- ✓ **Objective #5** – What are the recent methods with advantages and drawbacks?

OPTIMIZATION UNDER UNCERTAINTIES IN NOT ONLY A NUMERICAL PROBLEM

Outline of my presentation

1. Introduction / motivation / illustrations

1. Introduction / motivation / illustrations

1.1 Bibliography

1.2 Optimization without uncertainties vs with uncertainties

1.3 Applications

Bibliography on general considerations

NOT ONLY A NUMERICAL PROBLEM …

- ✓ Beyer HG., Sendhoff B., **Robust optimization - A comprehensive survey. Computer Methods in Applied Mechanics and Engineering**, 196:3190-3218, 2007.

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- ✓ Göhler SM., Eifler T., Howard TJ., **Robustness Metrics: Consolidating the multiple approaches to Quantify Robustness**, Journal of Mechanical Design, Vol 138, Nov. **2016**.

About robustness : "*… consumers will often consider it to be synonymous with strength or durability.* "

About metrics : "*From the 108 relevant publications found, 38 metrics were determined to be conceptually different from one another.*"

Robustness: 38 metrics !

#	Name	Mathematical expression	Necessary information entities			Level of complexity		Robustness metric class	Reference
			Model/ experiment	Functional limits	Expected/ measured variation	Independent variables (single/multiple)	Dependent variables (single/multiple)		
5	Normalized partial derivative/sensitivity coefficient	$S_{i\ \text{mean}} = \frac{\partial f}{\partial x_i}(X) \cdot \frac{x_i}{f(X)}$	✓	—	—	single	single	Sensitivity	[10,20,40]
		$S_{i\ \text{Std}} = \frac{\partial f}{\partial x_i}(X) \cdot \frac{\sigma(x_i)}{\sigma(f)}$	✓	—	(✓)	single	single	Sensitivity	[10,13,40]
6	Importance factor	$I_i = \frac{\left(\frac{\partial f}{\partial x_i}(X)\right)^2}{\sum_{j=1}^{N}\left(\frac{\partial f}{\partial x_j}(X)\right)^2}$	✓	—	—	single	single	Sensitivity	[42]
7	FAST Index	$S_{\text{col}}^{(i)} = \frac{\sum_p \left(\lvert A_{\text{pcol}}^{(i)}\rvert^2 + \lvert B_{\text{pcol}}^{(i)}\rvert^2\right)}{\sum_j \left(\lvert A_j^{(i)}\rvert^2 + \lvert B_j^{(i)}\rvert^2\right)}$	✓	—	—	multiple	single	Sensitivity	[9,43,44]
8	Regression coefficients	$\beta_i = \frac{\sum_j \left[\left((x_{i_j} - \mu_{x_i}) \cdot (y - \mu_y)\right)\right]^2}{\sum_j (x_{i_j} - \mu_{x_i})^2}$	✓	—	(✓)	single/ (multiple)	single	Sensitivity	[13,19,32]
9	Standardized regression coefficients	$\text{SRC}(y, x_i) = \beta_i \frac{\sigma_{x_i}}{\sigma_y}$	✓	—	(✓)	single	single	Sensitivity	[45]
10	Spearman robustness index	$\text{SRI} = \min\left\lvert \frac{1}{\rho_{X_i} \cdot \beta_{X_i} \cdot \mu_{X_i}} \right\rvert$	✓	—	(✓)	single	single	Sensitivity	[19,46]
11	Spearman robustness index 2	$\text{SRI} = \frac{1}{\sigma_\rho \sigma_{\beta_{X_i} \cdot \mu_{X_i}}}$	✓	—	(✓)	single	single	Sensitivity	[46]
12	Robustness index	$\eta = \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_1, \ldots, x_i \cdot (1+\Delta), \ldots, x_n) - f(X)}{f(X)}$	✓	—	—	single	single	Sensitivity	[47]
13	Euclidean norm of Jacobian	$\lVert J \rVert_2 = \sqrt{\lambda_{\max}(A^T A)}$	✓	—	—	multiple	single	Sensitivity	[48,49]
14	Frobenius norm of Jacobian	$\lVert J \rVert_F = \left(\sum_{i,j} \lvert a_{ij}\rvert^2\right)^{\frac{1}{2}}$	✓	—	—	multiple	single	Sensitivity	[48]
15	Condition number	$\kappa = \lVert J \rVert_2 \, \lVert J^{-1} \rVert_2$	✓	—	—	multiple	single	Sensitivity	[48–50]
16	Objective robustness index	$\max_{\Delta p} R(\Delta p) = \left[\sum_{i=1}^{n} \left\lvert \frac{f_i(X_0 + \Delta) - f_i(X)}{\Delta f_{i,\text{limit}}} \right\rvert\right]^{\frac{1}{2}}$	✓	—	—	single	single	Sensitivity	[51]
17	Euclidean distance (robustness radius)	$r_E = \min_{X_j : (f_{ij}(X_j) = f_{\max}) \vee (f_{ij}(X_j) = f_{\min})} \sqrt{(X_j - X_{\text{nom}}) D^{-1} (X_j - X_{\text{nom}})^T}$	✓	✓	—	multiple	multiple	Feasible design space	[52,53,54]

Robustness: 38 metrics !

#	Name	Mathematical expression	Necessary information entities: Model/ experiment	Necessary information entities: Functional limits	Necessary information entities: Expected/ measured variation	Level of complexity: Independent variables (single/multiple)	Level of complexity: Dependent variables (single/multiple)	Robustness metric class	Reference
18	Mahalanobis distance	$r_M = \min_{X_j:(f_0(X_j)=f_{\max}) \vee (f_0(X_j)=f_{\min})} \sqrt{(X_j - X_{\text{nom}})\Sigma^{-1}(X_j - X_{\text{nom}})^T}$	✓	✓	—	multiple	multiple	Feasible design space	[29,52,55]
19	Feasible volume	$\text{Vol}(n, A, b) = \frac{1}{n}\sum_{p=1}^{m}\frac{b_p}{\lvert A_{p,q}\rvert}\cdot \text{Vol}(n-1, \bar{A}, \bar{b})$	✓	✓	—	multiple	multiple	Feasible design space	[30,53,56]
20	Min–max interval	$\text{MMI} = f_{\max} - f_{\min}$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[8,57]
21	Sensitivity index (2)	$\text{SI} = \frac{f_{\max} - f_{\min}}{f_{\max}}$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[32]
22	Percentile difference	$\Delta y_{5\%}^{95\%} = y^{95\%} - y^{5\%}$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[58]
23	Variance	$V(y) = \int (f(X) - E(y))^2 \cdot p(X)dX$ $V(y) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 V(X_i)$, for independent X_i $V(y) = \sum_{i=1}^{n} V_i + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} V_{ij} + \cdots + V_{12\ldots n}$ (variance decomposition (HDMR))	✓	—	✓	multiple	single	Functional expectancy and dispersion	[8,9,13,19,59]
24	Standard deviation	$\sigma = \sqrt{V} = \sqrt{\int (f(X) - E(y))^2 \cdot p(X)dX}$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[9]
25	Conditional variance	$V_{i1\ldots is} = V_{X_{i1\ldots is}}(E_{X_{\sim i1\ldots is}}(y \mid X_{i1\ldots is}))$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[9,43,44]
26	Sensitivity index/Sobol index	$S_{i1\ldots is} = \frac{V_{i1\ldots is}}{V}$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[9,43]
27	Uncertainty importance	$I_i = \sqrt{V(y) - E[V(y \mid x_i)]}$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[9]
28	Design preference index	$\text{DPI} = E[P(y)] = \int_{y-\Delta y}^{y+\Delta y} P(y)f(y)dy$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[60]

Robustness: 38 metrics !

			Necessary information entities			Level of complexity			
#	Name	Mathematical expression	Model/ experiment	Functional limits	Expected/ measured variation	Independent variables (single/multiple)	Dependent variables (single/multiple)	Robustness metric class	Reference
29	Function robustness	$f^R = \frac{1}{N}\sum_{i=1}^{N}\frac{\sigma_f}{\sigma_{x_i}}$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[61]
30	Importance index	$\mathrm{II}_i = \frac{\sigma^2_{x_i}}{\sigma^2_y}$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[32,62]
31	Expectancy measure	$F(x) = \int f(X).p(\mathrm{X})dX$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[8,63]
32	Quality loss function	$L(y)_{\mathrm{NTB}} = \frac{A_0}{\Delta_0^2}(y-m)^2$ $L(y)_{\mathrm{STB}} = \frac{A_0}{\Delta_0^2}(y)^2$ $L(y)_{\mathrm{LTB}} = A_0\Delta_0^2\left(\frac{1}{y}\right)^2$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[1,2,12]
33	Mean square deviation	$\mathrm{MSD}_{\mathrm{NTB}} = \sigma_a^2 + (\mu - m)^2$ $\mathrm{MSD}_{\mathrm{STB}} = \sigma_a^2 + \mu^2$ $\mathrm{MSD}_{\mathrm{LTB}} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{y_i}\right)^2$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[1,2,12]
34	Signal-to-noise ratio	$\mathrm{SNR}_{\mathrm{NTB}} = 10\log_{10}\frac{\mu^2}{\sigma^2}$ $\mathrm{SNR}_{\mathrm{STB}} = -10\log_{10}(\sigma^2 + \mu^2)$ $\mathrm{SNR}_{\mathrm{LTB}} = -10\log_{10}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{y_i}\right)^2\right]$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[1,2,12]
35	Weighted sum robustness	$R_w = w_1 \cdot \lvert\mu_y - m\rvert + w_2 \cdot \sigma_y$	✓	—	✓	multiple	single	Functional expectancy and dispersion	[64]
36	Probabilistic robustness threshold	$\Pr[\mathrm{LSL}_i < f_i < \mathrm{USL}_i]$	✓	✓	✓	multiple	multiple	Probability of functional compliance	[8,58]
37	Design capability indices/error margin index	$C_{dl} = \frac{\mu - \mathrm{LRL}}{3\sigma}$; $C_{du} = \frac{\mathrm{URL} - \mu}{3\sigma}$; $C_{dk} = \mathrm{EMI} = \min\{C_{dl}, C_{du}\}$	✓	✓	✓	multiple	single	Probability of functional compliance	[65,66]
38	Information content	$I = \log\left(\frac{1}{p}\right)$	✓	✓	✓	multiple	multiple	Probability of functional compliance	[24]

Bibliography

Short bibliography at the end of this presentation (section 7.).

Two interesting PhD:

- ✓ Dubourg V., Adaptive surrogate models for reliability analysis and reliability-base design optimization, PhD thesis, Clermont Auvergne University, 2012.
- ✓ Maliki M., Adaptive surrogate models for the reliable lightweight design of automotive body structures, Clermont Auvergne University, 2016.

1. Introduction / motivation / illustrations

1.1 Bibliography

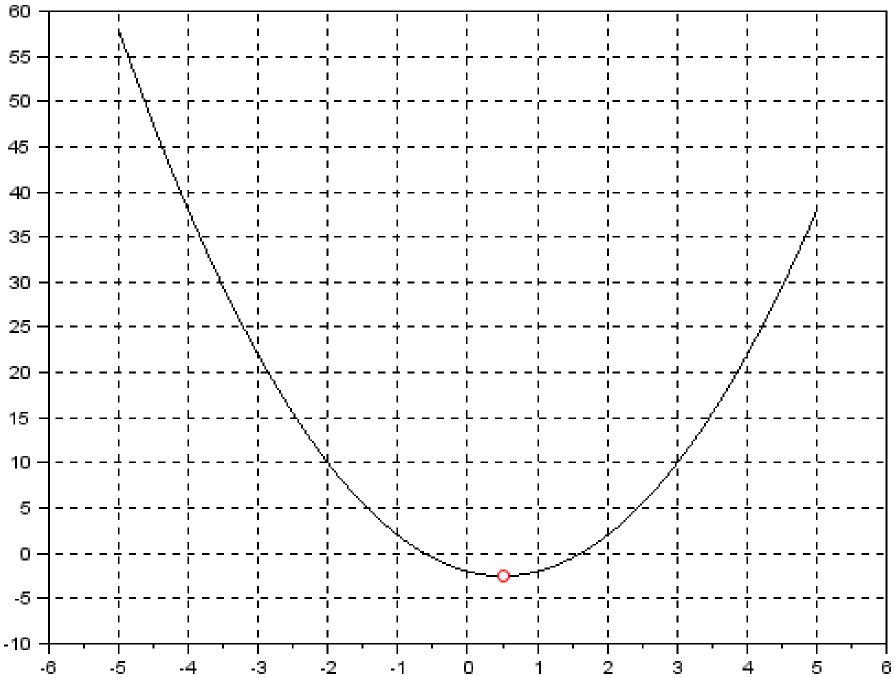
1.2 Optimization without uncertainties vs with uncertainties

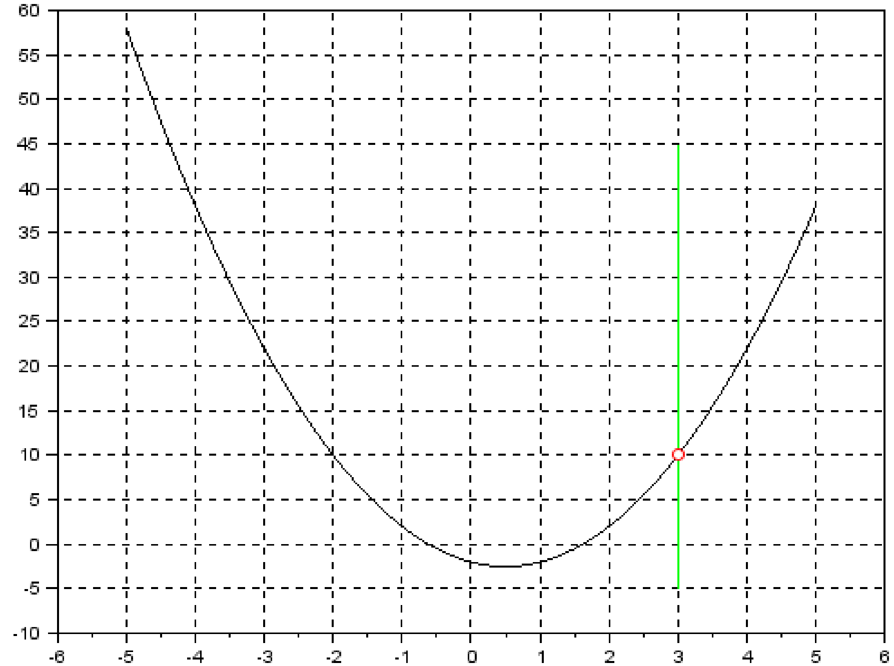
1.3 Applications

Short illustration

$$x^* = \min_{x \in \mathbb{R}} \; f(x) = 2x^2 - 2x - 2$$

$$x^* = \min \; f(x) = 2x^2 - 2x - 2$$
$$\text{under} \quad g(x) = -x + 3 \leq 0$$





Short illustration

$$x^*(\omega) = \min_{x \in \mathbb{R}} \; f(x) = a(\omega)x^2 + b(\omega)x + c(\omega)$$

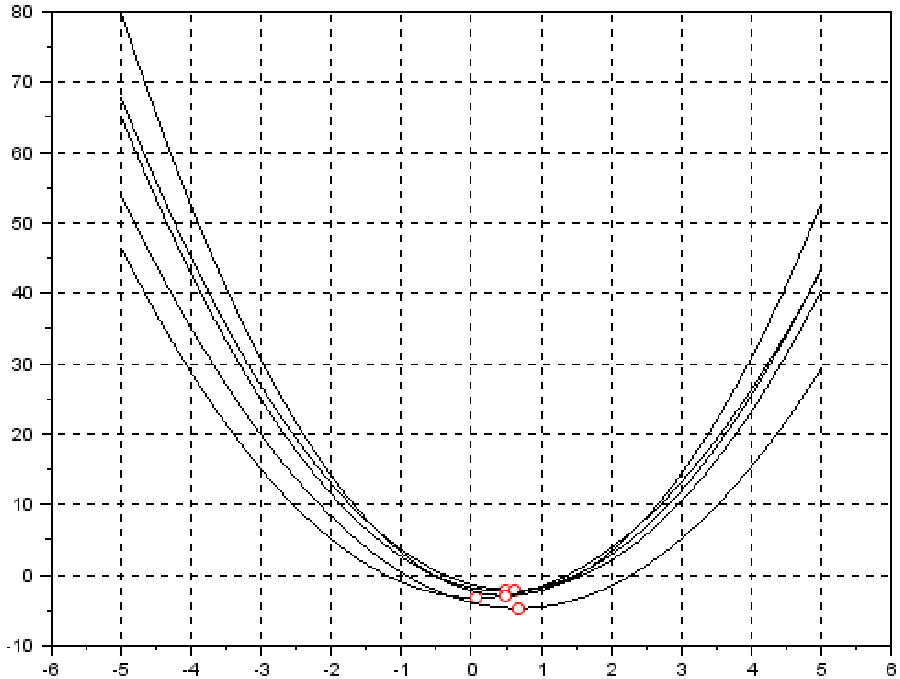
$$a \to N(2,1), b \to N(-2,1), c \to N(-2,1)$$

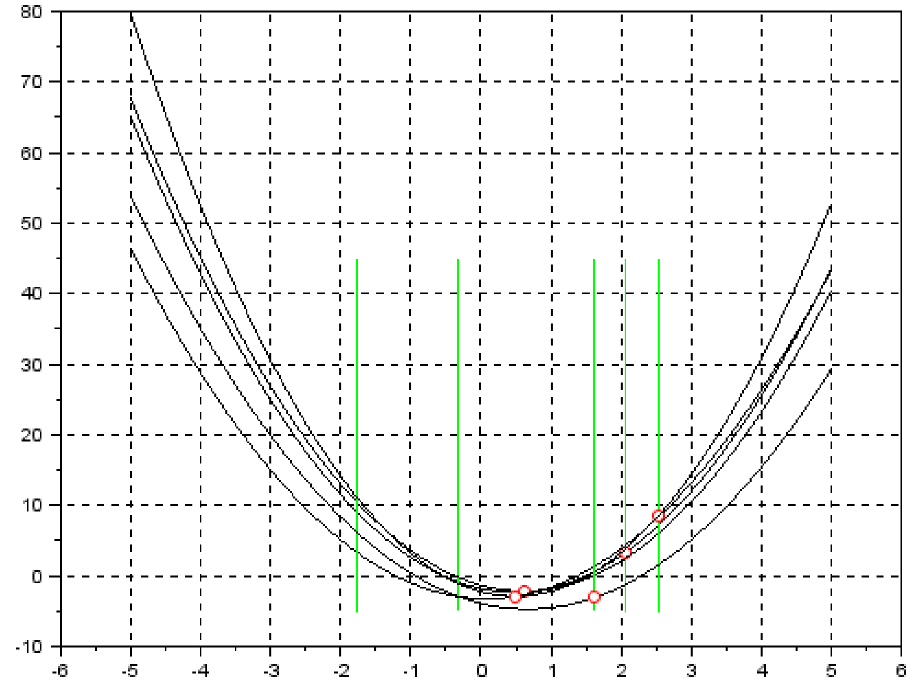
$$x^*(\omega) = \min \; f(x) = a(\omega)x^2 + b(\omega)x + c(\omega)$$

under $\quad g(x) = -x + d(\omega) \le 0$

$$a \to N(2,1), b \to N(-2,1), c \to N(-2,1)$$

$$d \to U(-3,3)$$





Design requirements

A mechanical system can be characterized by two different output functions

✓ **Objective-type functions**

→ to be maximized, minimized, quantify the performance of the system

$$f(x) \quad x \in \mathbb{R}^n$$

✓ **Constraint-type functions**

→ must be satisfied in all operating conditions

$$g_j(x) \quad x \in \mathbb{R}^n \quad j = 1, \dots, m$$

Admissible space : $\left\{ x \,/\, \forall j = 1, \dots, m \quad g_j(x) \leq 0 \right\}$

In optimization!

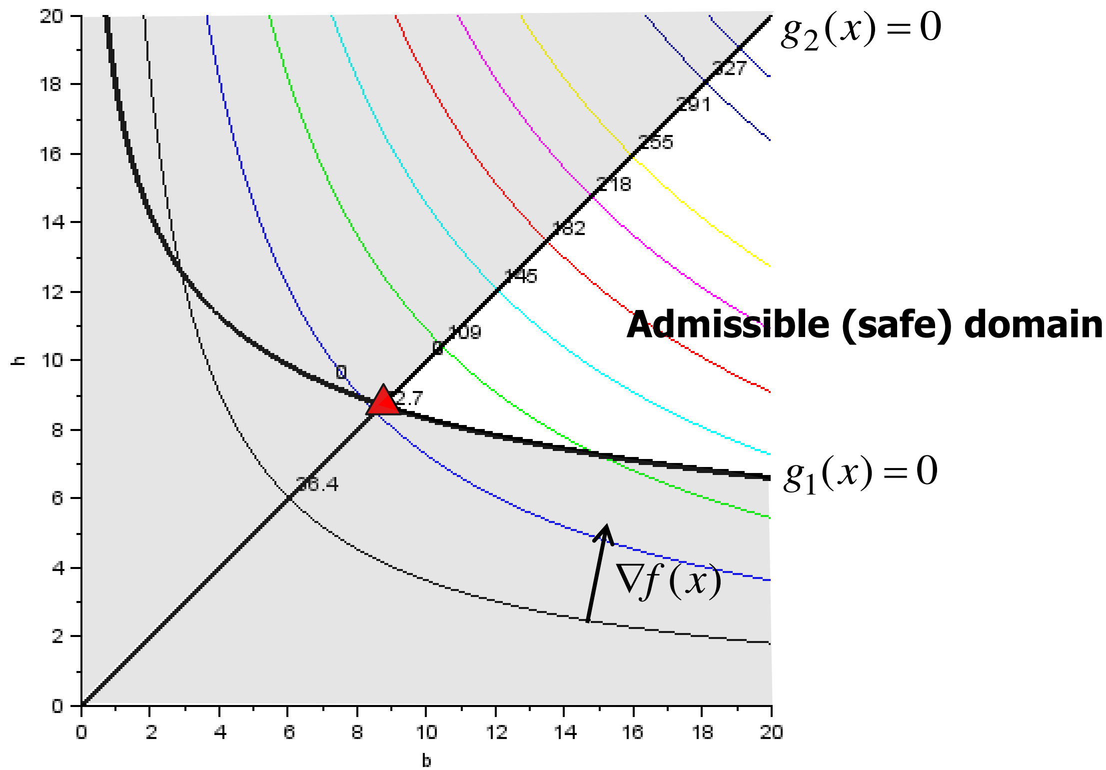
Admissible (safe) if ≥ 0 in reliability

Deterministic optimization

Find $x^* \in \mathbb{R}^n$ such that : $x^* = \textbf{Argmin} \quad f(x)$ $\quad f : \mathbb{R}^n \to \mathbb{R}$

$\textbf{under} \quad g_j(x) \leq 0$ $\quad g_j : \mathbb{R}^n \to \mathbb{R}$

$j = 1, \dots, m$



Optimization <u>under</u> uncertainties

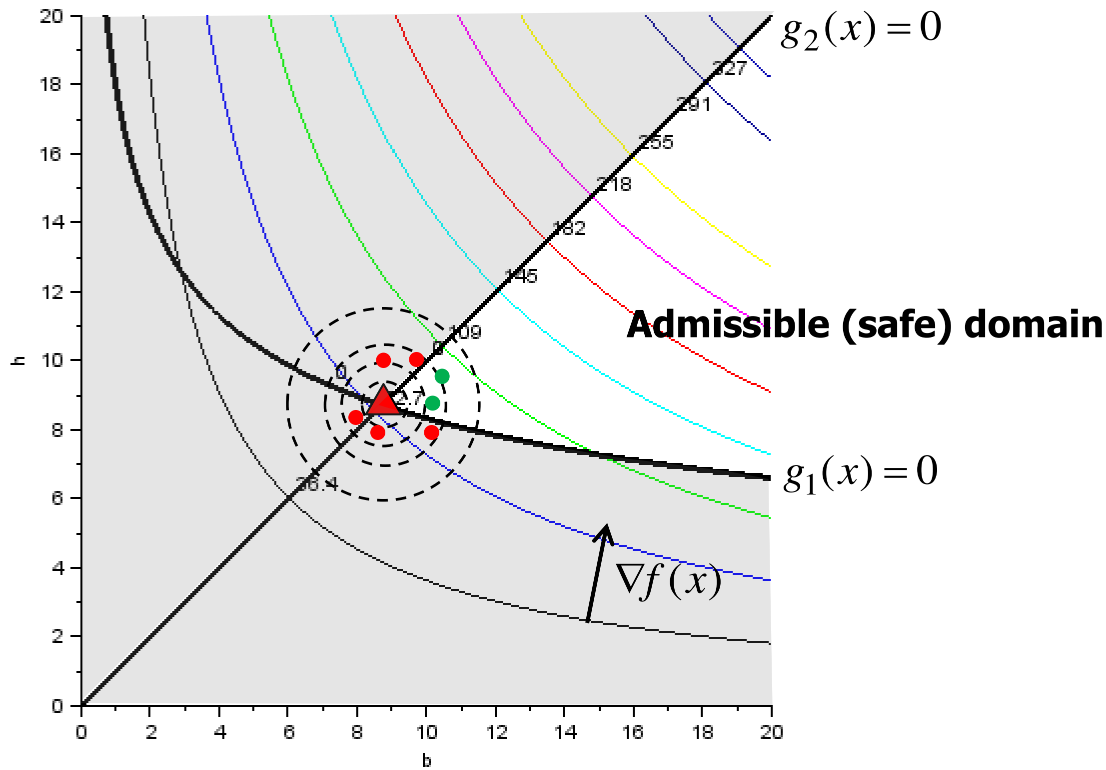
$x(\omega)$?

Find $x^* \in \mathbb{R}^n$ such that : $x^* = \textbf{Argmin} \quad f(x)$ $\qquad f : \mathbb{R}^n \rightarrow \mathbb{R}$

Signification?

under $\quad g_j(x) \leq 0$ $\qquad g_j : \mathbb{R}^n \rightarrow \mathbb{R}$

? $x(\omega)$ $\quad j = 1,\ldots,m \quad$ (m **can be greater than** n)



1. Introduction / motivation / illustrations

1.1 Bibliography

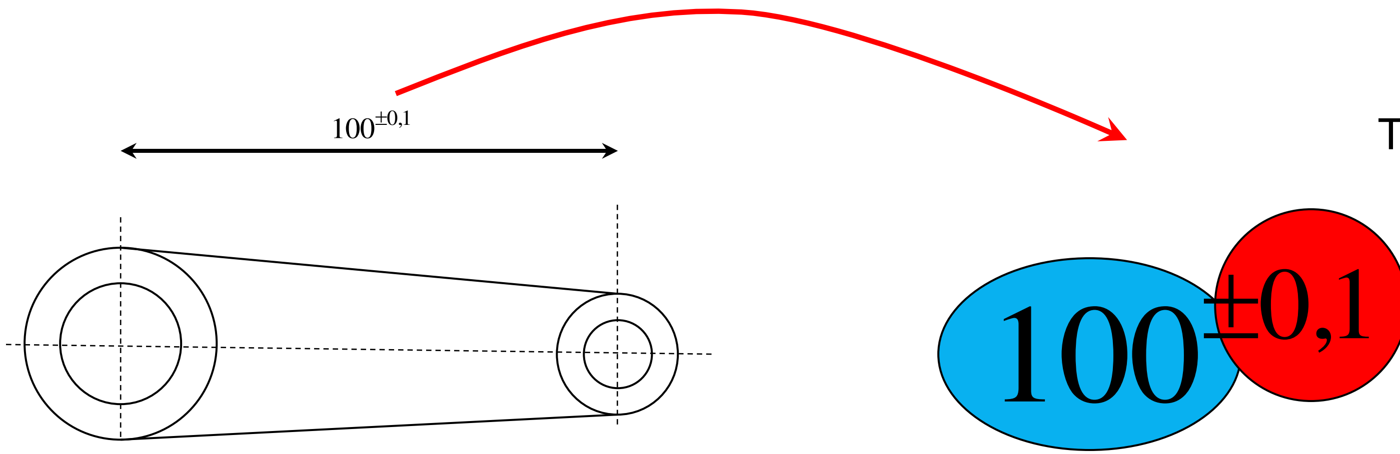
1.2 Optimization without uncertainties vs with uncertainties

1.3 Applications

Industrial problem in engineering

Main engineering problem :

Find nominal design and / or tolerance interval (allowable uncertainties)



Tolerance due to manufacturing
t

Nominal value
Call « Target »
T

Nominal variable / tolerance

Type II variables

Must be nominal value and/or tolerances

$x(\omega)$?

Find $x^* \in \mathbb{R}^n$ such that : $x^* = \textbf{Argmin} \; f(x)$ $\quad f : \mathbb{R}^n \to \mathbb{R}$

under $g_j(x) \leq 0$ $\quad g_j : \mathbb{R}^n \to \mathbb{R}$

? $x(\omega)$ $\quad j = 1, \dots, m \quad$ (m **can be greater than** n)

Industrial problem in engineering

Tolerance due to manufacturing
t

100 ±0,1

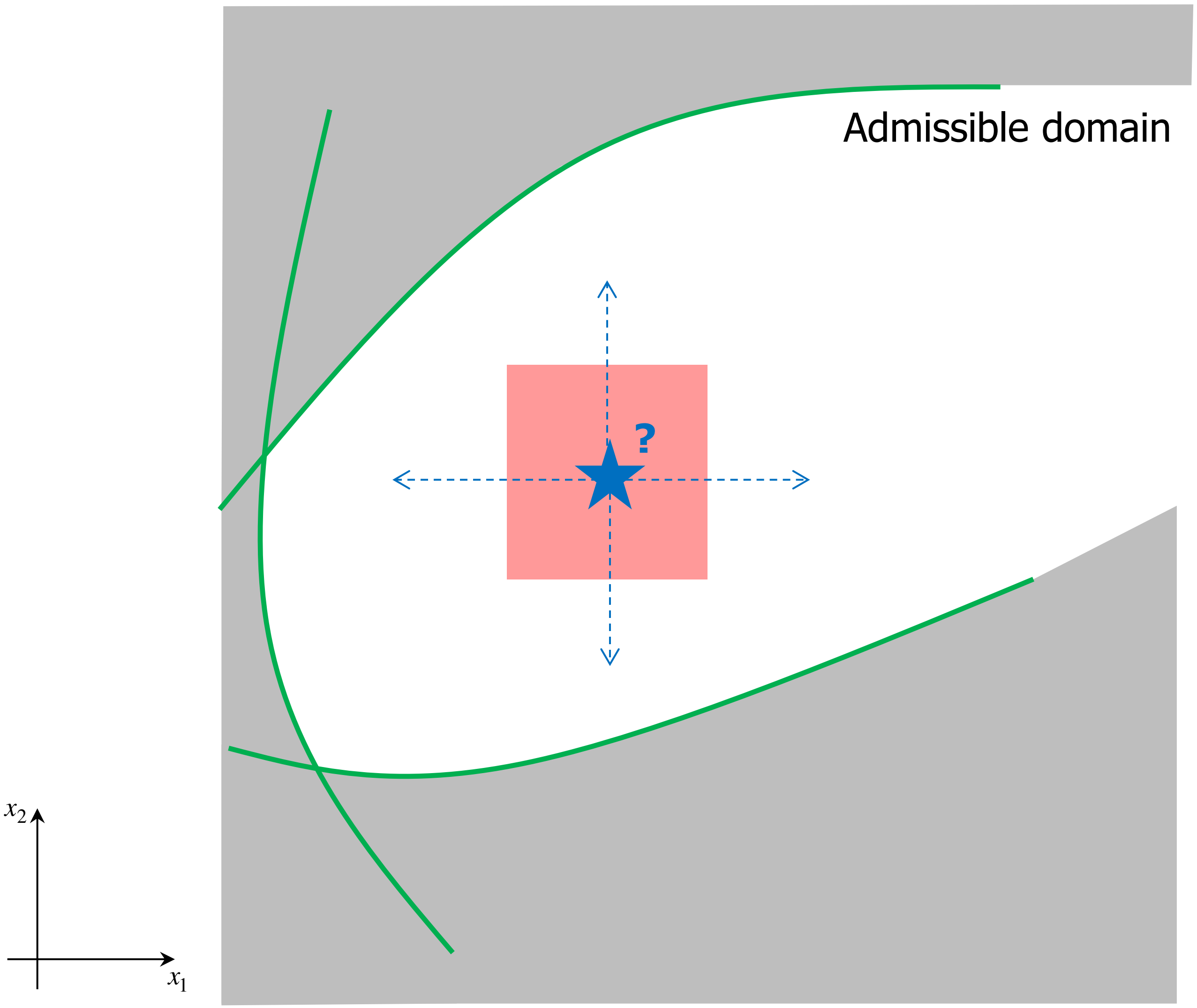
Nominal value
Call « Target »
T

- **t** ↗ manufactruring cost ↘
- **t** ↗ Reliability ↘
- **t** ↗ Robustness ↘

T linked to the production cost, weight, nominal margins, ...

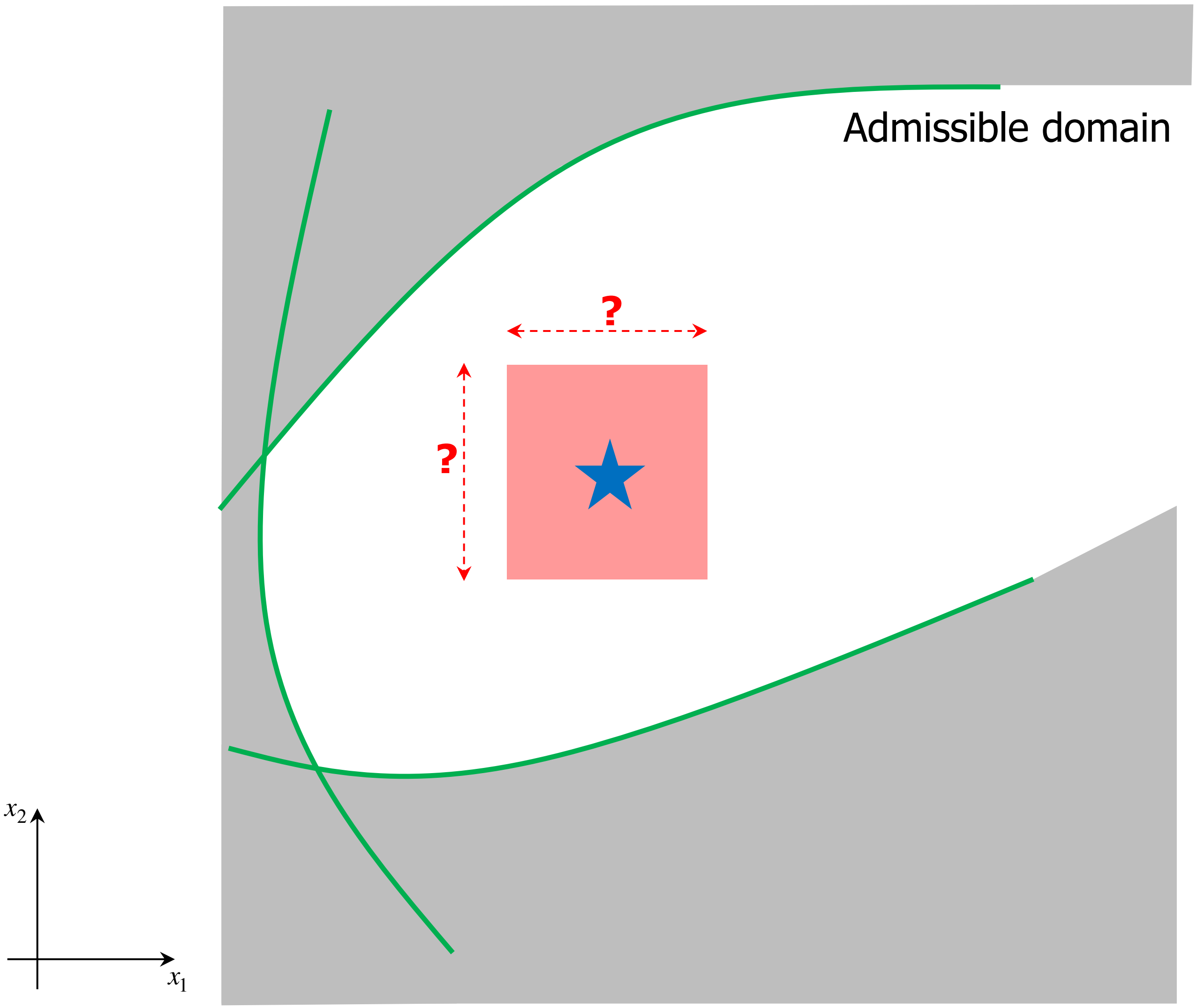
Industrial problem in engineering #1

Find Target values with known and non alterable uncertainties such that ...



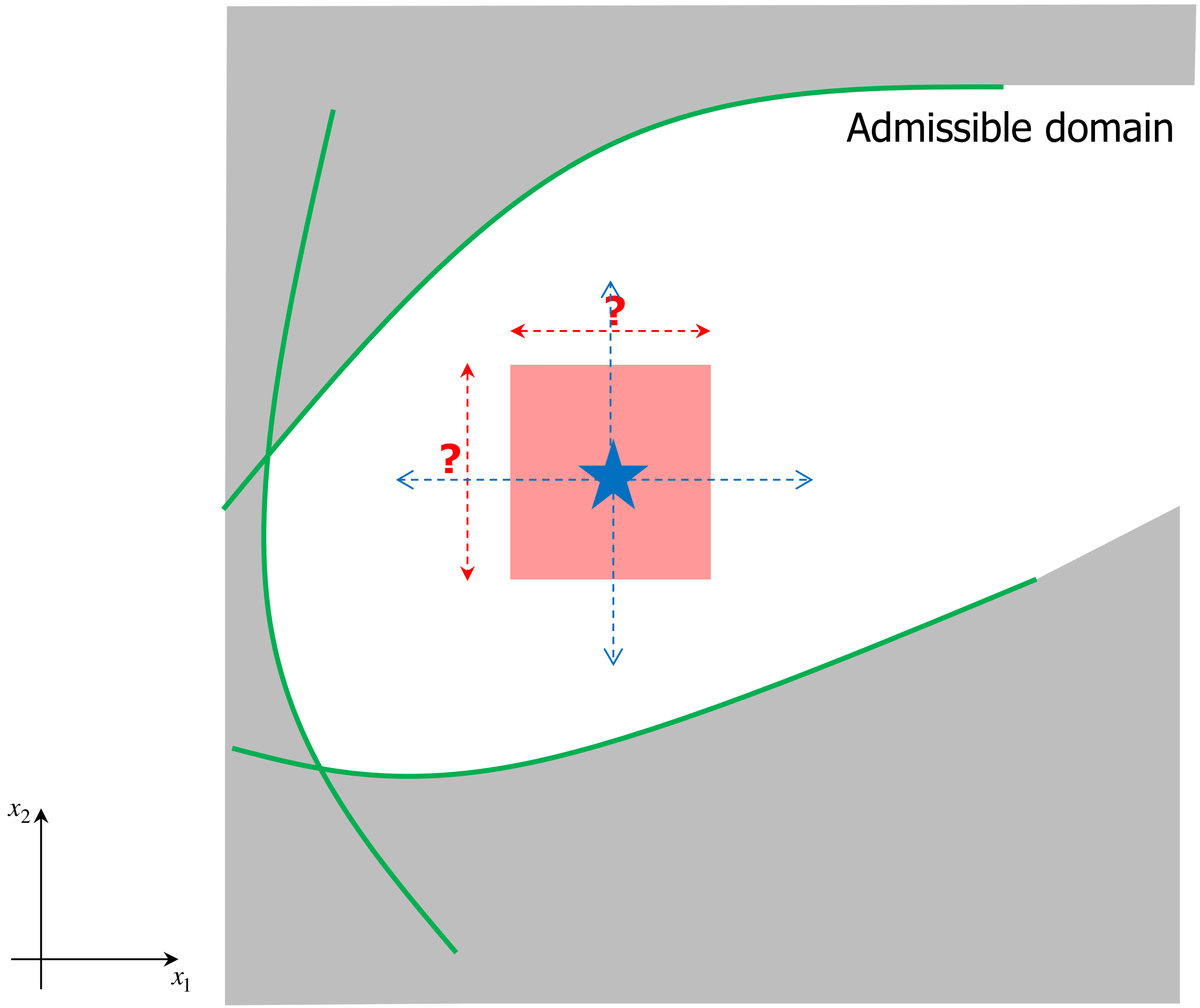
Industrial problem in engineering #2

Find Tolerances (uncertainties) from knowm and fixed Target values such that ...



Industrial problem in engineering #3

Find Target values and Tolerance values such that …

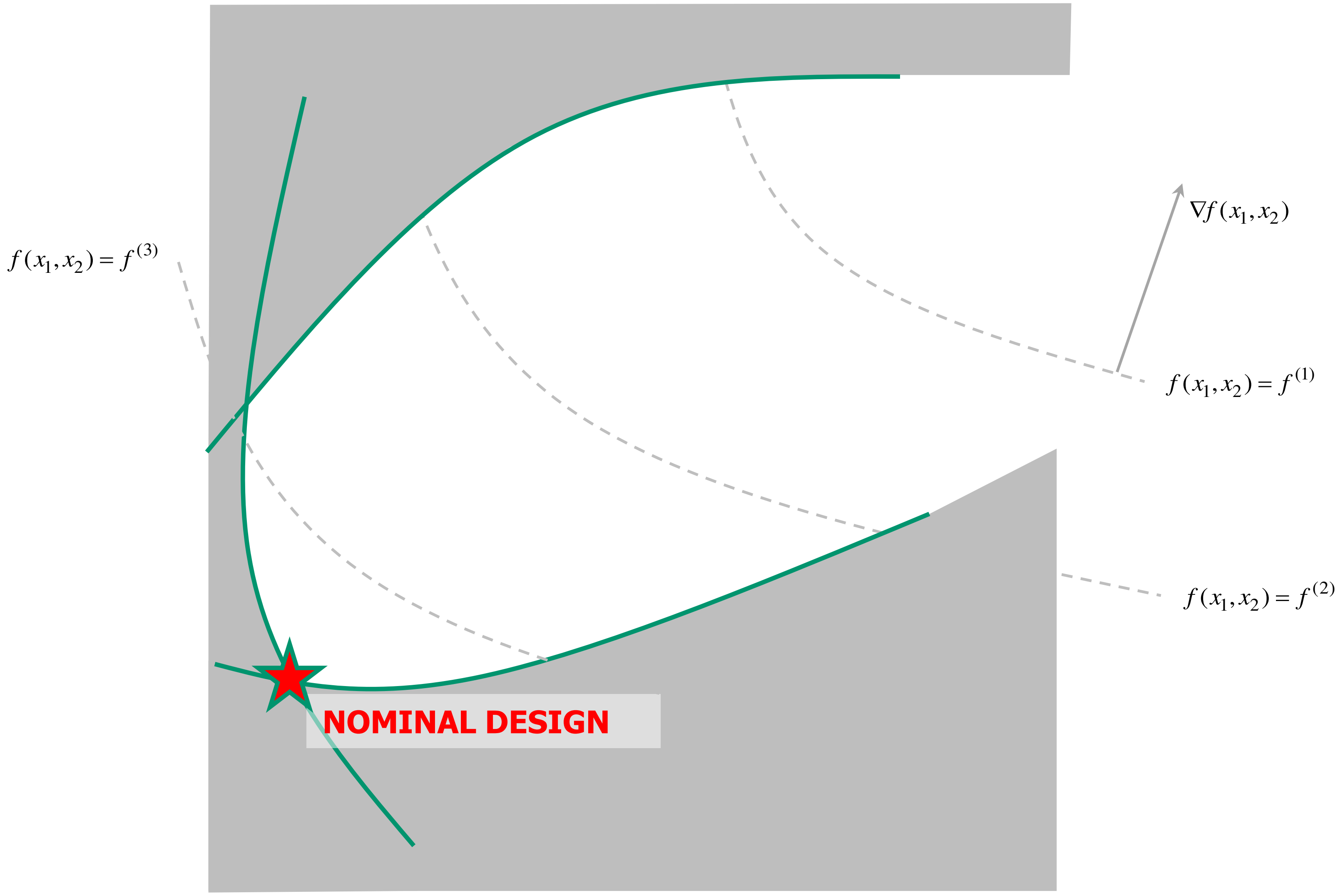


One way to deal with uncertainties in optimization

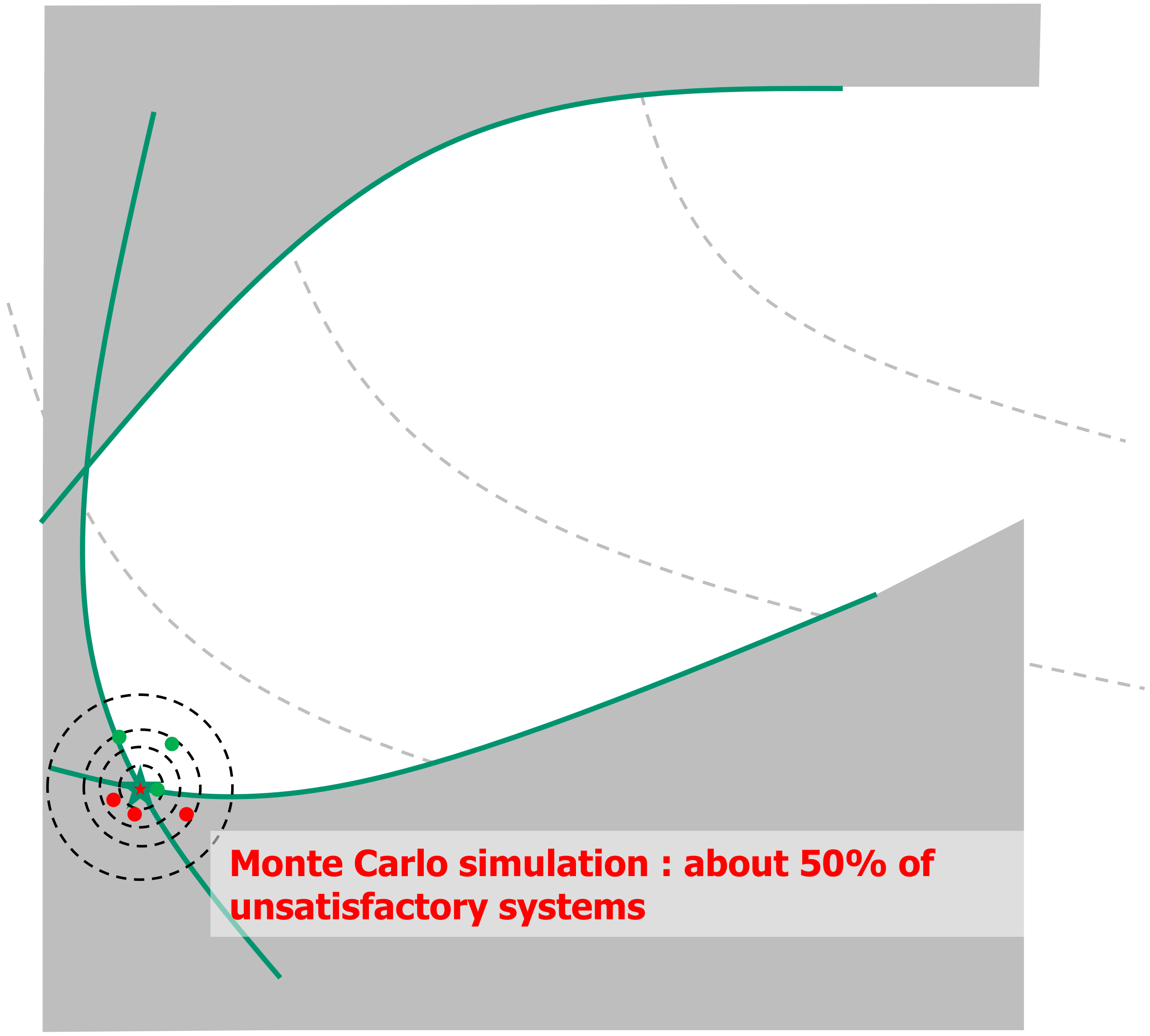
Steps to take uncertainties into account in optimization

- ✓ Deterministic optimization
- ✓ Add uncertainties to quantify the failure probability
- ✓ Modify the solution to reach a target failure probability

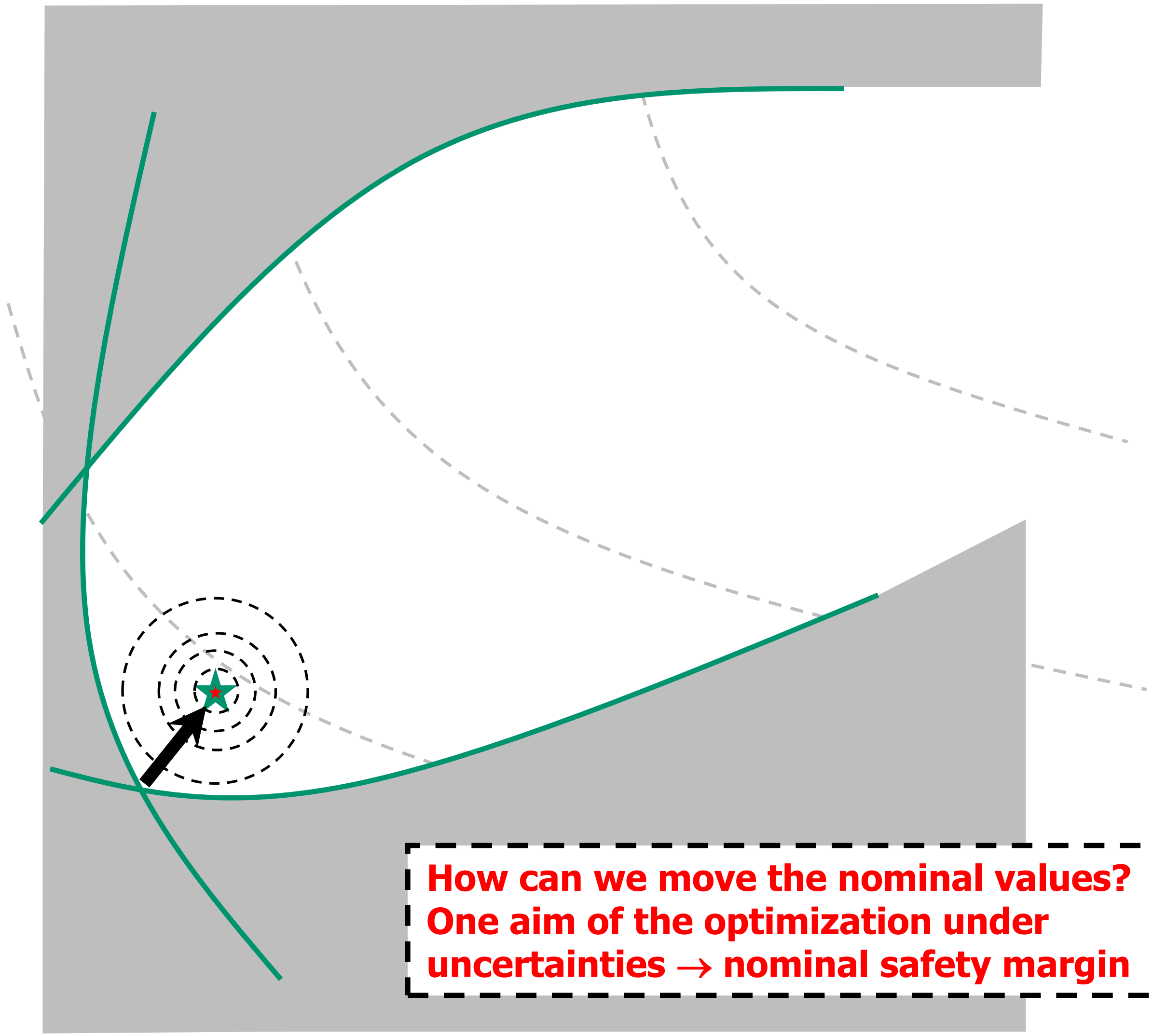
One way to deal with uncertainties in optimization



One way to deal with uncertainties in optimization



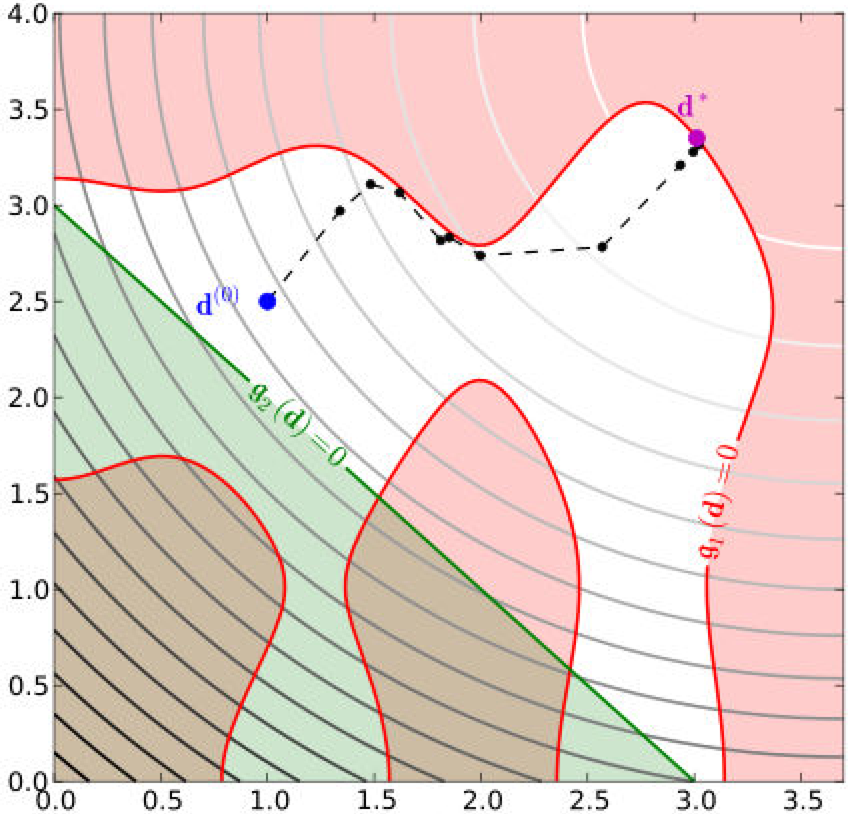
One way to deal with uncertainties in optimization



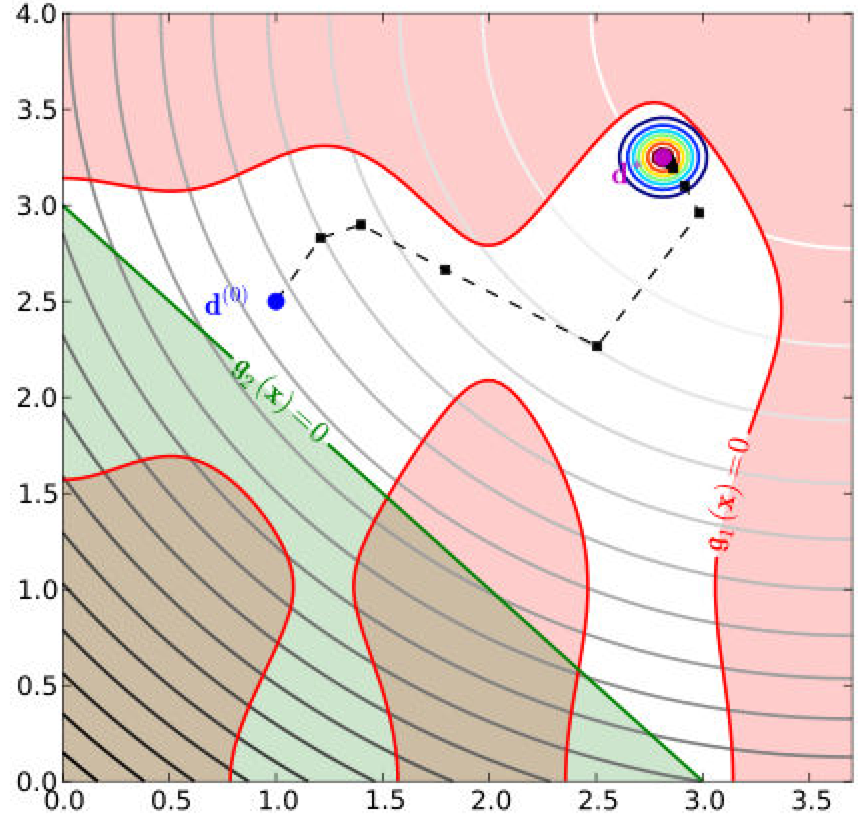
How can we move the nominal values?
One aim of the optimization under uncertainties → nominal safety margin

Deterministic optimization vs optimization under uncertainty

[Dubourg, 2012]



The solution satisfy the limit state function:
→ No nominal safety margin



Introduction of a safety margin between the solution and the limit state function

2. Deterministic optimization

2. Deterministic optimization

2.1 Deterministic optimization without constraints

2.2 Deterministic optimization with equality constraints

2.3 Deterministic optimization with inequality constraints

2.4 What about uncertainties

Deterministic optimization <u>without</u> constraints

Problem formulation

Find $x^* \in \mathbb{R}^n$ such that : $f^* = \mathbf{min} \quad f(x) \qquad x \in \mathbb{R}^n$

Illustration $n = 1$

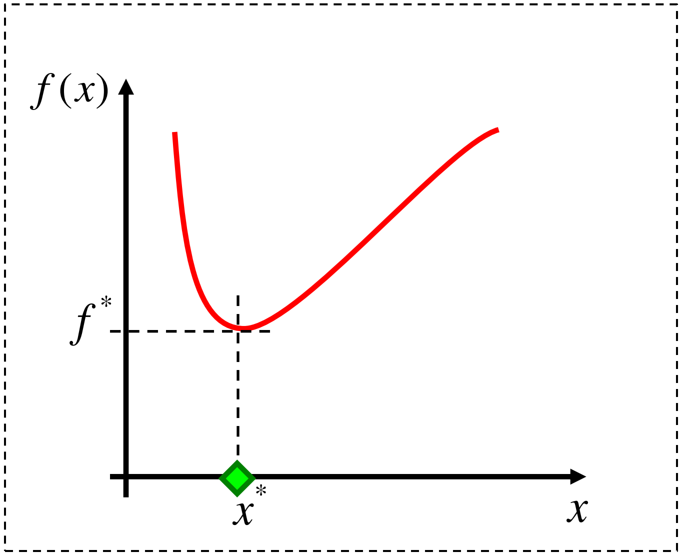
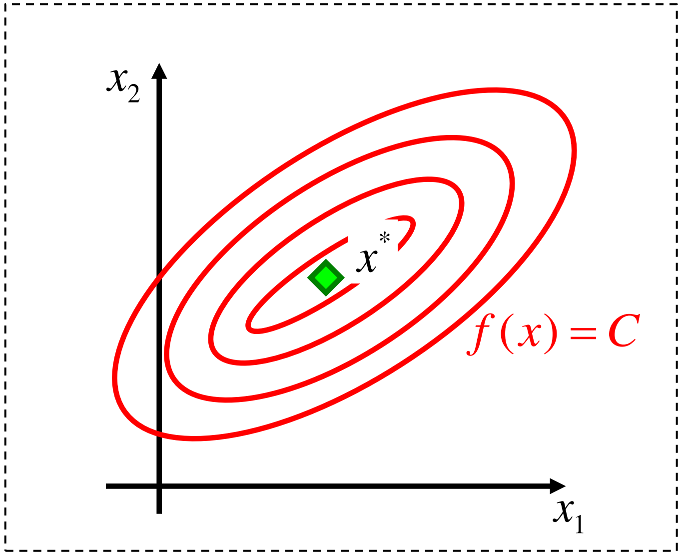


Illustration $n = 2$



Deterministic optimization <u>without</u> constraints

Necessary conditions of optimality - x^* solution if at least:

$\nabla f(x^*) = 0$	Gradiant null
$\nabla^2 f(x^*)$	Hessian matrix positive definite

Resolution methods:

- ✓ 0-order methods: simplex, genetic algorithm, simulated annealing, …
- ✓ 1-ordrer method: gradient-based method, …
  $\rightarrow$ Need the first derivative of the objective function
- ✓ 2-order method: Newton, quasi-Newton, …
  $\rightarrow$ Need the second derivative of the objective function

Important property of the gradient

$s \in \mathbb{R}^n$ is a **slope direction** for function $f:\mathbb{R}^n \rightarrow \mathbb{R}$ in $x^{(0)}$ if $\exists \eta > 0 \in \mathbb{R}$ such that:

$$f(x^{(0)} + rs) \leq f(x^{(0)}) \quad \forall r \in]0,\eta]$$

$s \in \mathbb{R}^n$ is a slope direction only if $s^t \nabla f(x_0) < 0$:

$$f(x^{(0)} + rs) \approx f(x^{(0)}) + rs^t \nabla f(x^{(0)})$$

$$\leq f(x^{(0)}) \quad \text{if} \quad s^t \nabla f(x^{(0)}) < 0$$

$s = -\nabla f(x^{(0)})$ is the direction of highest slope in $x^{(0)}$

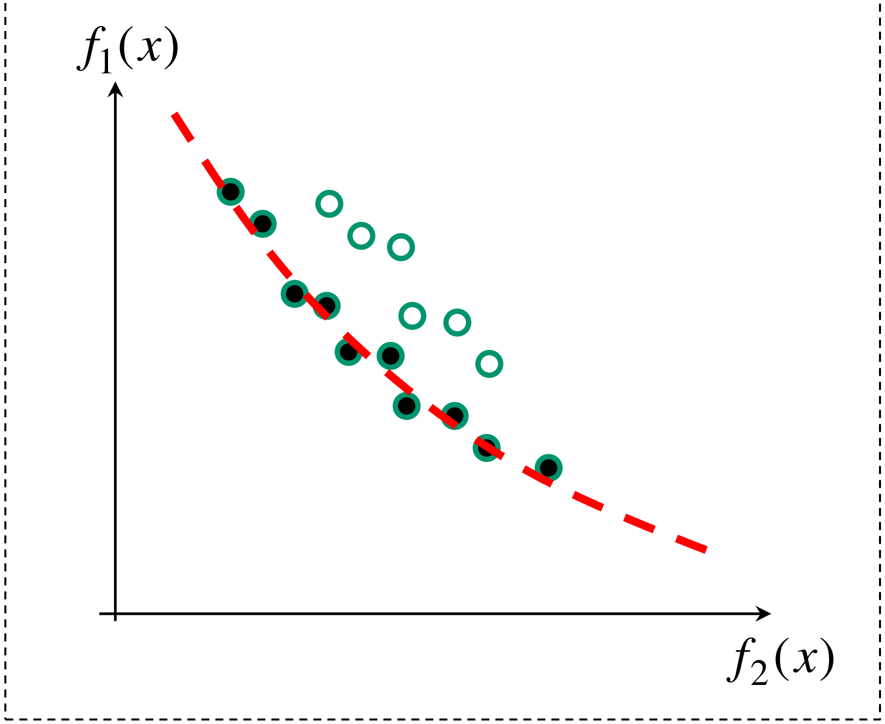
Pareto Front

Problem formulation

Find $x^* \in \mathbb{R}^n$ such that : $x^* = \textbf{Argmin} \quad f(x)$ $\quad f : \mathbb{R}^n \to \mathbb{R}^p$

- ✓ A compromise is needed
- ✓ Pareto front construction using genetic algorithms
- ✓ …

Illustration $p = 2$



2. Deterministic optimization

2.1 Deterministic optimization without constraints

2.2 Deterministic optimization with equality constraints

2.3 Deterministic optimization with inequality constraints

2.4 What about uncertainties

Deterministic optimization with equality constraints

Problem formulation

Find $x^* \in \mathbb{R}^n$ such that :

$$x^* = \mathbf{Argmin} \quad f(x) \qquad f : \mathbb{R}^n \to \mathbb{R}$$

$$\mathbf{under} \quad g_j(x) = 0 \qquad g_j : \mathbb{R}^n \to \mathbb{R}$$

$$j = 1, \ldots, m \quad (m \leq n)$$

Find $x^* \in \mathbf{D} \subset \mathbb{R}^n$ such that : $x^* = \mathbf{Argmin} \quad f(x) \qquad f : \mathbb{R}^n \to \mathbb{R}$

$$\mathbf{D} \equiv \left\{ x \in \mathbb{R}^n \,/\, g_j(x) = 0 \,;\, j = 1, \ldots, m \right\}$$

$$f^* = \mathbf{min} \quad f(x) \qquad x \in \mathbf{D} \subset \mathbb{R}^n$$

Admissible domain

Deterministic optimization with equality constraints

Illustration $n=1, m=1$

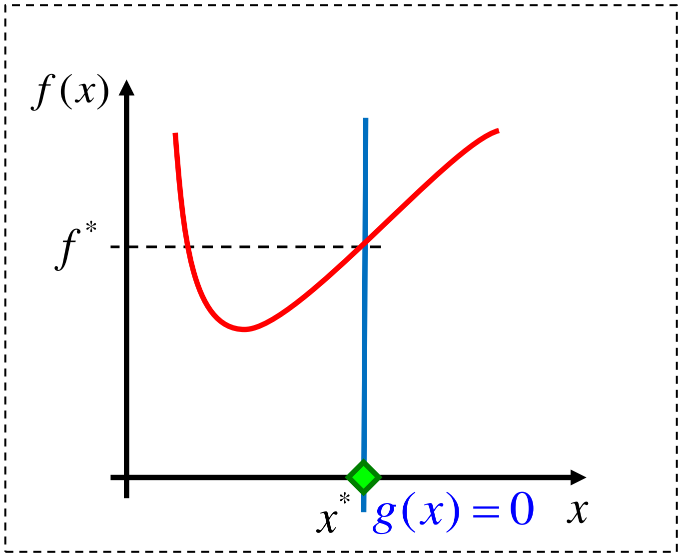
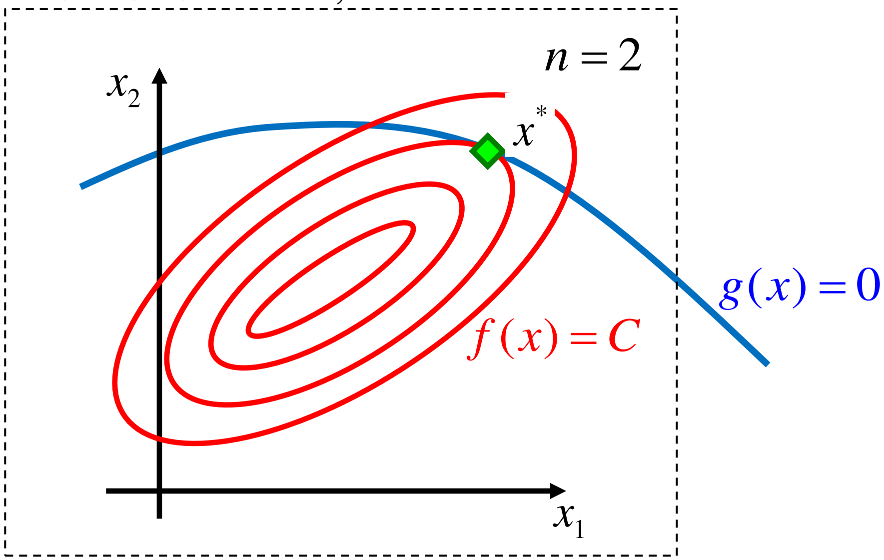


Illustration $n=2, m=1$



Deterministic optimization with equality constraints

Necessary conditions of optimality - x^* solution if at least:

$$g_j(x^*) = 0 \quad j = 1,\dots,m$$

$$\nabla f(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla g_j(x^*) = 0$$

$\Leftrightarrow$

$$\nabla L(x,\lambda) = 0$$

$$L(x,\lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x)$$

Lagrangian coefficients

Resolution methods:

- ✓ Dimension reduction by integration of the constraints inside the objective function.
- ✓ Lagrangian methods (Newton, SQP, …)

  → need the first and/or second derivative of objective function and/or constraints

Deterministic optimization with equality constraints

Lagrangian methods : iterative scheme converging towards the solution

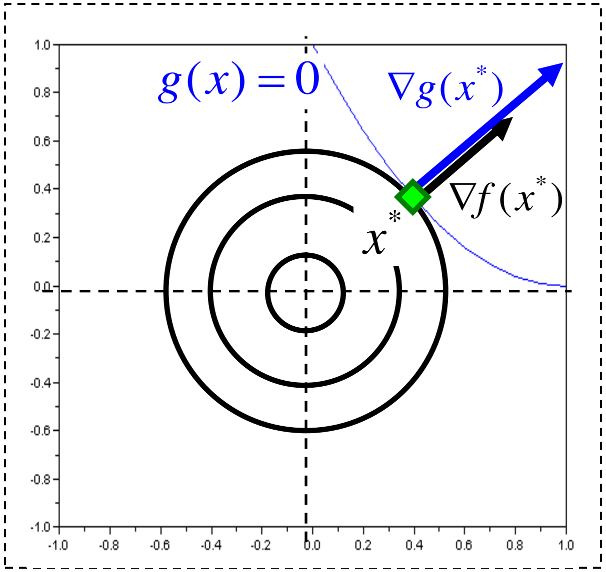
$$\begin{bmatrix} \nabla^2 f(x^{(k)}) & \nabla g_1(x^{(k)}) & \dots & \nabla g_m(x^{(k)}) \\ \nabla g_1(x^{(k)})^t & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 \\ \nabla g_m(x^{(k)})^t & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^{(k+1)} - x^{(k)} \\ \lambda_1 \\ \dots \\ \lambda_m \end{bmatrix} = \begin{bmatrix} -\nabla f(x^{(k)}) \\ -g_1(x^{(k)}) \\ \dots \\ -g_m(x^{(k)}) \end{bmatrix}$$

"FORM" illustration … Find $x^* \in \mathbb{R}^2$ the closest point to the origin such as

$$g(x_1, x_2) = x_2 - (x_1 - 1)^2 = 0$$

→ Rackwitz - Fiessler algorithm

$\lambda^* \approx -0.7$
$x_1^* \approx 0,4$
$x_2^* \approx 0,35$



2. Deterministic optimization

Deterministic optimization with inequality constraints

Problem formulation

Find $x^* \in \mathbb{R}^n$ such that : $x^* = \textbf{Argmin} \quad f(x)$ $\quad f : \mathbb{R}^n \rightarrow \mathbb{R}$

$\textbf{under} \quad g_j(x) \leq 0$ $\quad g_j : \mathbb{R}^n \rightarrow \mathbb{R}$

$j = 1, \ldots, m \quad (m \textbf{ can be greater than } n)$

Find $x^* \in \mathbf{D} \subset \mathbb{R}^n$ such that : $x^* = \textbf{Argmin} \quad f(x)$ $\quad f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$D \equiv \left\{ x \in \mathbb{R}^n \,/\, g_j(x) \leq 0 \,;\, j = 1, \ldots, m \right\}$$

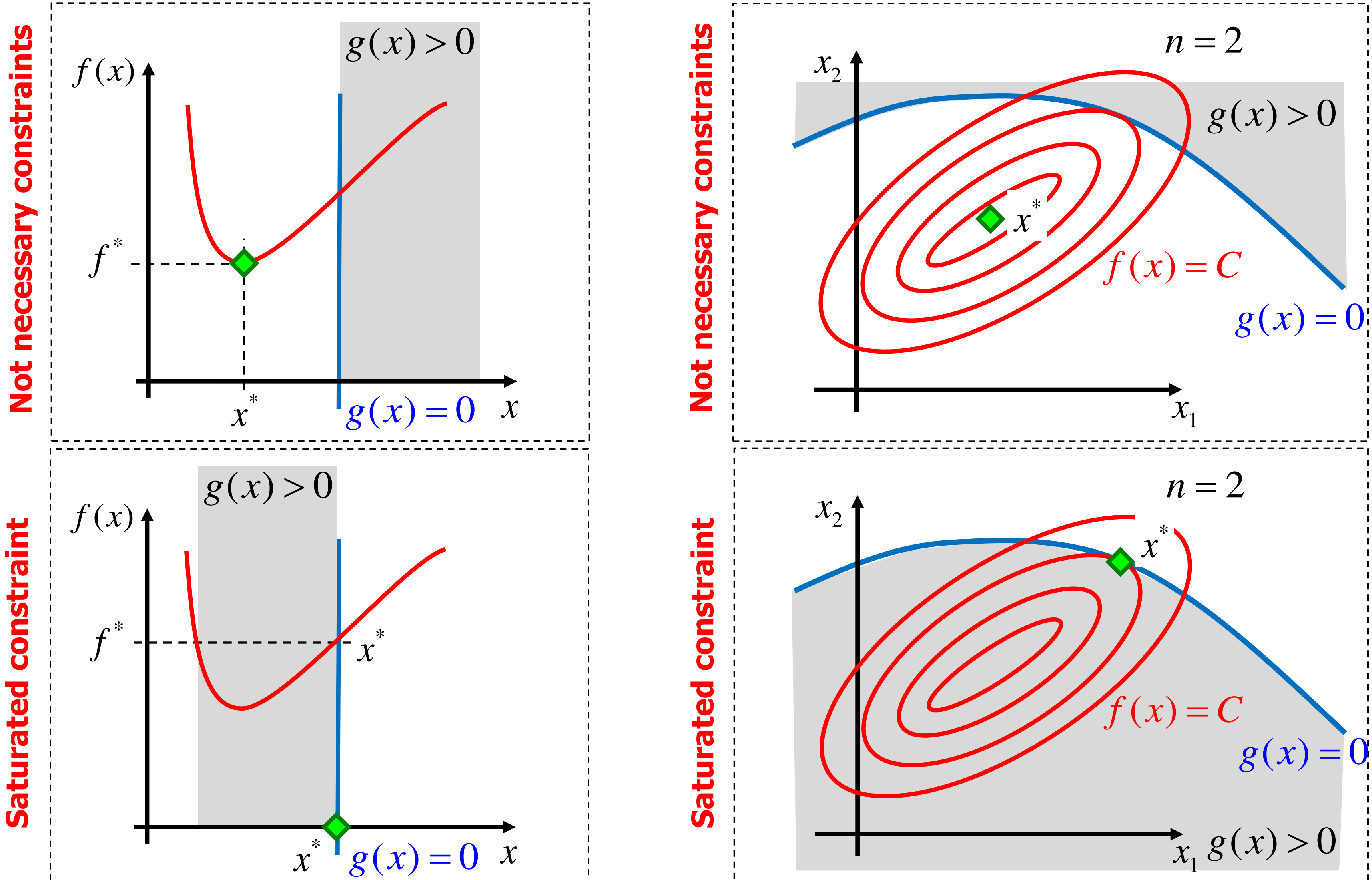
$$f^* = \textbf{min} \quad f(x) \qquad x \in \mathbf{D} \subset \mathbb{R}^n$$

Admissible domain

Be carful : a point is considered admissible if $g_j(x) \leq 0$ while in reliability analysis a point is considered safe if the performance function is negative … Can be problematic

Deterministic optimization with inequality constraints

Illustration $n = 1, m = 1$ Illustration $n = 2, m = 1$



Deterministic optimization with inequality constraints

Necessary conditions of optimality - x^* solution if at least (KKT conditions) :

$$g_j(x^*) \leq 0 \quad j = 1,\ldots,m$$

$$\nabla f(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla g_j(x^*) = 0$$

$$\lambda_j^* \geq 0$$

$$\lambda_j^* g_j(x^*) = 0$$

Resolution methods :

- ✓ Heuristic methods (0-order)
- ✓ Penality function
- ✓ Lagrangian methods (Newton, SQP, …) …

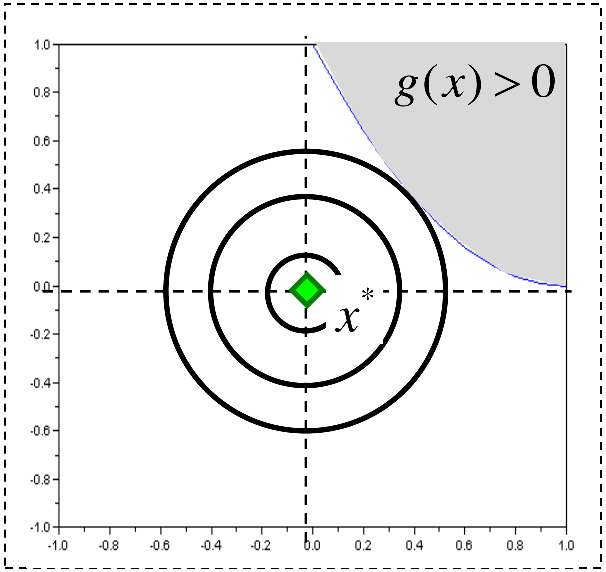
Deterministic optimization with inequality constraints

Come back to the "FORM" illustration …

Find $x^* \in \mathbb{R}^2$ the closest point to the origin such as

$$g(x_1, x_2) = x_2 - (x_1 - 1)^2 \leq 0$$

Solution is zero: this is the optimum without constraints because it is admissible



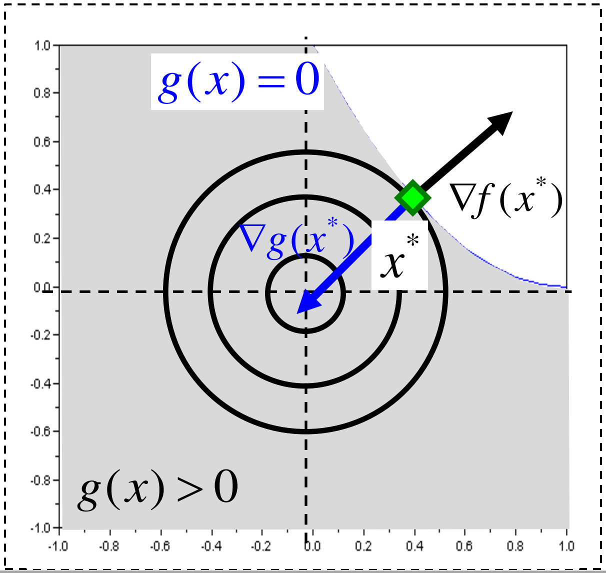
Come back to the "FORM" illustration …

Find $x^* \in \mathbb{R}^2$ the closest point to the origin such as

$$g(x_1, x_2) = -x_2 + (x_1 - 1)^2 \leq 0$$

Solution saturates the constraint

$\lambda^* \approx +0.7$
$x_1^* \approx 0{,}4$
$x_2^* \approx 0{,}35$

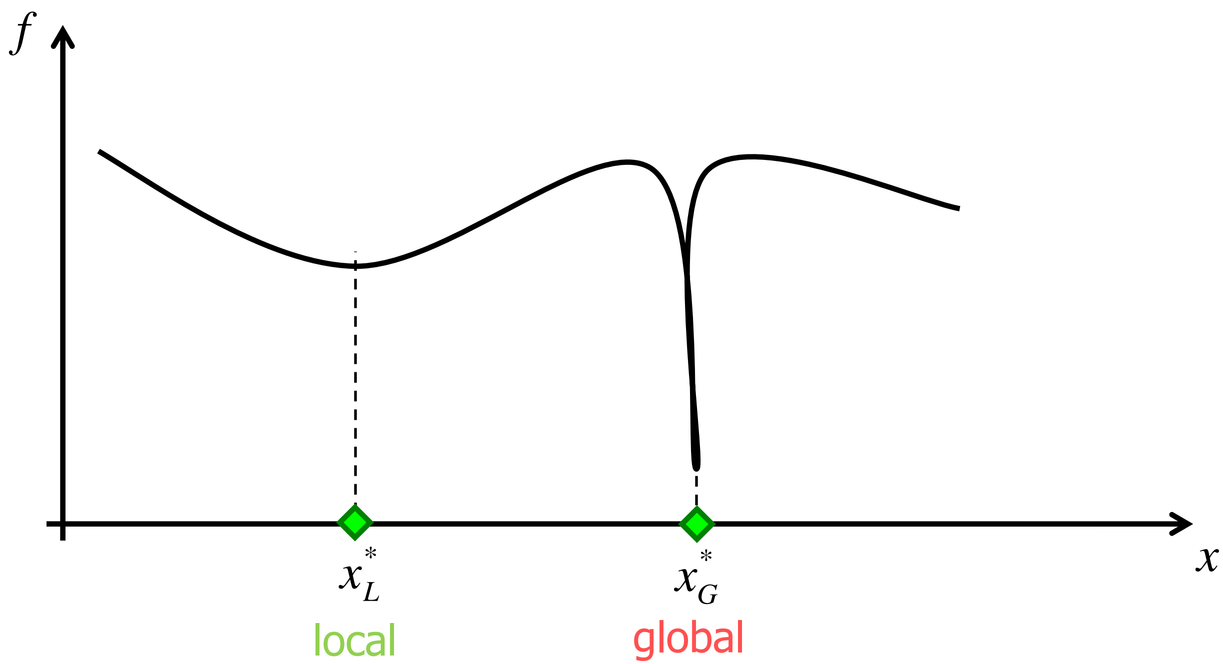


Be carful : before using Rackwith – Fiessler algorithm, check that the mean point is safe.

2. Deterministic optimization

What about uncertainties?

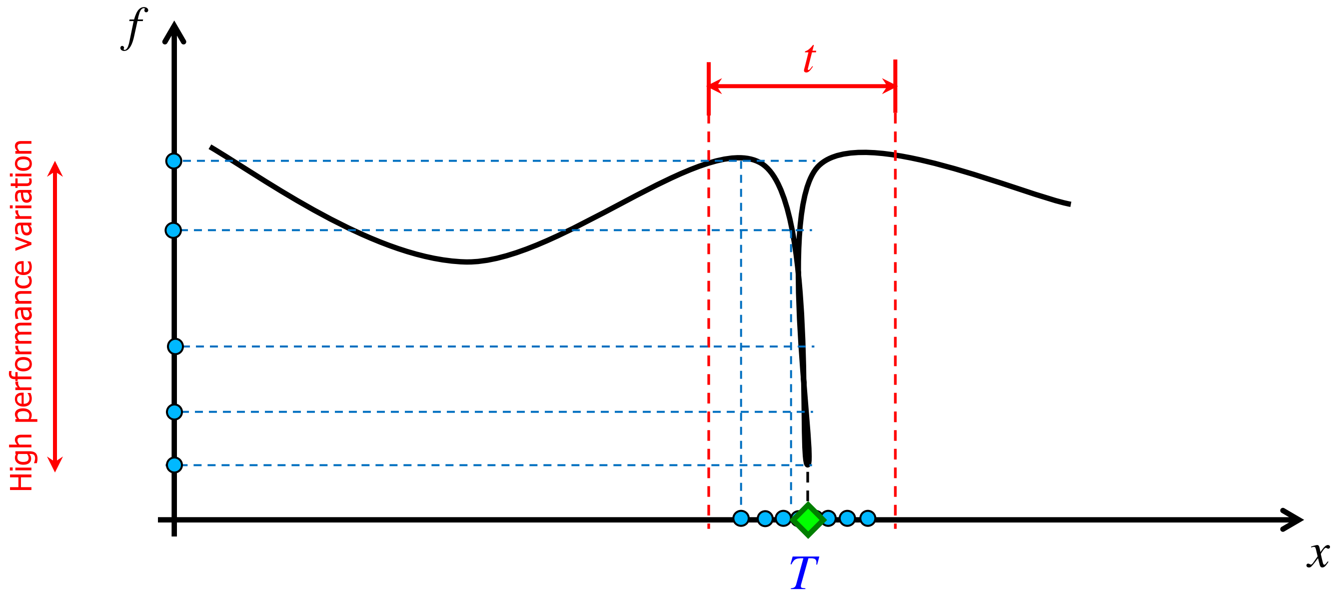
Let $f(x)$ be a 1D-objective function to minimize



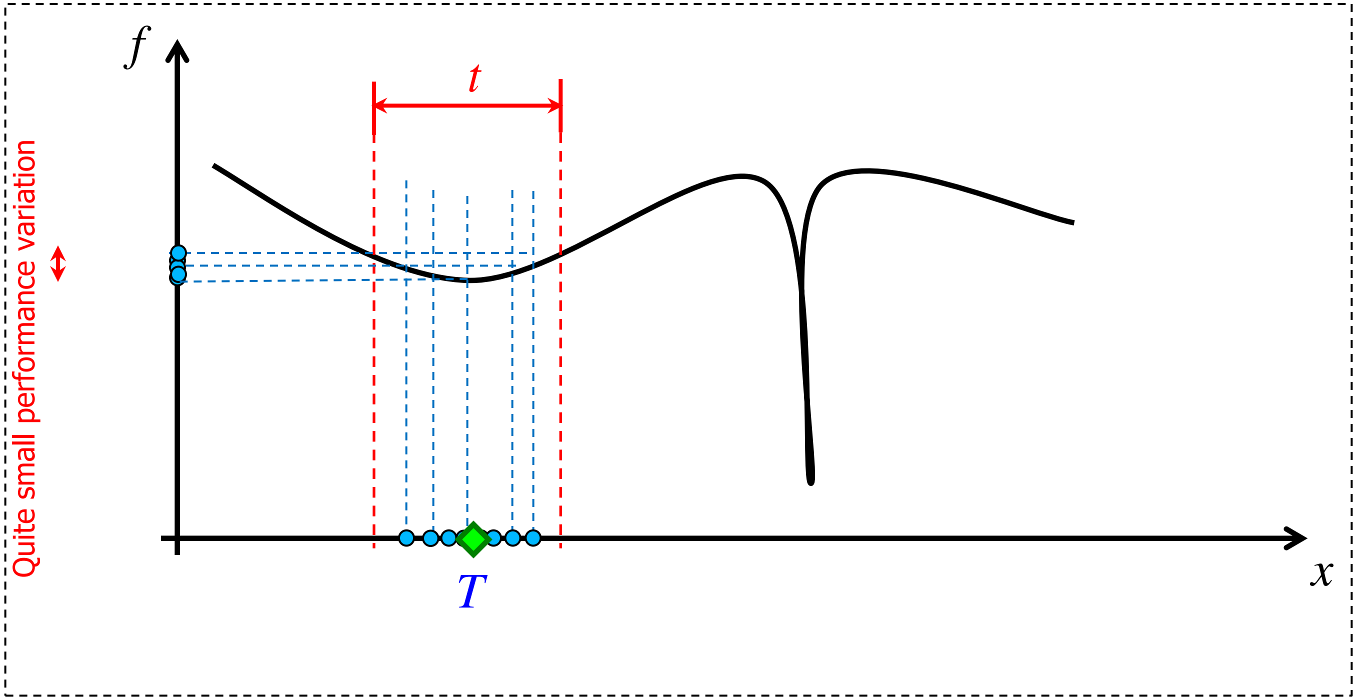
Which target value would you choose? x_L^* or x_G^* ?

A deterministic formulation and an efficient algorithm lead to choose $\boxed{x_G^*}$

What about uncertainties?



What about uncertainties ?



3. Optimization under uncertainties – Definition, classification and mathematical formulation of problems

Introduction

Section coming from [Lelievre et al., 2016] and [Beyer et al., 2007]

Mathematical formulation often confused!!

What is Reliability? Robustness? … Everybody has his own definition
→ **The solution depends on the formulation**

Objectives of this section:
→ **Clarify the formulations and propose a classification**

3. Optimization under uncertainties – Definition, classification and mathematical formulation of problems

Sources of uncertainties and classification

First classification of uncertainties by the origin:

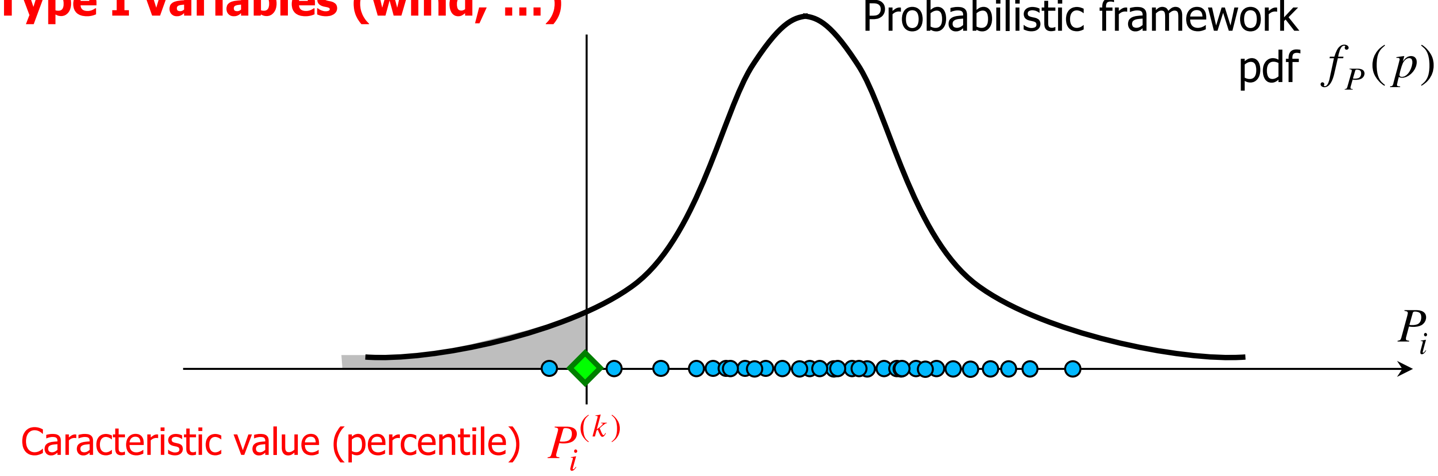
- ✓ **Objective or random uncertainties** : irreducible due to uncontrollable physical phenomena (wind, seism, manufacturing conditions, …), also called inherent uncertainties.
- ✓ **Epistemological uncertainties** : due to a lack of knowledge, can be reduced by experiments …

Second classification of uncertainties

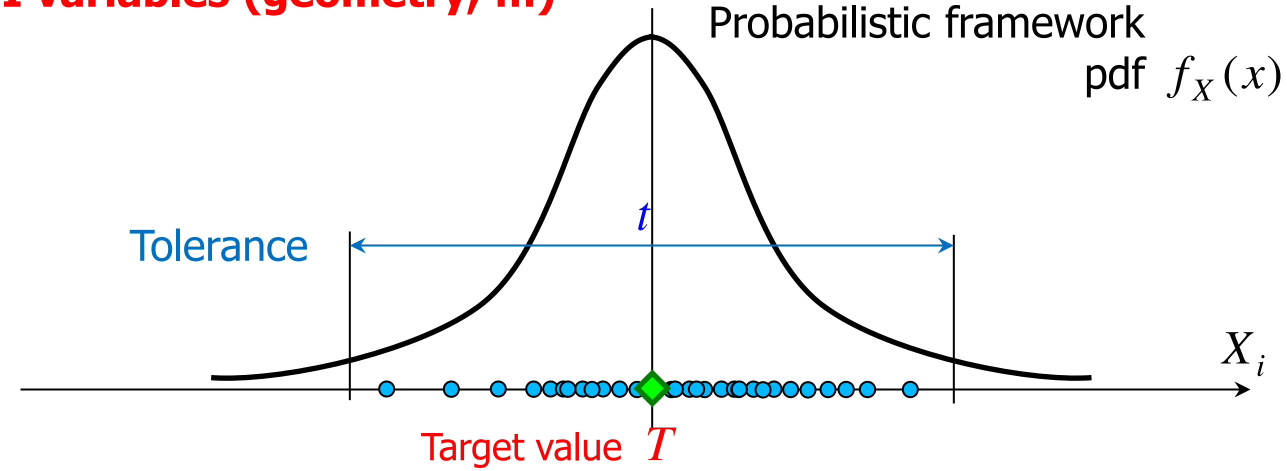
- ✓ **Type I**: random uncertainties linked to the environment and conditions of use. The designer has to "live with". Noted $\mathbf{P}(\omega)$ hereafter function of the randomness ω .
- ✓ **Type II**: random uncertainties, the designer can adjust by the mean of the Target value. Noted $\mathbf{X}(\mathbf{T},\omega)$ hereafter, function of the randomness and target value $\mathbf{T}$.
- ✓ **System function uncertainties**: model uncertainties linked to the performance of the system (can be grouped in type I).
- ✓ **Feasibility uncertainties**: model uncertainties linked to the constraints applied on the system (can be grouped in type I).

Approaches for describing uncertainties in design

✓ **Type I variables (wind, …)**



✓ **Type II variables (geometry, …)**



Design requirements

A mechanical system can be characterized by two different output functions

✓ **Objective-type function**

$\rightarrow$ to be maximized, minimized, quantify the performance of the system

$$f\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right)$$

✓ **Constraint-type function**

$\rightarrow$ must be satisfied in all operating conditions

$$g_j\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right) \quad j=1,\ldots,m$$

Admissible (safe) space: $\left\{X,P \,/\, \forall j=1,\ldots,m \quad g_j\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right) \geq 0\right\}$

Be careful

Approaches for describing uncertainties in design

Two approaches to consider uncertainty in design

✓ **Worst case approach** : uncertainties are considered with deterministic values

Type I : $\mathbf{P}(\omega) \rightarrow \mathbf{P}_k$

Type II : $\mathbf{X}(\mathbf{T},\omega) \rightarrow \mathbf{T}$

Find $\mathbf{T}^* \in \mathbb{R}^n$ such that : $\mathbf{T}^* = \mathbf{Argmin} \quad f(\mathbf{T},\mathbf{P}_k)$ $\qquad f : \mathbb{R}^n \rightarrow \mathbb{R}$

$\mathbf{under} \quad g_j(\mathbf{T},\mathbf{P}_k) \geq 0$ $\qquad g_j : \mathbb{R}^n \rightarrow \mathbb{R}$

$j = 1,\ldots,m$

Safety Margin? Failure probability? Robustness of the design?

Approaches for describing uncertainties in design

✓ **Probabilistic framework** : uncertainties are considered using pdf

Type I : $\mathbf{P}(\omega) \quad \rightarrow \quad f_P(p)$

Type II : $\mathbf{X}(\mathbf{T},\omega) \quad \rightarrow \quad f_x(x,T)$

Find $\mathbf{T}^* \in \mathbb{R}^n$ such that … reliable / robust system … **lots of possibilities**

3. Optimization under uncertainties – Definition, classification and mathematical formulation of problems

Proposal - Reliability / Robustness

My proposal …

ROBUSTNESS

$x(\omega)$?

Find $x^* \in \mathbb{R}^n$ such that : $x^* = \textbf{Argmin} \; f(x)$ $\quad f : \mathbb{R}^n \rightarrow \mathbb{R}$

under $g_j(x) \geq 0$ $\quad g_j : \mathbb{R}^n \rightarrow \mathbb{R}$

? $x(\omega)$ $\quad j = 1, \ldots, m$ $\quad$ (m **can be greater than** n)

RELIABILITY

RELIABILITY

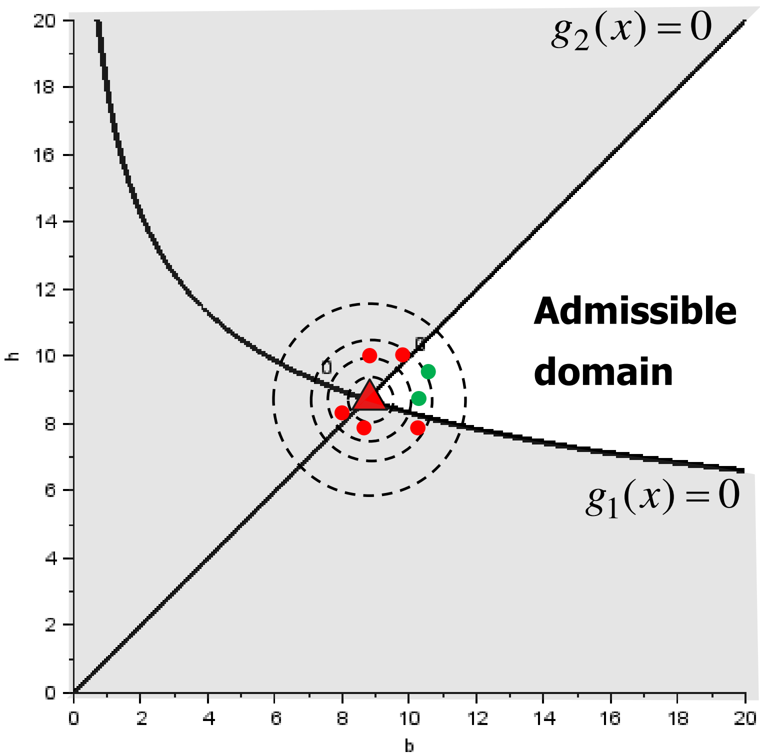
→ CONSIDERATION OF UNCERTAINTIES ON CONSTRAINT FUNCTIONS

ROBUSTNESS

→ CONSIDERATION OF UNCERTAINTIES ON OBJECTIVE FUNCTIONS

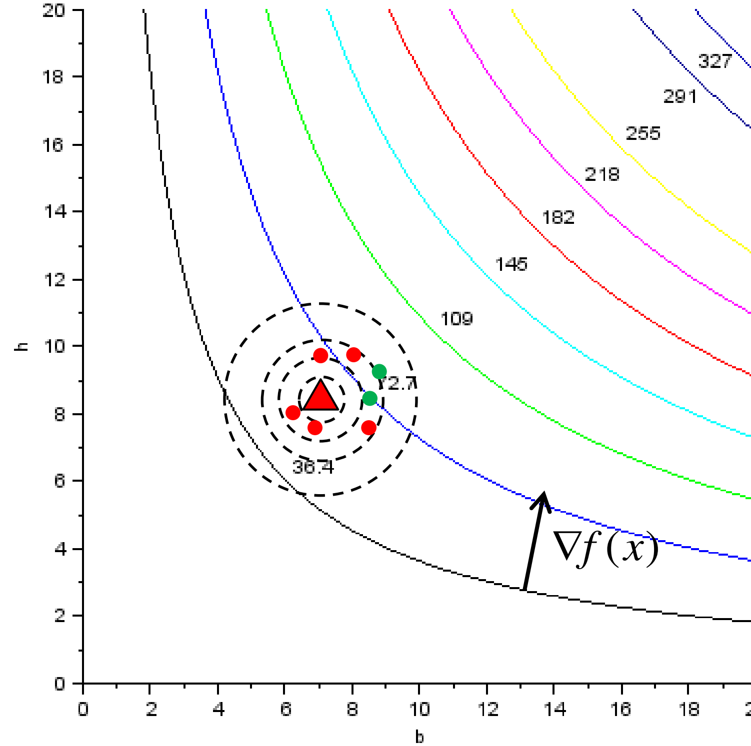
Proposal - Reliability / Robustness

RELIABILITY



Answer to the question "Is this design reliable?": YES / NO

ROBUSTNESS



No Answer to the question "Is this design robust?"

Design classification

[Lelievre et al., 2016]

ROBUSTNESS →

RELIABILITY ↑

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	Optimal design	Robust design
Constraint function with X, P deterministic	Admissible design	Optimal and admissible design	Robust and admissible design
Constraint function with X, P uncertain	Reliable design	Optimal and reliable design	Robust and reliable design

RELIABILITY

→ CONSIDERATION OF UNCERTAINTIES ON CONSTRAINT FUNCTIONS

ROBUSTNESS

→ CONSIDERATION OF UNCERTAINTIES ON OBJECTIVE FUNCTIONS

Problem classification

[Lelievre et al., 2016]

ROBUSTNESS →

RELIABILITY ↑

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	determinitic optimization without constraints	Optimization of the robutness
Constraint function with X, P deterministic	Sizing / dimensionning	determinitic optimization under constraints	Optimization of the robutness under determinitic constraint
Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robutness under uncertain constraint

RELIABILITY

→ CONSIDERATION OF UNCERTAINTIES ON CONSTRAINT FUNCTIONS

ROBUSTNESS

→ CONSIDERATION OF UNCERTAINTIES ON OBJECTIVE FUNCTIONS

Optimal design

ROBUSTNESS →

RELIABILITY ↑

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	**OPTIMAL DESIGN**	Robust design
Constraint function with X, P deterministic	Admissible design	Optimal and admissible design	Robust and admissible design
Constraint function with X, P uncertain	Reliable design	Optimal and reliable design	Robust and reliable design

$$\text{Find} \quad \mathbf{T}_{OPT} = \underset{\mathbf{T}}{\mathbf{Argmin}} \quad f(\mathbf{T}, \mathbf{P}_k)$$

Admissible design

ROBUSTNESS →

RELIABILITY ↑

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	Optimal design	Robust design
Constraint function with X, P deterministic	**ADMISSIBLE DESIGN**	Optimal and admissible design	Robust and admissible design
Constraint function with X, P uncertain	Reliable design	Optimal and reliable design	Robust and reliable design

$$\textbf{Find} \quad \mathbf{T}_{Adm} \,/\, g\left(\mathbf{T}_{Adm}, \mathbf{P}_k\right) \geq 0$$

Optimal and admissible design

ROBUSTNESS

RELIABILITY

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	Optimal design	Robust design
Constraint function with X, P deterministic	Admissible design	**OPTIMAL AND ADMISSIBLE DESIGN**	Robust and admissible design
Constraint function with X, P uncertain	Reliable design	Optimal and reliable design	Robust and reliable design

$$\textbf{Find} \quad \mathbf{T}_{OptAdm} = \underset{\mathbf{T}}{\textbf{Argmin}}\ f\left(\mathbf{T},\mathbf{P}_k\right)$$

$$\textbf{under} \quad g\left(\mathbf{T},\mathbf{P}_k\right) \geq 0$$

Reliable design

ROBUSTNESS →

RELIABILITY ↑

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	Optimal design	Robust design
Constraint function with X, P deterministic	Admissible design	Optimal and admissible design	Robust and admissible design
Constraint function with X, P uncertain	**RELIABLE DESIGN**	Optimal and reliable design	Robust and reliable design

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{\mathrm{Re}\,l} \quad / \quad \textbf{Prob}\left(g(\mathbf{X}(\tilde{\mathbf{T}}_{\mathrm{Re}\,l},\omega),\mathbf{P}(\omega)) \leq 0\right) \leq \mathrm{Pr}_{Target}$$

Other possible formulation based on quantiles : $g_k\left(\mathbf{X}(\mathbf{T}_{\mathrm{Re}\,l},\omega),\mathbf{P}(\omega)\right) \geq g_{Target}$

Robust design

ROBUSTNESS

RELIABILITY

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	Optimal design	**ROBUST DESIGN**
Constraint function with X, P deterministic	Admissible design	Optimal and admissible design	Robust and admissible design
Constraint function with X, P uncertain	Reliable design	Optimal and reliable design	Robust and reliable design

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{Rob} = \underset{\mathbf{T}}{\textbf{Argmin}} \quad \Psi(\mathbf{T})$$

$$\Psi(\mathbf{T}) \quad ?$$

Robust design

$\Psi\left(\mathbf{T}\right)$?

- ✓ **Agregation function :**

$$\Psi\left(\mathbf{T}\right) = \lambda \mathbf{E}\left[f\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right)\right] + (1-\lambda)\mathbf{Var}\left[f\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right)\right]$$

[Papadrakakis et al. 2005]

Robust design

$\Psi(\mathbf{T})$?

✓ **Tagushi approach:**

Mean square deviation :

Need a target value for the objective !

$$\Psi(\mathbf{T}) = \mathbf{E}\left[\left(f\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right) - f_{target}\right)^2\right]$$

$$= \mathbf{E}\left[\left(f\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right) - \mathbf{E}\left[f\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right)\right]\right)^2\right]$$

$$+ \mathbf{E}\left[\left(\mathbf{E}\left[f\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right)\right] - f_{target}\right)^2\right]$$

$$= \sigma_f^2(\mathbf{T}) + \delta_f^2(\mathbf{T})$$

Standard deviation of the objective function

Mean shift of the objective function

[Trosset, 1997]

Robust design

$\Psi(\mathbf{T})$?

✓ **Multi-objective formulation (Pareto Front) :**

$\mathbf{E}\left[f\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right)\right]$ to minimize [Rathod et al. 2009]

$\mathbf{Var}\left[f\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right)\right]$ to minimize …

Robust design

ROBUSTNESS →

RELIABILITY ↑

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	Optimal design	Robust design
Constraint function with X, P deterministic	Admissible design	Optimal and admissible design	Robust and admissible design
Constraint function with X, P uncertain	Reliable design	**OPTIMAL AND RELIABLE DESIGN**	Robust and reliable design

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{OptRel} = \underset{\mathbf{T}}{\mathbf{Argmin}}\; f\left(\mathbf{T},\mathbf{P}_k\right)$$

$$\textbf{under} \quad \mathbf{Prob}\left(g\left(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)\right)\le 0\right)\le \Pr_{Target}$$

Currently called a RBDO problem

Robust design

ROBUSTNESS

RELIABILITY

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	Optimal design	Robust design
Constraint function with X, P deterministic	Admissible design	Optimal and admissible design	**ROBSUT AND ADMISSIBLE DESIGN**
Constraint function with X, P uncertain	Reliable design	Optimal and reliable design	Robust and reliable design

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{RobAdm} = \underset{\mathbf{T}}{\textbf{Argmin}} \quad \Psi(\mathbf{T})$$

$$\textbf{under} \quad g(\mathbf{T}, \mathbf{P}_k) \geq 0$$

Robust design

ROBUSTNESS →

RELIABILITY ↑

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	Optimal design	Robust design
Constraint function with X, P deterministic	Admissible design	Optimal and admissible design	Robust and admissible design
Constraint function with X, P uncertain	Reliable design	Optimal and reliable design	**ROBSUT AND RELIABLE DESIGN**

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{RobRel} = \underset{\mathbf{T}}{\textbf{Argmin}} \quad \Psi(\mathbf{T})$$

$$\textbf{under} \quad \textbf{Prob}\left(g(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)) \leq 0\right) \leq \Pr_{Target}$$

3. Optimization under uncertainties – Definition, classification and mathematical formulation of problems

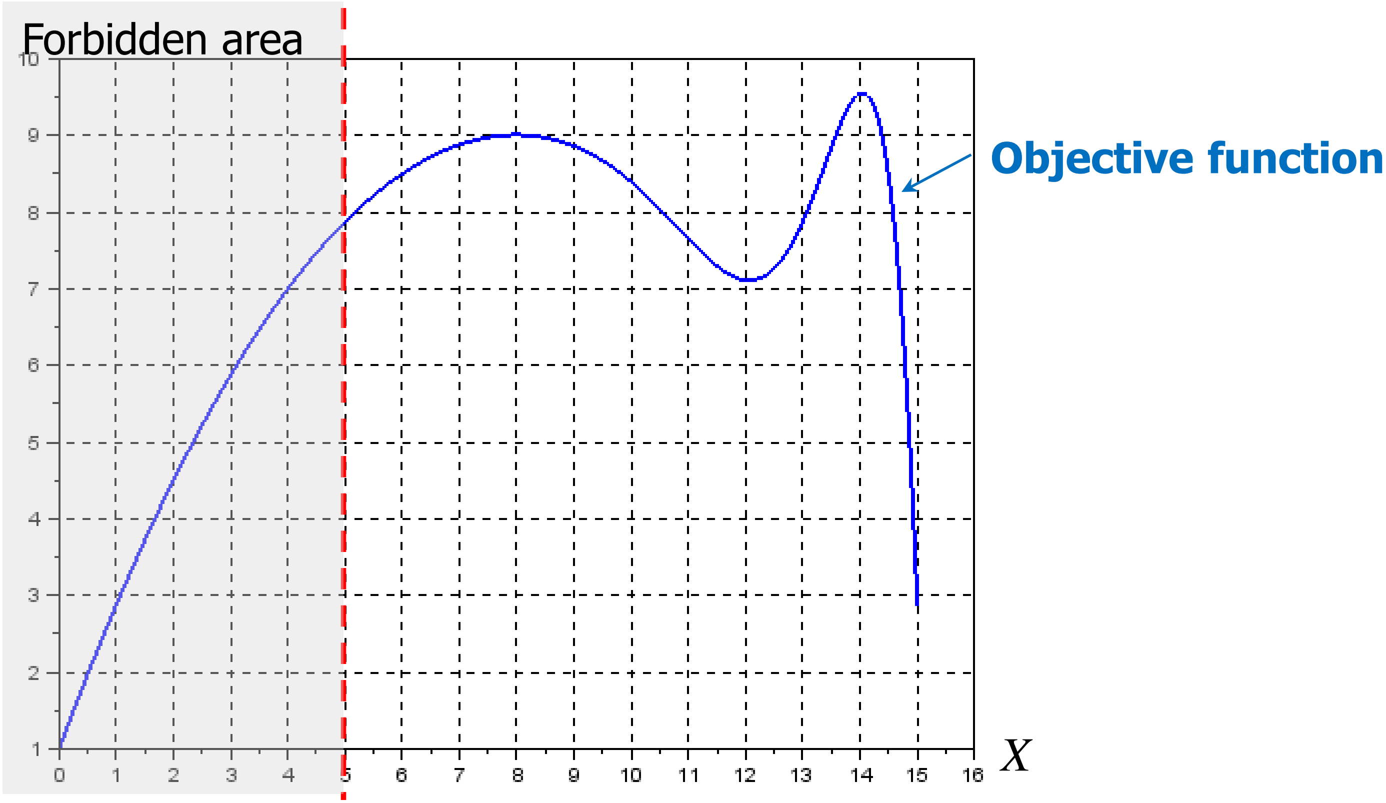
1D illustration

Objective function :
To be maximized

$$f(X) = 9 + 6\exp(-14 + X)\sin(15 - X) - \frac{1}{8}(8 - X)^2$$

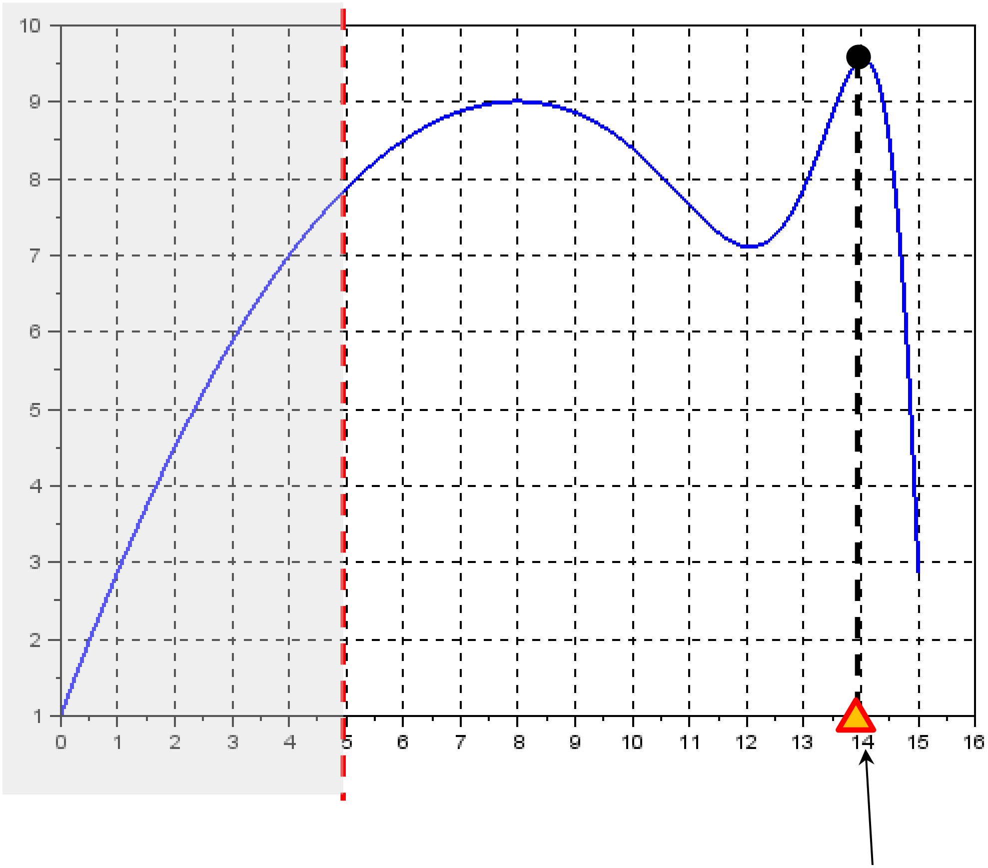
Constraint function :

$$X \geq 5 \qquad g(X) = X - 5 \geq 0$$



1D illustration

WITHOUT UNCERTAINTIES

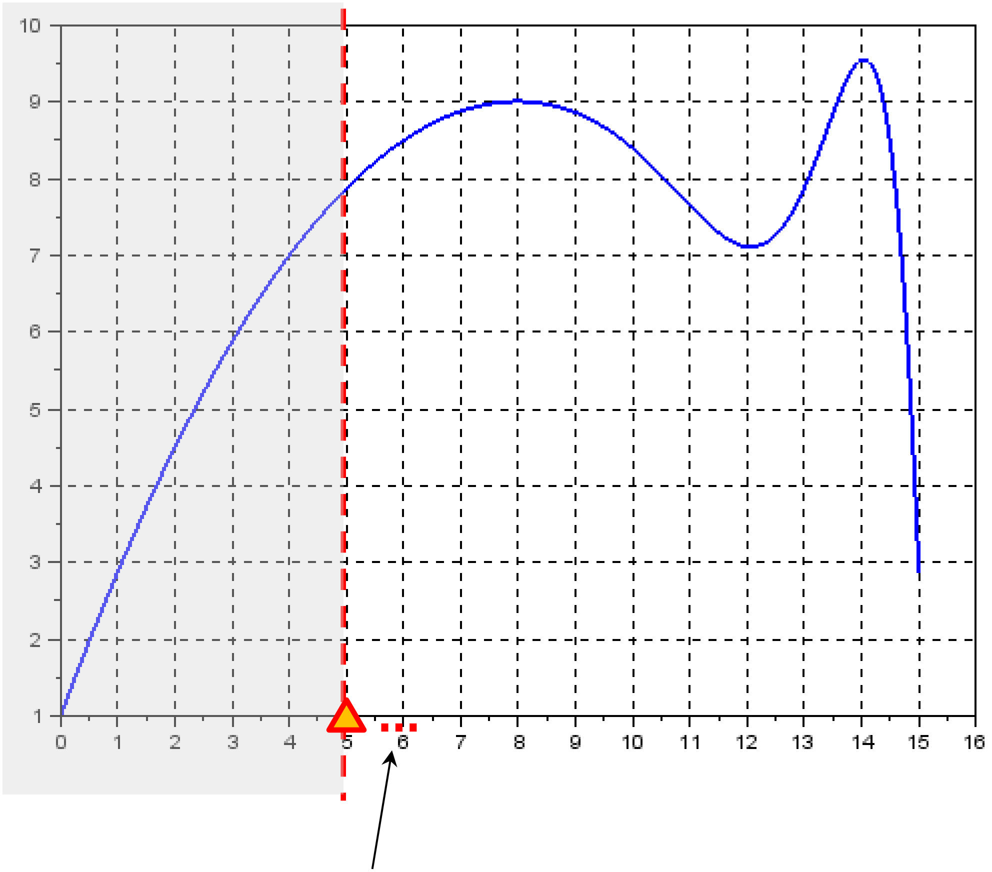


Find $\mathbf{T}_{OPT} = \underset{\mathbf{T}}{\mathbf{Argmax}} \; f(\mathbf{T})$

Optimal design

1D illustration

WITHOUT UNCERTAINTIES

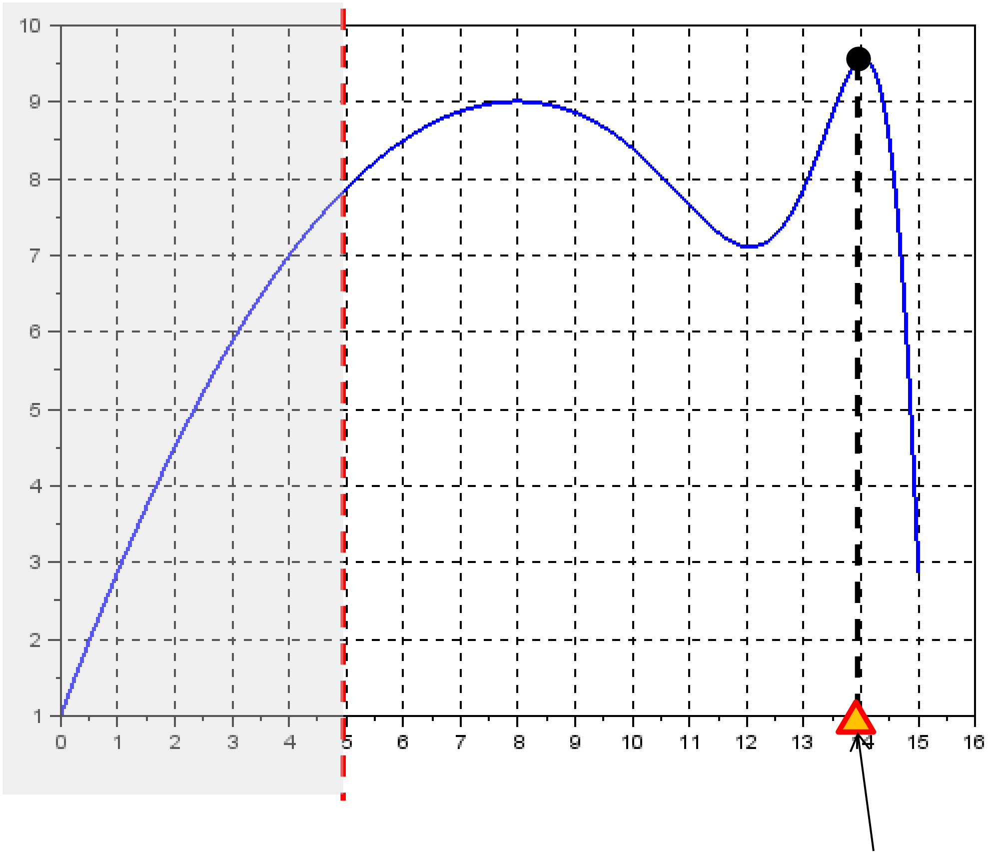


$$\textbf{Find} \quad \mathrm{T}_{Adm} \,/\, g\left(\mathrm{T}_{Adm}\right) \geq 0$$

Admissible design

1D illustration

WITHOUT UNCERTAINTIES



Optimal and Admissible design :

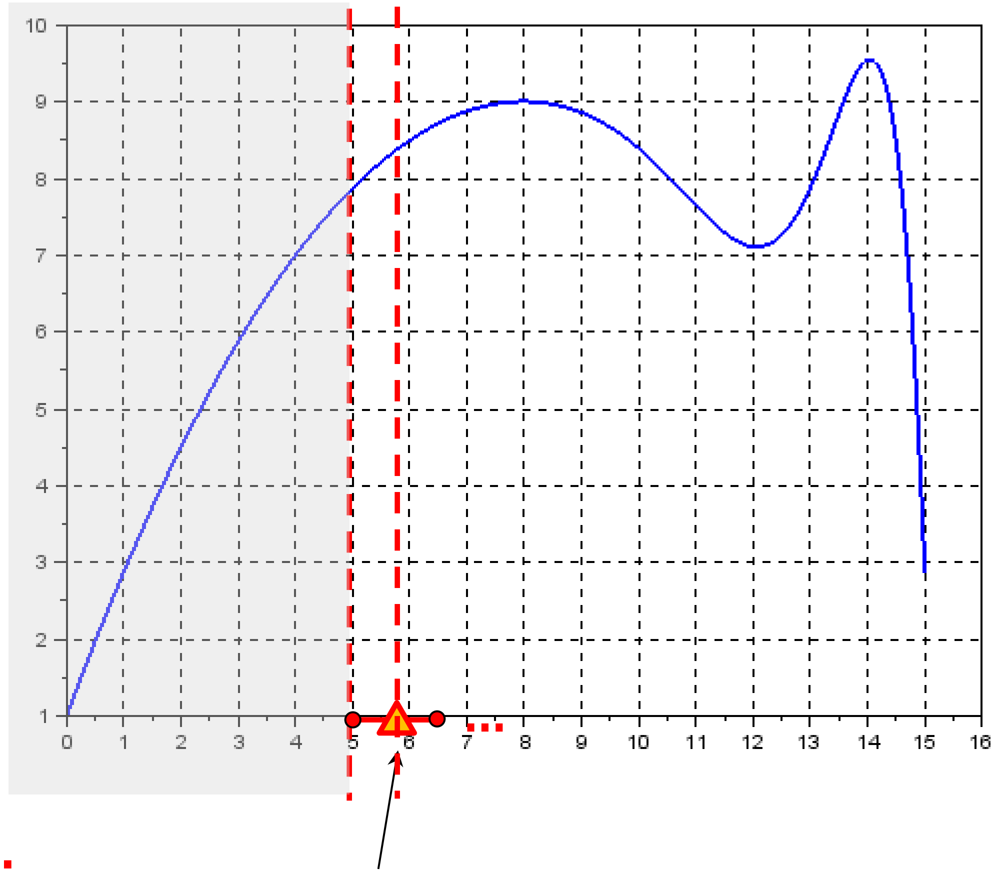
$$\textbf{Find} \quad T_{OptAdm} = \textbf{Argmax}\ f\left(T\right)$$

$$\textbf{under} \quad g\left(T\right) \geq 0$$

1D illustration

WITH UNCERTAINTIES

$$X \rightarrow N(T, \sigma = 1)$$



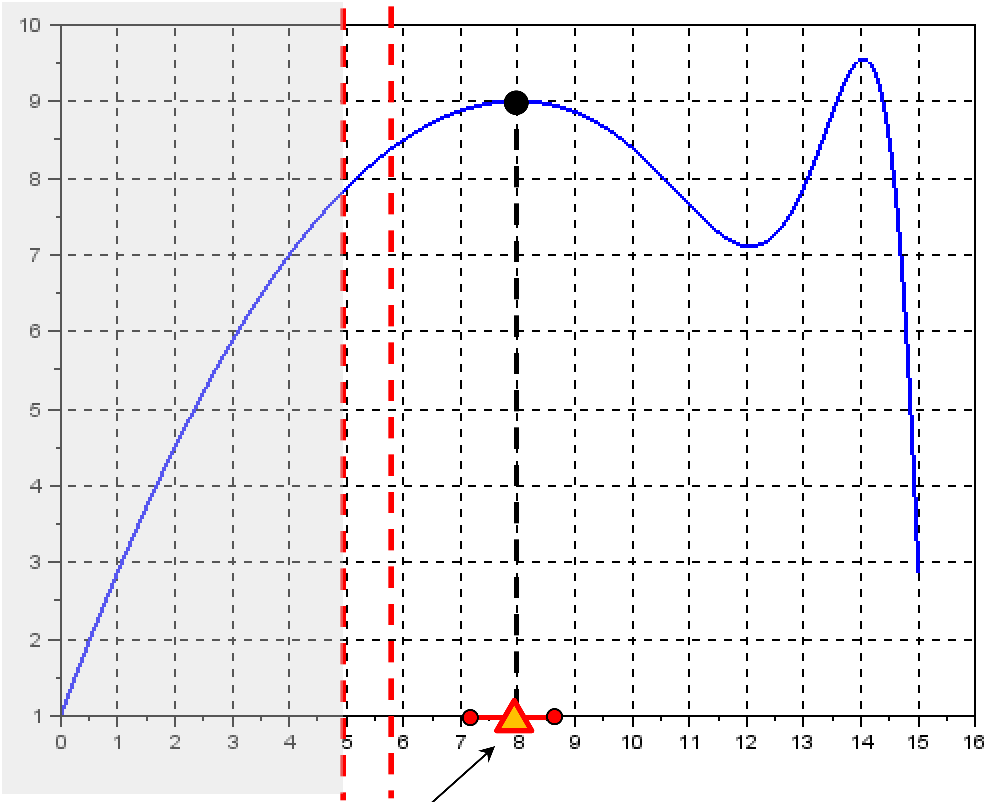
Reliable design :

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{\mathrm{Re}l} \quad / \quad \textbf{Prob}\left(g(\mathbf{X}(\tilde{\mathbf{T}}_{\mathrm{Re}l}, \omega)) \leq 0 \right) \leq 0{,}1$$

1D illustration

WITH UNCERTAINTIES

$$X \rightarrow N(T, \sigma = 1)$$



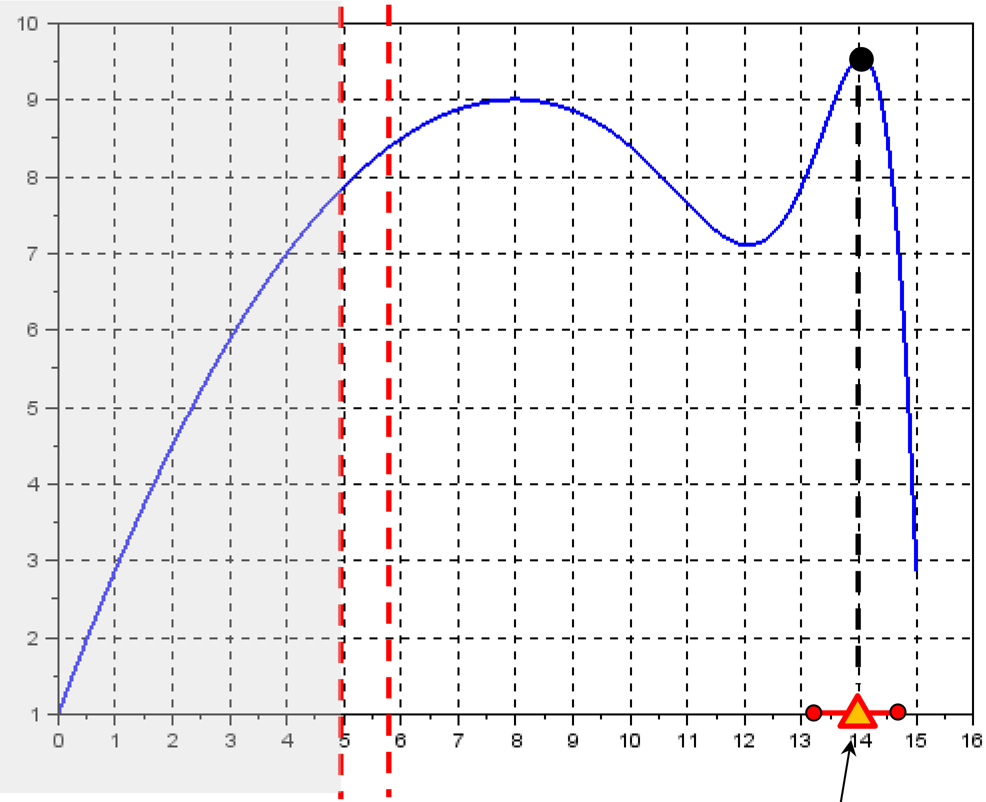
Robust design :

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{Rob} = \underset{\mathbf{T}}{\textbf{Argmin}} \quad \sigma_f^2(\mathbf{T})$$

1D illustration

WITH UNCERTAINTIES

$$X \rightarrow N(T, \sigma = 1)$$



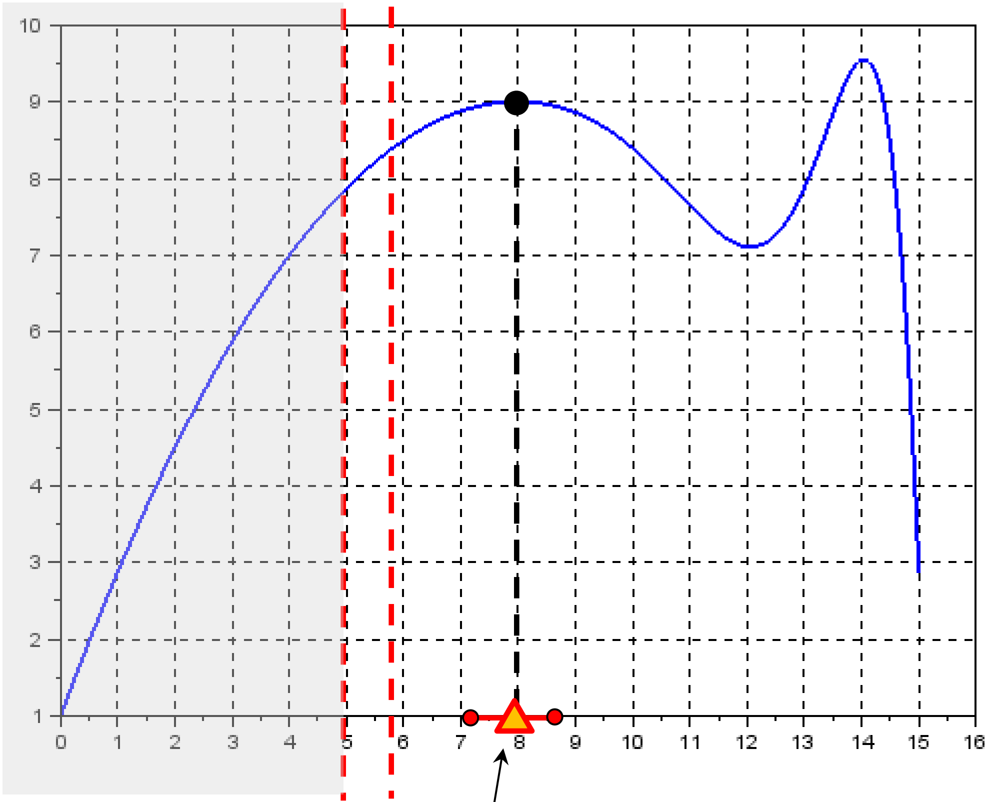
Optimal and reliable design : **Find** $\tilde{\mathbf{T}}_{OptRel} = \underset{\mathbf{T}}{\mathbf{Argmax}}\, f(\mathbf{T})$

under $\mathbf{Prob}\left(g(\mathbf{X}(\tilde{\mathbf{T}}_{OptRel}, \omega)) \leq 0 \right) \leq 0{,}1$

1D illustration

WITH UNCERTAINTIES

$X \rightarrow N(T, \sigma = 1)$



Robust and admissible design :

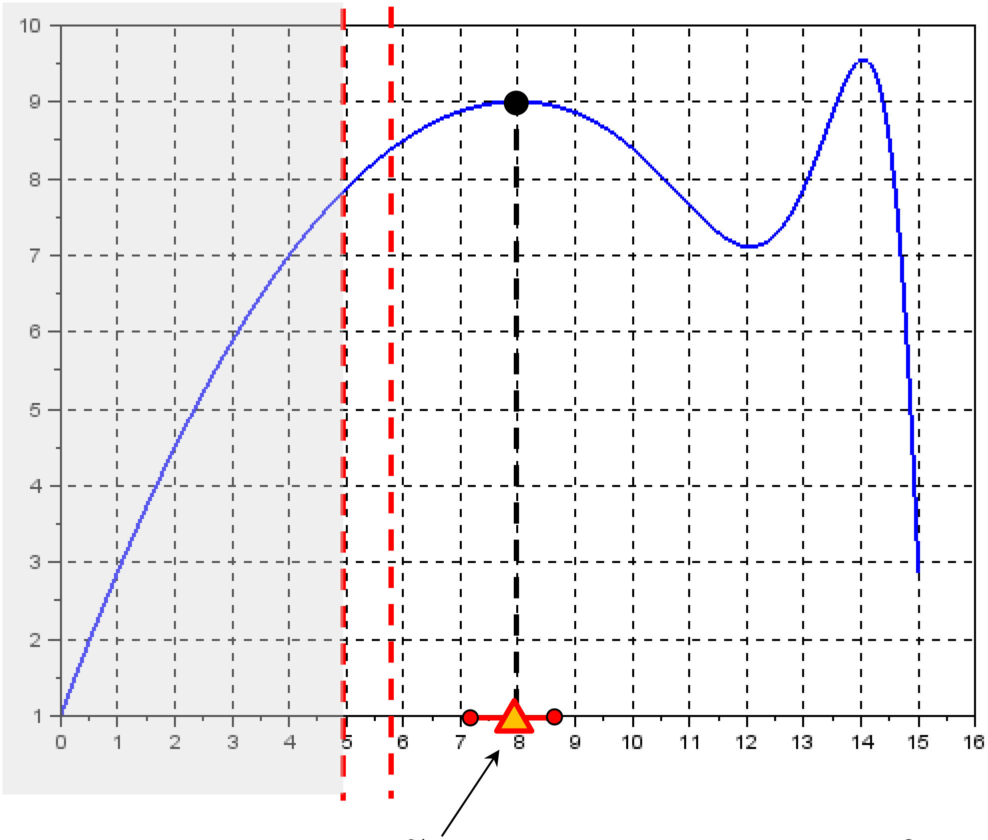
Find $\tilde{\mathbf{T}}_{RobAdm} = \underset{\mathbf{T}}{\mathbf{Argmin}}\ \sigma_f^2(\mathbf{T})$

under $g(\mathbf{T}) \leq 0$

1D illustration

WITH UNCERTAINTIES

$$X \rightarrow N(T, \sigma = 1)$$



Robust and reliable design :

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{RobRel} = \underset{\mathbf{T}}{\textbf{Argmin}} \quad \sigma_f^2(\mathbf{T})$$

$$\textbf{under} \quad \textbf{Prob}\left(g(\mathbf{X}(\tilde{\mathbf{T}}_{RobRel}, \omega) \leq 0 \right) \leq 0{,}1$$

1D illustration - synthesis

	No objective function	Objective function with X deterministic	Objective function with X uncertain
No constraint function	X	Optimal design T=14,04 / f=9,56	Robust design T=8 / f=9
Constraint function with X, P deterministic	Admissible design T≥5	Optimal and admissible design T=14,04 / f=9,56	Robust and admissible design T=8 / f=9
Constraint function with X, P uncertain	Reliable design T≥5,7	Optimal and reliable design T=14,04 / f=9,56	Robust and reliable design T=8 / f=9

2D illustration – Application to a container

Objective function : To be minimized

$$f(R,h) = 2\pi Rh + 2\pi R^2$$

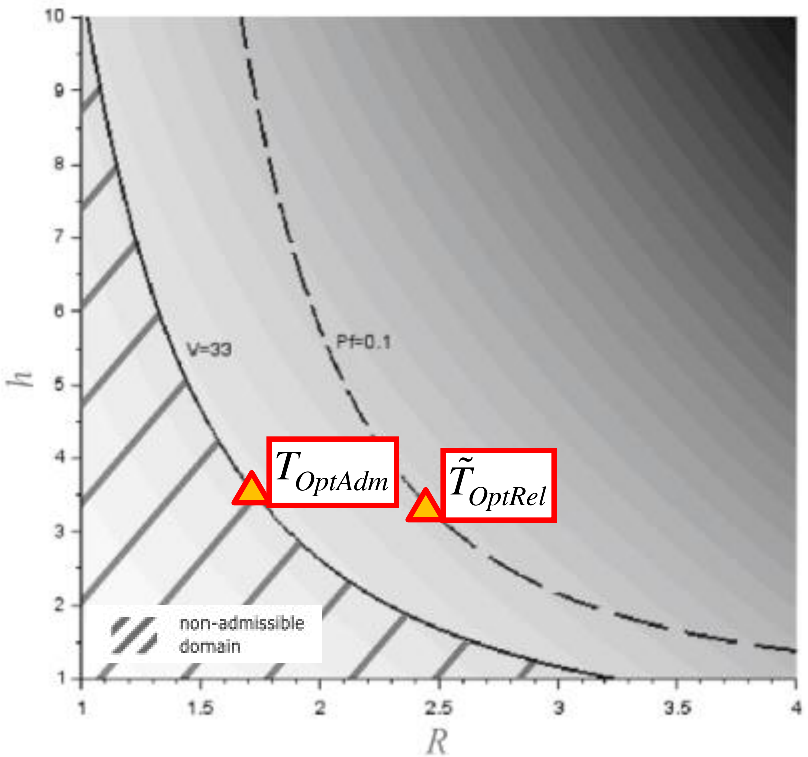
$$T_R \in [1;4] \quad T_h \in [1;10]$$

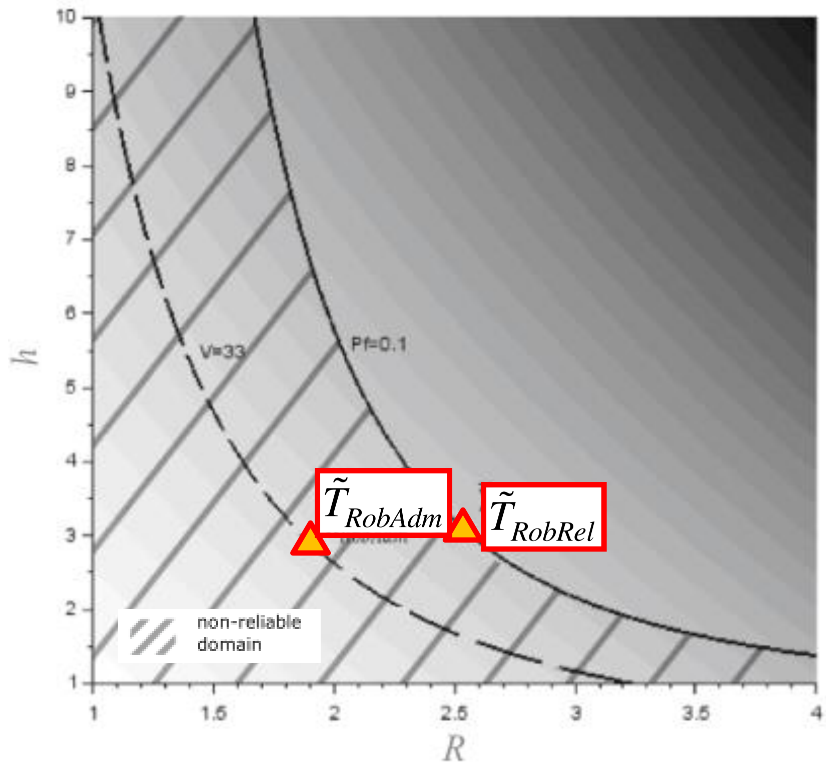
Constraint function :

$$g(R,h) = \pi R^2 h - 33 \geq 0$$

$$R \to N(T_R;0{,}5)$$

$$h \to N(T_h;0{,}5)$$





Will be detailed at the end in section 10 …

3. Optimization under uncertainties – Definition, classification and mathematical formulation of problems

Classification limits

Failure probability can be in the objective function (see [Moses, 1997]):

$$f(d) = \mathbf{E}(C_I(d, X(\omega))) + \mathbf{E}(C_F(d, X(\omega)))$$

Initial construction cost

Failure cost: $\mathbf{E}(C_F(d, X(\omega))) = C_f P_f(d)$

↑ Difficult to assess

One very important issue ...

VERY IMPORTANT

The optimization variables must be variables known by the engineers :

- ✓ Nominal values (or characteristic values)
- ✓ Tolerance intervals

BUT : the assessment of reliability and/or robustness needs to know the distribution of random variables.

What is the link between engineering variables and random variables ?

$$\mathbf{T} \xrightarrow{??} \mathbf{X}(\mathbf{T}, \omega)$$

Crucial assumption acting on optimal results

One very important issue ...

Standard assumption [Lafon et al.]:

- ✓ Gaussian distribution of random variables
- ✓ Mean value = Nominal value
- ✓ Standard deviation = constant / coefficient of variation = constant

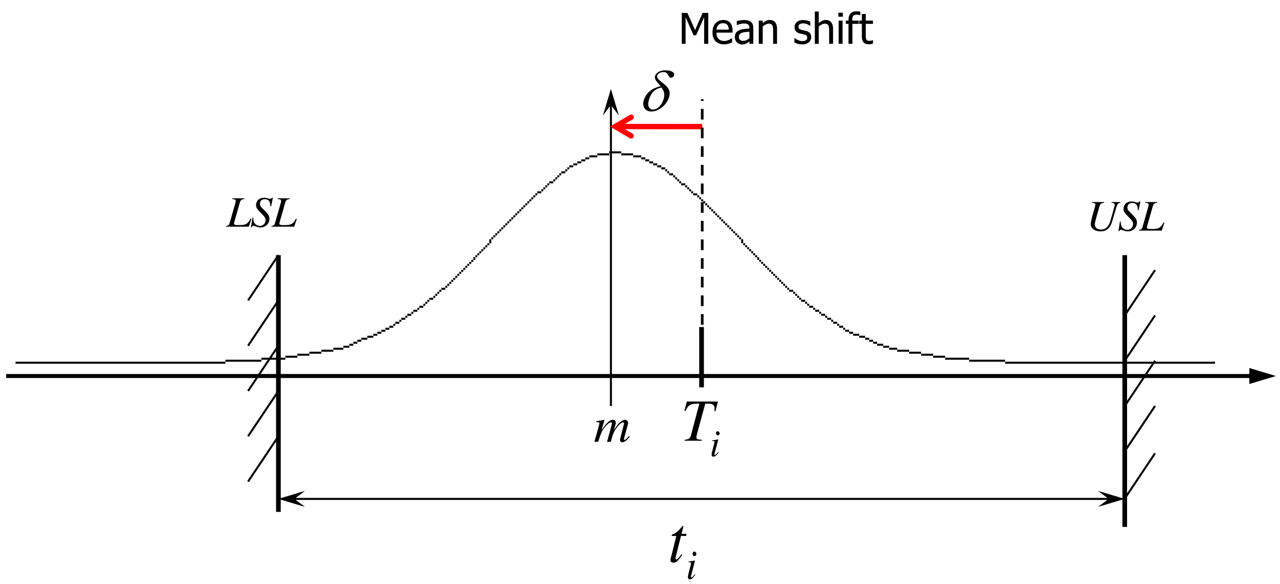
Alternatives (6-sigma approach)

- ✓ Gaussian distribution of random variables
- ✓ Mean value = Nominal value
- ✓ Standard deviation = tolerance / 6

One very important issue ...

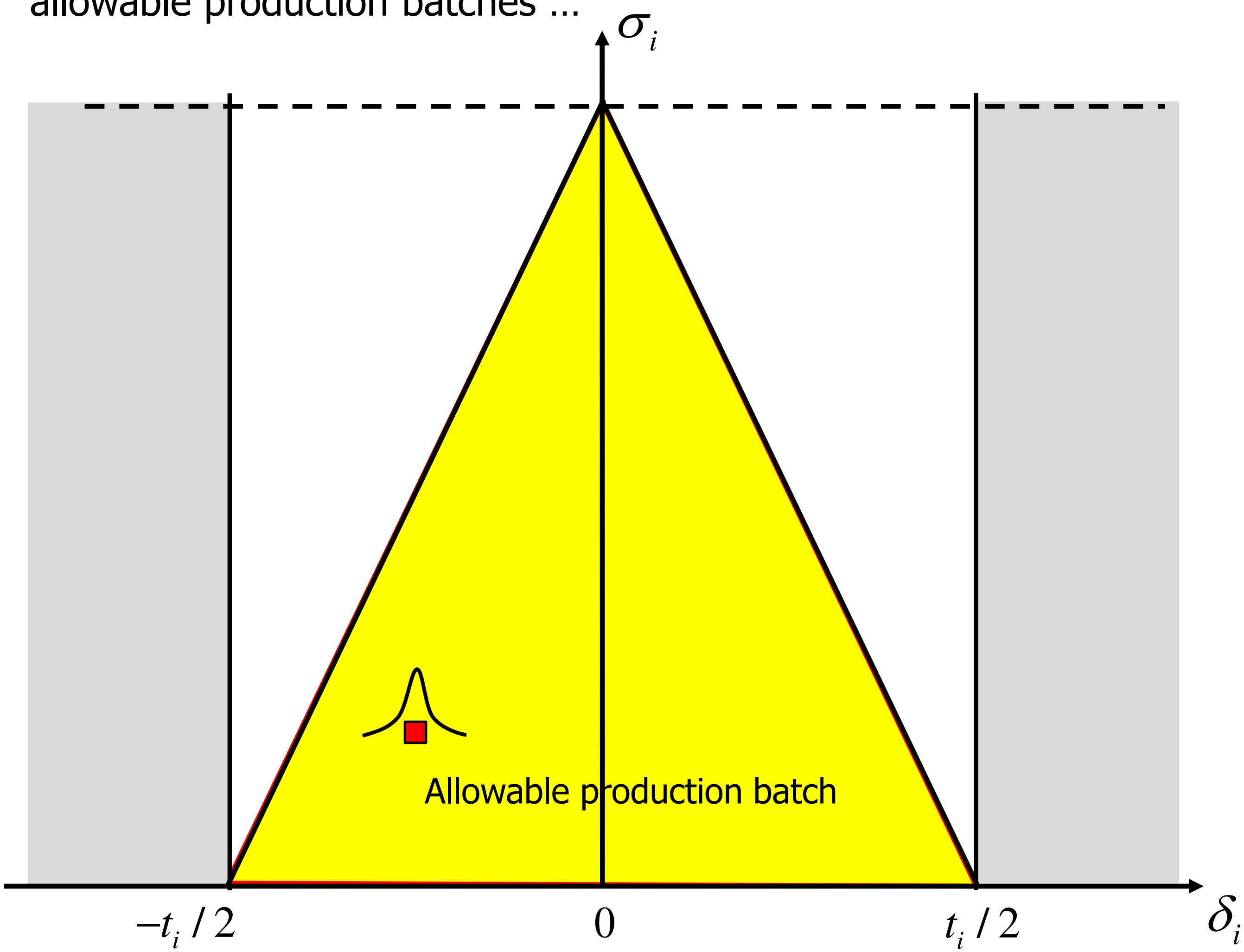
But :

- ✓ A mean shift can exist, in mass production, it exists and varies
- ✓ A tolerance increase, increase allowable mean shift



One very important issue ...

In mass production, for a unique nominal / tolerance pair, it exists a wide range of allowable production batches …



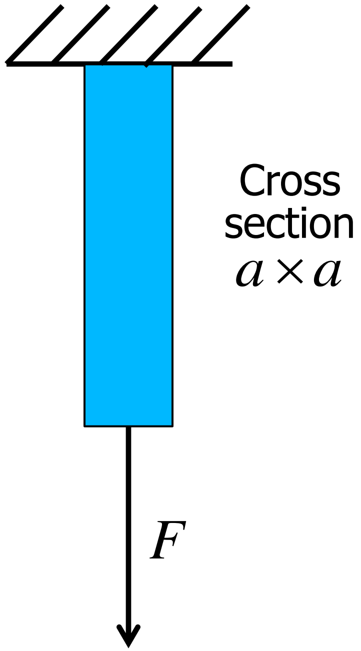
To go in depth : the APTA method

[Gayton et al. 2011]

One very important issue ... illustration

Failure mode: plasticity reached in the beam

$$P_f = \mathbf{Prob}\left(f_y(\omega) - \frac{F(\omega)}{a(\omega)^2} \le 0\right)$$



Hypothesis:

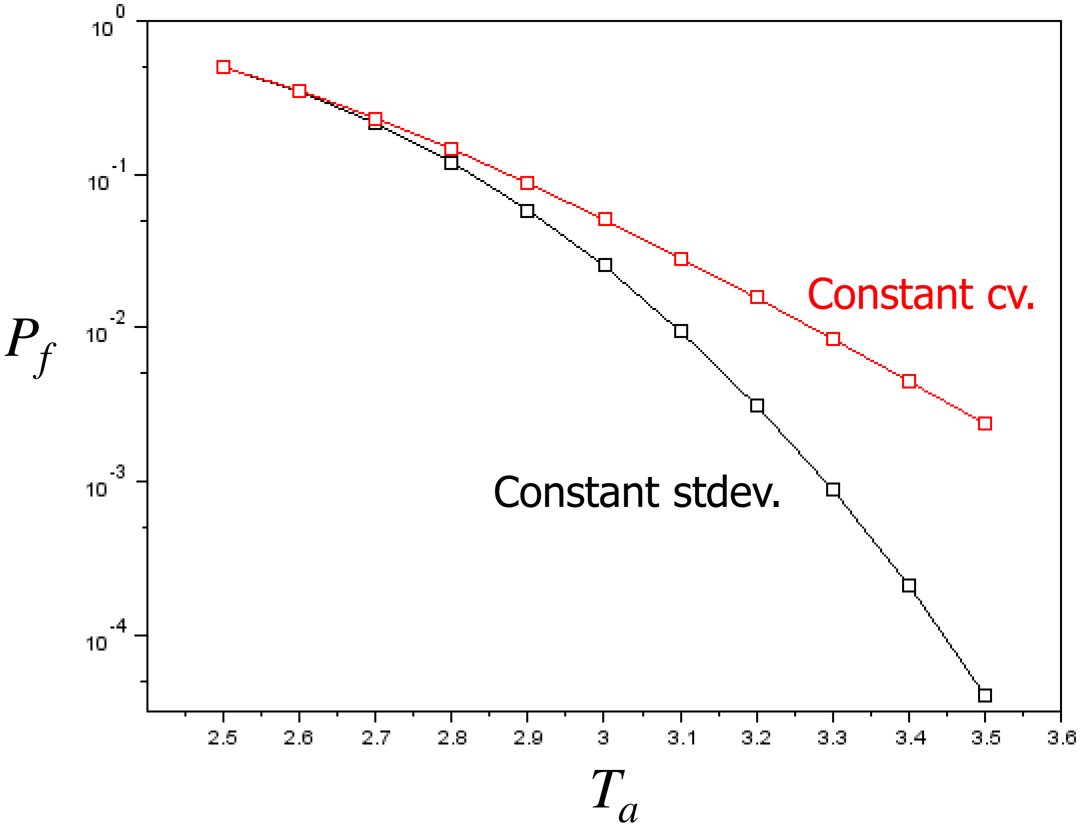
- ✓ a, f_y, F are Gaussian random variables

  $m_{fy} = 80000 MPa \quad \sigma_{fy} = 5000 MPa$

  $m_F = 10000 N \quad \sigma_F = 500 N$

- ✓ Hypothesis #1: $m_a = T_a \quad \sigma_a = 0{,}5$
- ✓ Hypothesis #2: $m_a = T_a \quad \sigma_a = 0{,}05 T_a \quad ie. \quad c_a = 5\%$

One very important issue ... illustration



4. Reliability index and failure probability local sensitivity to parameters

Failure probability local derivatives

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{RobAdm} = \underset{\mathbf{T}}{\textbf{Argmin}} \quad \Psi(\mathbf{T})$$

$$\textbf{under} \quad c(\mathbf{T}) = \textbf{Prob}\big(g(\mathbf{X}(\mathbf{T},\omega),\mathbf{P}(\omega)) \le 0\big) - \mathrm{Pr}_{Target} = 0$$

Lagrangian iterative method with $\Psi(\mathbf{T}) = \sigma_f^2 + \delta_f^2$:

$$\frac{\partial^2 m_f(\mathbf{T})}{\partial^2 T_i} \qquad \frac{\partial^2 \sigma_f(\mathbf{T})}{\partial^2 T_i} \qquad \qquad \frac{\partial\, m_f(\mathbf{T})}{\partial\, T_i} \qquad \frac{\partial\, \sigma_f(\mathbf{T})}{\partial\, T_i}$$

$$\begin{bmatrix} \nabla^2\Psi(\mathbf{T}^{(k)}) & \nabla c(\mathbf{T}^{(k)}) \\ \nabla c(\mathbf{T}^{(k)})^t & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{T}^{(k+1)} - \mathbf{T}^{(k)} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -\nabla\Psi(\mathbf{T}^{(k)}) \\ -c(\mathbf{T}^{(k)}) \end{Bmatrix}$$

$$\frac{\partial\, P_f(\mathbf{T})}{\partial\, T_i}$$

Failure probability local derivatives

$\rightarrow$ Need the evaluation of the gradient and hessien regarding T that can be an hyperparameter of the density function or a model parameter

$$\frac{\partial m_f(\mathbf{T})}{\partial T_i} \quad \frac{\partial^2 m_f(\mathbf{T})}{\partial^2 T_i}$$

$$\frac{\partial \sigma_f(\mathbf{T})}{\partial T_i} \quad \frac{\partial^2 \sigma_f(\mathbf{T})}{\partial^2 T_i}$$

$$\frac{\partial P_f(\mathbf{T})}{\partial T_i} \quad \frac{\partial^2 P_f(\mathbf{T})}{\partial^2 T_i}$$

- ✓ **Local sensitivity to hyper-parameters**
- ✓ **Local sensitivity to model parameters**

Failure probability local derivatives

$$\frac{\partial P_f(\mathbf{T})}{\partial T_i}$$

Can always be computable by finite differences

$$\left.\frac{\partial P_f(\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^{(0)}} \approx \frac{1}{h}\left(P_f(\mathbf{T}^{(0)}+he_i)-P_f(\mathbf{T}^{(0)})\right)$$

But:

- ✓ Problem of seed (need to keep the same one)
- ✓ Choice of increment (always delicate)
- ✓ Two MC simulations (twice more time consuming)

4. Reliability index and failure probability local sensitivity to parameters

4.1 Local sensitivity to hyper-parameters

4.2 Local sensitivity to model parameters

Reliability index local derivatives

$$\frac{\partial P_f(\mathbf{T})}{\partial T_i} \quad ?$$

$$\left.\frac{\partial P_f(\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^{(0)}} \approx \left.\frac{\partial \Phi(-\beta(\mathbf{T}))}{\partial T_i}\right|_{\mathbf{T}^{(0)}} = -\phi\left(-\beta(\mathbf{T}^{(0)})\right)\left.\frac{\partial \beta(\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^{(0)}}$$

Reliability index sensitivity w.r.t. a distribution parameter $\beta(\mathbf{T}) = \sqrt{\mathbf{u}^*(\mathbf{T})^t . \mathbf{u}^*(\mathbf{T})}$

$$\left.\frac{\partial \beta(\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^{(0)}} = -\left.\frac{\partial \mathbf{u}^*(\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^{(0)}} . \boldsymbol{\alpha}$$

(Hohenbichler et al., 1986)

$$\left.\frac{\partial \beta(\mathbf{T})}{\partial \mathbf{T}}\right|_{\mathbf{T}^{(0)}} = -J(\mathbf{u}^*, \mathbf{T}^{(0)}).\boldsymbol{\alpha}$$

Jacobian matrix of the isoprobabilistic transformation

Failure probability local derivatives

[Rubinstein and al.]

$$Q(T) = \int_{\mathbb{R}^n} h(\mathbf{X}) f_{\mathbf{X}}(\mathbf{X}, \mathbf{T})\, \mathbf{dX}$$

Failure probability

$h(\mathbf{X})$: failure indicator function (0 / 1) $\mathbf{I}_f(\mathbf{X})$

$f_X(\mathbf{X})$: Joint density function

Esperance

$h(\mathbf{X})$: considered function $Q(T) = \mathbf{E}\left[h(\mathbf{X})\right]$

$f_X(\mathbf{X})$: Joint density function

Variance

$h(\mathbf{X}) \rightarrow \left(h(\mathbf{X}) - m_h(\mathbf{T})\right)^2$ Contains $T \rightarrow$ *More complex (can be written)*

$f_X(\mathbf{X})$: Joint density function

Failure probability local derivatives to hyperparameters

$$\left.\frac{\partial\, Q(T)}{\partial\, T_i}\right|_{\mathbf{T}^*} = \frac{\partial}{\partial T_i} \cdot \int_{\mathbb{R}^n} h(\mathbf{X})\, f_{\mathbf{X}}(\mathbf{X},\mathbf{T})\, \mathbf{dX}$$

$$= \int_{\mathbb{R}^n} h(\mathbf{X}) \left.\frac{\partial f_{\mathbf{X}}(\mathbf{X},\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^*} \mathbf{dX}$$

$$= \int_{\mathbb{R}^n} h(\mathbf{X}) \frac{\left.\frac{\partial f_{\mathbf{X}}(\mathbf{X},\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^*}}{f_{\mathbf{X}}(\mathbf{X},\mathbf{T}^*)} f_{\mathbf{X}}(\mathbf{X},\mathbf{T}^*)\, \mathbf{dX}$$

$$= \int_{\mathbb{R}^n} h(\mathbf{X}) \left.\frac{\partial \ln f_{\mathbf{X}}(\mathbf{X},\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^*} f_{\mathbf{X}}(\mathbf{X},\mathbf{T}^*)\, \mathbf{dX}$$

$$= \int_{\mathbb{R}^n} h(\mathbf{X})\, s_i(\mathbf{X},\mathbf{T}^*)\, f_{\mathbf{X}}(\mathbf{X},\mathbf{T}^*)\, \mathbf{dX}$$

Score function : the score function is the derivative of the log of the joint density with respect to the parameters

Assessment using simulation :

$$\left.\frac{\partial\, Q(T)}{\partial\, T_i}\right|_{\mathbf{T}^*} \approx \frac{1}{N} \sum_{j=1}^{N} h(\mathbf{X}^{(j)})\, s_i(\mathbf{X}^{(j)},\mathbf{T}^*)$$

Failure probability local derivatives to hyperparameters

Score function assessment

$$s_i(\mathbf{X},\mathbf{T}^*) = \left.\frac{\partial \ln f_{\mathbf{X}}(\mathbf{X},\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^*}$$

For independent random variables :

$$s_i(\mathbf{X},\mathbf{T}^*) = \left.\frac{\partial \ln f_i(\mathbf{X},\mathbf{T})}{\partial T_i}\right|_{\mathbf{T}^*}$$

For gaussian and independent random variables :

$$s_i(\mathbf{X},\mathbf{T}^*) = \left.\frac{\partial \ln f_i(x_i,\mathbf{T})}{\partial m_i}\right|_{\mathbf{T}^*} = \frac{x_i - m_i}{\sigma_i^2}$$

$$s_i(\mathbf{X},\mathbf{T}^*) = \left.\frac{\partial \ln f_i(x_i,\mathbf{T})}{\partial \sigma_i}\right|_{\mathbf{T}^*} = -\frac{1}{\sigma_i} + \frac{(x_i - m_i)^2}{\sigma_i^3}$$

Failure probability local derivatives to hyperparameters

Confidence interval

$$\left.\frac{\partial Q(T)}{\partial T_i}\right|_{\mathbf{T}^*} = \mathbf{E}\left[h(\mathbf{X})s_i(\mathbf{X},\mathbf{T}^*)\right] = \mathbf{E}\left[g(\mathbf{X})\right]$$

$$\approx \frac{1}{N}\sum_{j=1}^{N} h(\mathbf{X}^{(i)})s_i(\mathbf{X}^{(j)},\mathbf{T}^*) = \tilde{g}_N$$

$\tilde{g}_N$ non biased estimator of $\mathbf{E}\left[g_N\right] \rightarrow \mathbf{E}\left[g_N\right] = \mathbf{E}\left[\tilde{g}_N\right]$

$$\mathbf{Var}\left[\tilde{g}_N\right] = \mathbf{Var}\left[\frac{1}{N}\sum_{j=1}^{N} g(\mathbf{X}^{(j)})\right]$$

$$= \int_{\mathbb{R}^n} g^2(\mathbf{X}) f_{\mathbf{X}}(\mathbf{X})\,\mathbf{dX} - \mathrm{E}\left[g(X)\right]^2$$

$$\approx \frac{1}{N}\sum_{j=1}^{N} g^2(\mathbf{X}^{(j)}) - \left(\frac{1}{N}\sum_{j=1}^{N} g(\mathbf{X}^{(j)})\right)^2 = \sigma_g^2$$

Failure probability local derivatives to hyperparameters

Confidence interval

Central limit theorem : the random variable $Z = \frac{\tilde{g}_N - \mathbf{E}\left[g(X)\right]}{\sigma_g / \sqrt{N}}$ follows a standard Gaussian random variable.

The confidence interval length associated to the risk α is :

$$CI(\alpha, N) = 2u_{1-\alpha/2} \frac{\sigma_g}{\sqrt{N}} = \frac{2u_{1-\alpha/2}}{N} \sqrt{\sum_{j=1}^{N} g^2(\mathbf{X}^{(j)}) - \frac{1}{N}\left(\sum_{j=1}^{N} g(\mathbf{X}^{(j)})\right)^2}$$

$$g(\mathbf{X}) = h(\mathbf{X}) s_i(\mathbf{X}, \mathbf{T}^*)$$

Failure probability local derivatives to hyperparameters

$$\left.\frac{\partial^2 Q(T)}{\partial T_i\,\partial T_j}\right|_{\mathbf{T}^*} = \frac{\partial^2}{\partial T_i\,\partial T_j}\cdot\int_{\mathbb{R}^n} h(\mathbf{X})\, f_{\mathbf{X}}(\mathbf{X},\mathbf{T})\,\mathbf{dX}$$

$$= \frac{\partial}{\partial T_j}\cdot\int_{\mathbb{R}^n} h(\mathbf{X})\, s_i(\mathbf{X},\mathbf{T}^*)\, f_{\mathbf{X}}(\mathbf{X},\mathbf{T})\,\mathbf{dX}$$

$$= \int_{\mathbb{R}^n} h(\mathbf{X})\, s_i(\mathbf{X},\mathbf{T}^*)\,\frac{\partial f_{\mathbf{X}}(\mathbf{X},\mathbf{T})}{\partial T_j}\,\mathbf{dX} + \int_{\mathbb{R}^n} h(\mathbf{X})\left.\frac{\partial s_i(\mathbf{X},\mathbf{T}^*)}{\partial T_j}\right|_{\mathbf{T}^*} f_{\mathbf{X}}(\mathbf{X},\mathbf{T})\,\mathbf{dX}$$

$$= \int_{\mathbb{R}^n} h(\mathbf{X})\, s_i(\mathbf{X},\mathbf{T}^*)\, s_j(\mathbf{X},\mathbf{T}^*)\,\mathbf{dX} + \int_{\mathbb{R}^n} h(\mathbf{X})\left.\frac{\partial s_i(\mathbf{X},\mathbf{T}^*)}{\partial T_j}\right|_{\mathbf{T}^*} f_{\mathbf{X}}(\mathbf{X},\mathbf{T})\,\mathbf{dX}$$

For independent variables:

$$\left.\frac{\partial^2 Q(T)}{\partial T_i\,\partial T_j}\right|_{\mathbf{T}^*} = \int_{\mathbb{R}^n} h(\mathbf{X})\, s_i(\mathbf{X},\mathbf{T}^*)\, s_j(\mathbf{X},\mathbf{T}^*)\,\mathbf{dX}$$

Failure probability local derivatives to hyperparameters - Illustration on a toy example

Function to be studied : $f(x_1) = 3x_1 + 1$ $\quad x_1 \to N(m_1;\sigma_1) \quad m_1 = 1;\sigma_1 = 2$

Exact expressions

$$m_f = 3m_1 + 1 = 4$$

$$\left.\frac{\partial m_f}{\partial m_1}\right|_{m_1=1;\sigma_1=2} = 3$$

$$\left.\frac{\partial m_f}{\partial \sigma_1}\right|_{m_1=1;\sigma_1=2} = 0$$

$$\mathbf{P}_f = \mathbf{Prob}(f(x_1(\omega)) \leq 0) = \Phi\left(-\frac{3m_1+1}{3\sigma_1}\right) = 0,25249$$

$$\left.\frac{\partial \mathbf{P}_f}{\partial m_1}\right|_{m_1=1;\sigma_1=2} = -\frac{1}{\sigma_1}\phi\left(-\frac{3m_1+1}{3\sigma_1}\right) = -0,1597$$

$$\left.\frac{\partial \mathbf{P}_f}{\partial \sigma_1}\right|_{m_1=1;\sigma_1=2} = \frac{3m_1+1}{3\sigma_1^2}\phi\left(-\frac{3m_1+1}{3\sigma_1}\right) = 0,1065$$

Computable using finite differences but:

- step size?
- need two MC simulations

Failure probability local derivatives to hyperparameters - Illustration on a toy example

$$m_f = \int_{\mathbb{R}} f(x_1) f_X(x_1) dx_1$$

$$\left.\frac{\partial m_f}{\partial m_1}\right|_{m_1=1;\sigma_1=2} = \int_{\mathbb{R}} f(x_1) s_{m_1}(x_1, m_1=1, \sigma_1=2) f_X(x_1) dx_1 \approx \frac{1}{N}\sum_{i=1}^{N} f(x_1^{(i)}) s_{m_1}(x_1^{(i)}, m_1=1, \sigma_1=2)$$

$$\left.\frac{\partial m_f}{\partial \sigma_1}\right|_{m_1=1;\sigma_1=2} = \int_{\mathbb{R}} f(x_1) s_{\sigma_1}(x_1, m_1=1, \sigma_1=2) f_X(x_1) dx_1 \approx \frac{1}{N}\sum_{i=1}^{N} f(x_1^{(i)}) s_{\sigma_1}(x_1^{(i)}, m_1=1, \sigma_1=2)$$

$$\mathbf{P}_f = \int_{\mathbb{R}} I_{Df}(x_1) f_X(x_1) dx_1$$

$$\left.\frac{\partial \mathbf{P}_f}{\partial m_1}\right|_{m_1=1;\sigma_1=2} = \int_{\mathbb{R}} I_{Df}(x_1) s_{m_1}(x_1, m_1=1, \sigma_1=2) f_X(x_1) dx_1 \approx \frac{1}{N}\sum_{i=1}^{N} I_{Df}(x_1^{(i)}) s_{m_1}(x_1^{(i)}, m_1=1, \sigma_1=2)$$

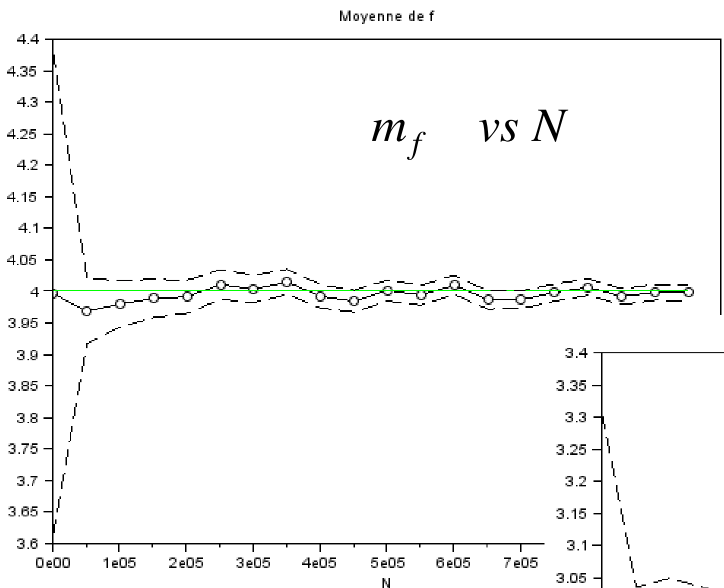
$$\left.\frac{\partial \mathbf{P}_f}{\partial \sigma_1}\right|_{m_1=1;\sigma_1=2} = \int_{\mathbb{R}} I_{Df}(x_1) s_{\sigma_1}(x_1, m_1=1, \sigma_1=2) f_X(x_1) dx_1 \approx \frac{1}{N}\sum_{i=1}^{N} I_{Df}(x_1^{(i)}) s_{\sigma_1}(x_1^{(i)}, m_1=1, \sigma_1=2)$$

Failure probability local derivatives to hyperparameters - Illustration on a toy example

$$s_{m_1}(x_1, m_1, \sigma_1) = \left.\frac{\partial \ln f_1(x_1, \mathbf{T})}{\partial m_1}\right|_{\mathbf{T}^*} = \frac{x_1 - m_1}{\sigma_1^2}$$

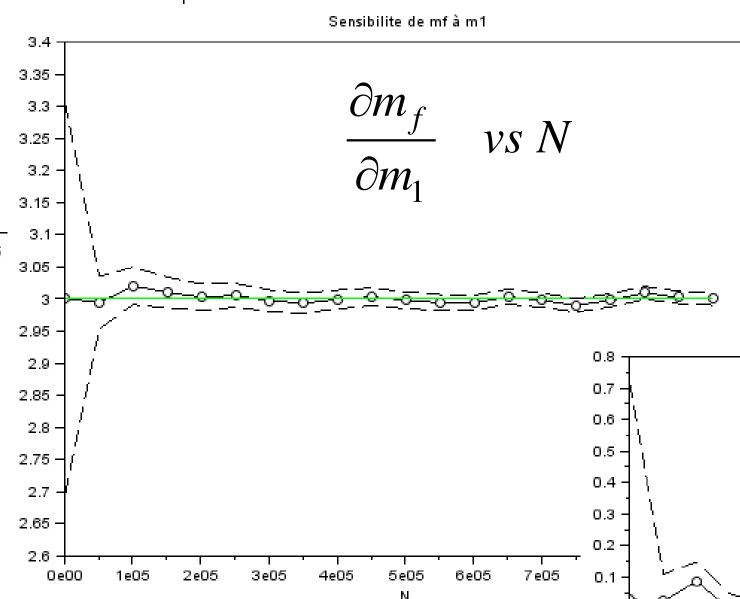
$$s_{\sigma_1}(x_1, m_1, \sigma_1) = \left.\frac{\partial \ln f_1(x_1, \mathbf{T})}{\partial \sigma_1}\right|_{\mathbf{T}^*} = -\frac{1}{\sigma_1} + \frac{\left(x_1 - m_1\right)^2}{\sigma_1^3}$$

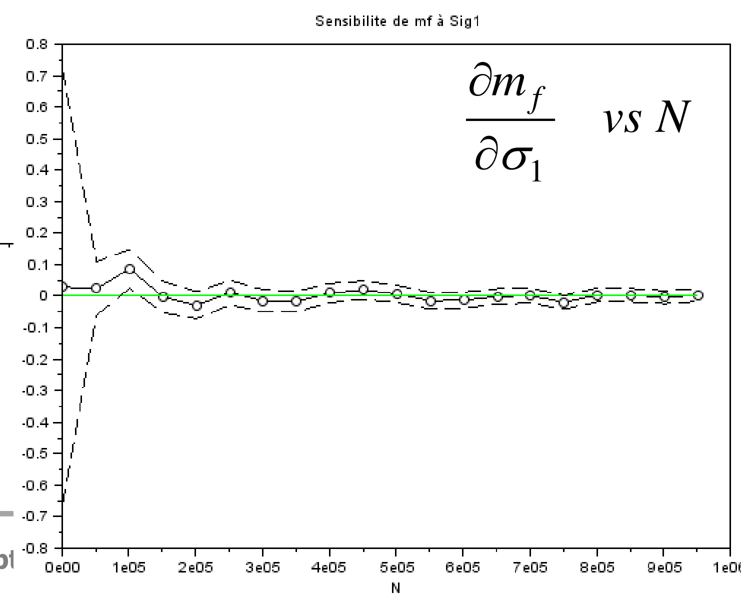
Failure probability local derivatives to hyperparameters - Illustration on a toy example



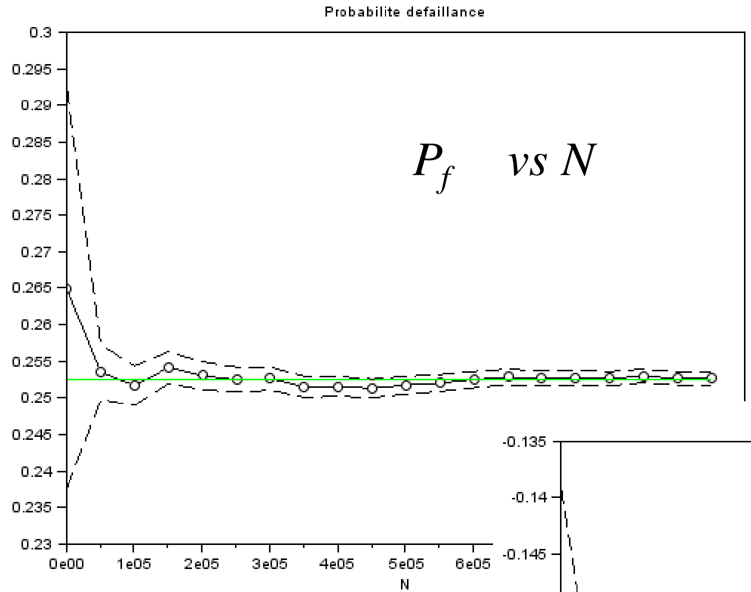
Results / mf

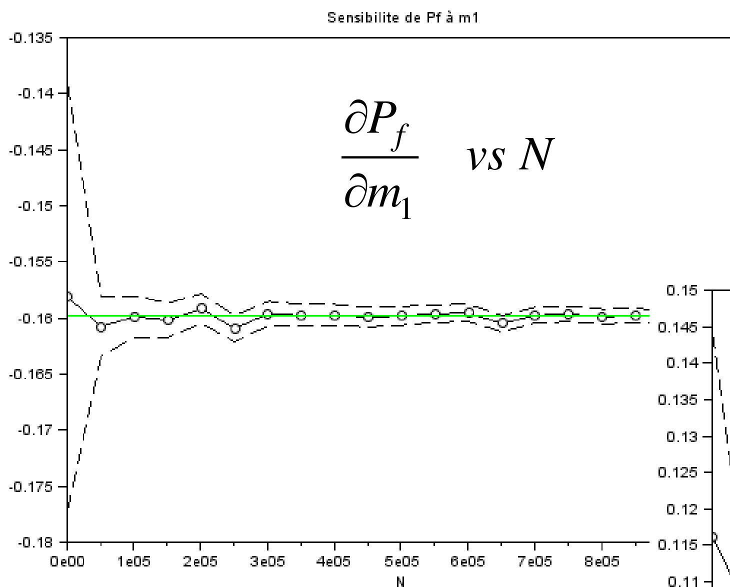
got from only one sample

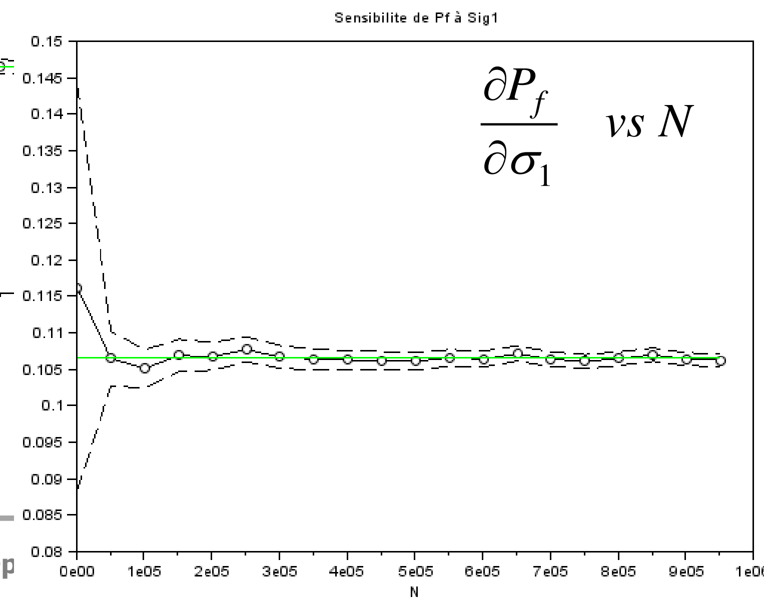




Failure probability local derivatives to hyperparameters - Illustration on a toy example







Results / Pf

got from only one sample

Application to tolerance optimization (RBDO problem)

Problem statement :

- ✓ $\mathbf{T}$ known nominal value of the design
- ✓ $g_j(\mathbf{X})\ j=1,\ldots,m$ set of functional requirements, must be >0 to be admissible
- ✓ $\mathbf{t}$ set of tolerances

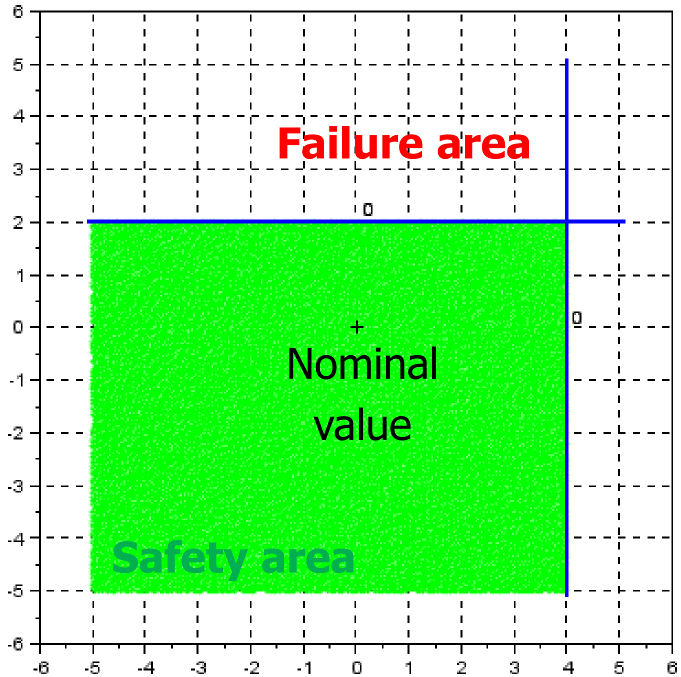
$\rightarrow$ Maximize the tolerance volume to facilitate the production with respect to a target reliability level.

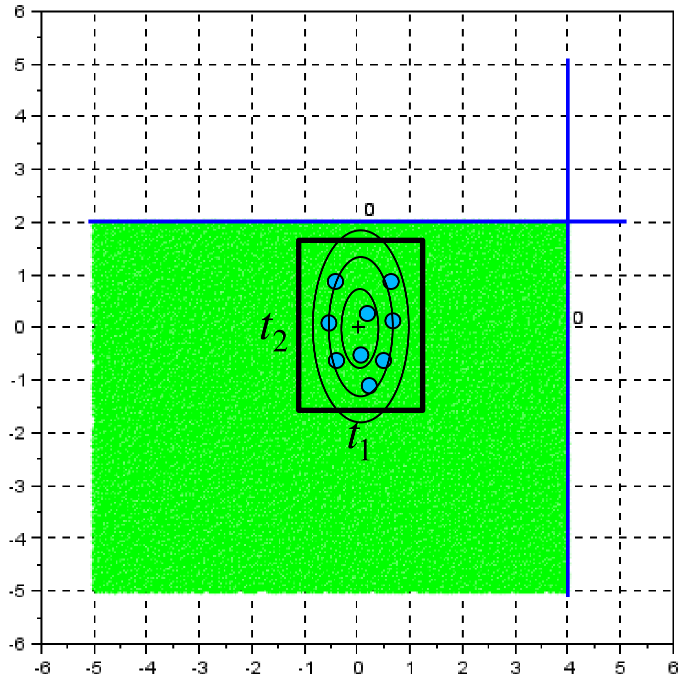
$$\mathbf{t}^* = \mathbf{Argmax} \prod_{i=1}^{n} t_i^{p_i} \qquad \mathbf{under} \quad P_f(\mathbf{t}) \le P_f^c$$

Weighted tolerance volume

Target failure probability

Application to tolerance optimization





Strong assumptions: geometrical variables follow gaussian lows and $m_i = T_i$ $\sigma_i = \frac{t_i}{6}$

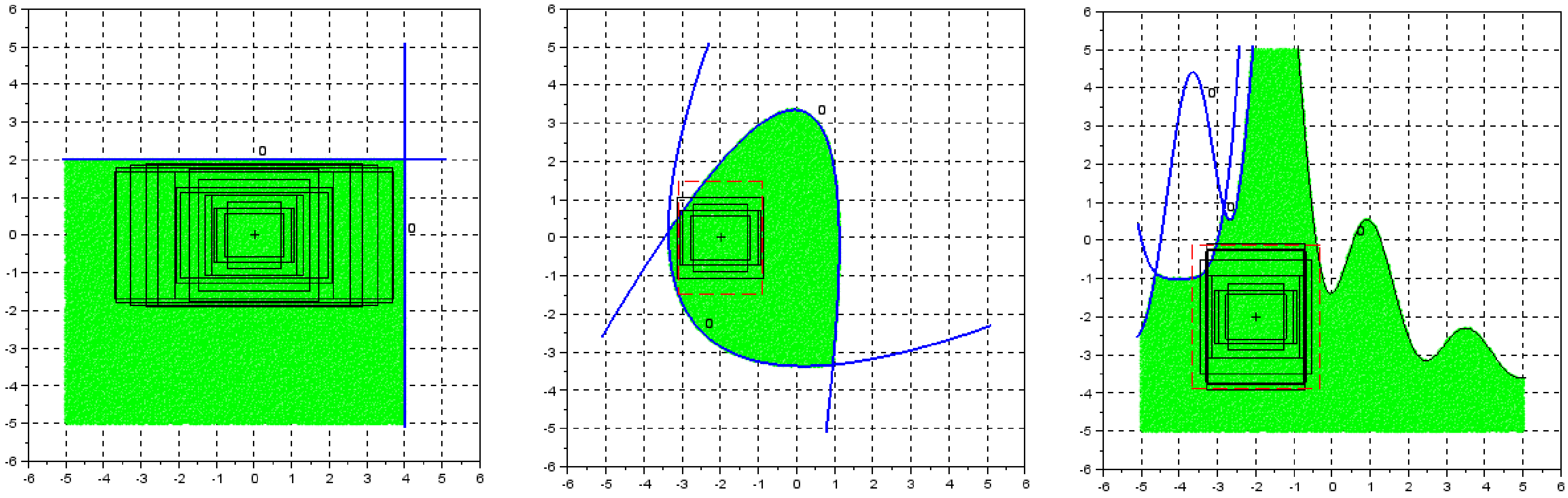
$$\sigma^* = \textbf{Argmax} \prod_{i=1}^{n} \sigma_i^{p_i} \qquad \textbf{under} \quad P_f(\sigma_i) \le P_f^c$$

and $t_i^* = 6\sigma_i^*$

Application to tolerance optimization – stochastic algorithm

Resolution using a meta-heuristic algorithm

Random growth of the tolerance until the constraint cannot be reached.



Application to tolerance optimization – stochastic algorithm

Resolution based on optimality conditions

$$\sigma^* = \textbf{Argmax} \quad f(\sigma_i) = \prod_{i=1}^{n} \sigma_i^{p_i} \qquad \textbf{under} \quad g(\sigma_i) = P_f(\sigma_i) - P_f^c \le 0$$

$$\begin{cases} g(\sigma_i^*) = P_f(\sigma_i^*) - P_f^c = 0 \\ \nabla f(\sigma_i^*) + \lambda \nabla P_f(\sigma_i^*) = 0 \end{cases} \Leftrightarrow \quad F(\mathbf{X}) = 0$$

Newton iterative resolution

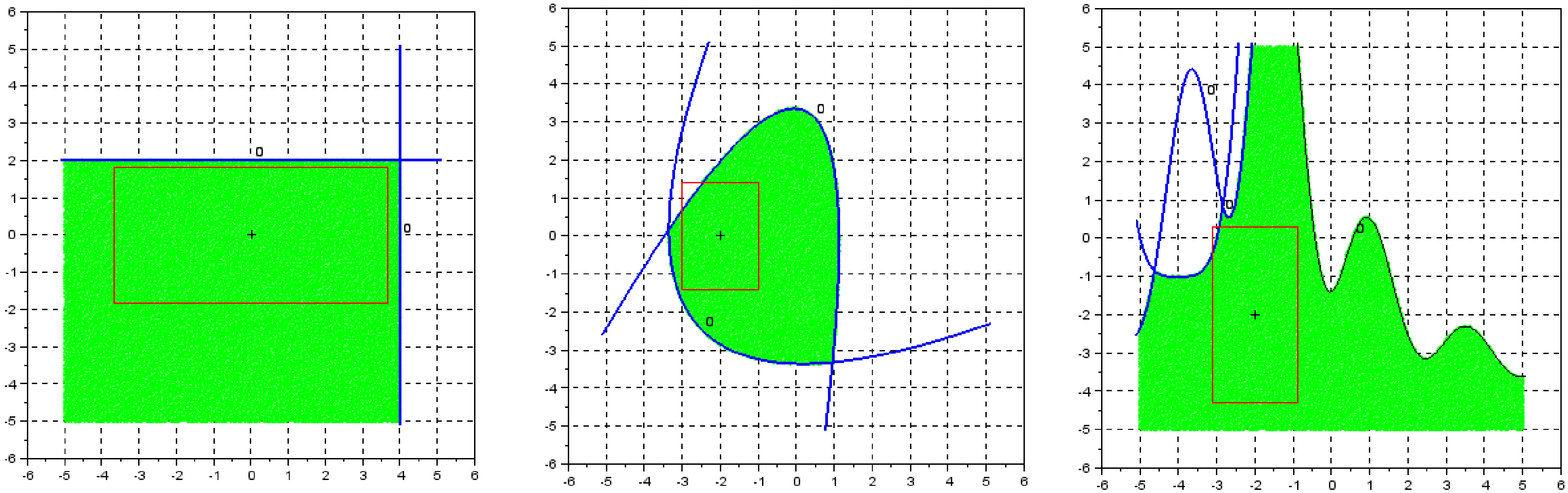
$$\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} - \left[J_F(\mathbf{X}^{(k)}) \right]^{-1} F(\mathbf{X}^{(k)})$$

Jacobian containing $\dfrac{\partial P_f}{\partial \sigma_i \partial \sigma_j} = \int_{Df} I_{Df}(\mathbf{X}) s_i(\mathbf{X}) s_j(\mathbf{X})\, \mathbf{dX}$

Free estimation using MC

Application to tolerance optimization – stochastic algorithm

Resolution based on optimality conditions



Time to show scilab subroutines?

4. Reliability index and failure probability local sensitivity to parameters

4.1 Local sensitivity to hyper-parameters

4.2 Local sensitivity to model parameters

Local sensitivity to model parameters

[Lacaze, 2015]

Failure probability as a function of a deterministic model parameter:

(**MORE COMPLEX PROBLEM, no score function**)

$$P_f(z) = \int_{\Omega_f(z)} f_{\mathbf{X}}(\mathbf{X}) \mathbf{d}\mathbf{X}$$

$$\Omega_f(z) : \left\{ X \in \mathbb{R}^n \, / \, \forall j = 1, \ldots, m \quad g_j(\mathbf{X}, z) \leq 0 \right\}$$

(j=1 in the following)

Deterministic model parameter

$$\left. \frac{\mathbf{d}P_f(z)}{\mathbf{d}z} \right|_{z^{(0)}}$$?

$$P_f(z) = \int_{\mathbb{R}^n} \mathbf{I}_{g(\mathbf{X},z) \leq 0} \, f_{\mathbf{X}}(\mathbf{X}) \mathbf{d}\mathbf{X}$$

$$\left. \frac{\mathbf{d}P_f(z)}{\mathbf{d}z} \right|_{z^{(0)}} = \int_{\mathbb{R}^n} \left. \frac{\mathbf{d}(\mathbf{I}_{g(\mathbf{X},z) \leq 0})}{\mathbf{d}z} \right|_{z^{(0)}} f_{\mathbf{X}}(\mathbf{X}) \mathbf{d}\mathbf{X}$$

$$\frac{\mathbf{d}(\mathbf{I}_{y \geq 0})}{\mathbf{d}y} = -\frac{\mathbf{d}(\mathbf{I}_{y \leq 0})}{\mathbf{d}y} = \delta_y = \begin{cases} +\infty & \textbf{if} \quad y = 0 \\ 0 & \textbf{otherwise} \end{cases}$$

Local sensitivity to model parameters

$$\left.\frac{\mathbf{d}P_f(z)}{\mathbf{d}z}\right|_{z^{(0)}} = -\int_{\mathbb{R}^n} \left.\frac{\mathbf{d}g(\mathbf{X},z)}{\mathbf{d}z}\right|_{z^{(0)}} \delta_{g(\mathbf{X},z^{(0)})}\, f_{\mathbf{X}}(\mathbf{X})\mathbf{d}\mathbf{X}$$

Monte Carlo estimation:

$$\left.\frac{\mathbf{d}P_f(z)}{\mathbf{d}z}\right|_{z^{(0)}} = -\frac{1}{N}\sum_{i=1}^{N} \left.\frac{\mathbf{d}g(\mathbf{X}^{(i)},z)}{\mathbf{d}z}\right|_{z^{(0)}} \delta_{g(\mathbf{X}^{(i)},z^{(0)})}$$

First problem, how to estimate ?

Second problem, a Dirac will numerically always be null ?

Dirac Gaussian approximation: $$\delta_y(\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma^2}\right)$$

Width parameter to be optimized: σ

Local sensitivity to model parameters

Sensitivity to threshold value:

$$P_f(z) = \int_{\Omega_f(z)} f_{\mathbf{X}}(\mathbf{X})\mathbf{dX}$$

$$\Omega_f(z) : \left\{ X \in \mathbb{R}^n \quad g_j(\mathbf{X}) \leq z \right\}$$

$$\left. \frac{\mathbf{d}P_f(z)}{\mathbf{d}z} \right|_{z=0} = \frac{1}{N} \sum_{i=1}^{N} \delta_{g(\mathbf{X}^{(i)})}$$

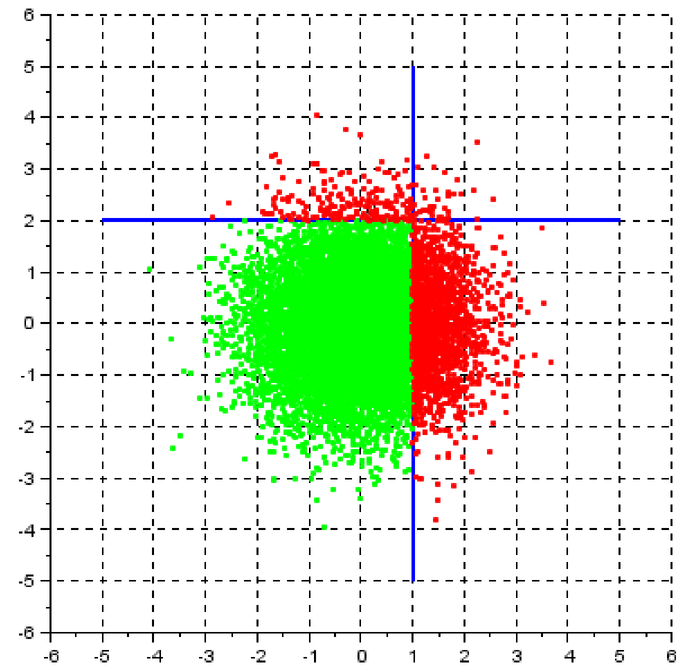
Trivial illustration, two performance functions in a standard Gaussian 2D space:

$$g_1(x_1, x_2) = -x_1 + 1$$

$$\rightarrow \quad \mathbf{Prob}(g_1(x_1, x_2) \leq 0) = \Phi(-1) = 0{,}1586$$

$$g_2(x_1, x_2) = -\frac{x_2}{2} + 1$$

$$\rightarrow \quad \mathbf{Prob}(g_2(x_1, x_2) \leq 0) = \Phi(-2) = 0{,}0227$$



Local sensitivity to model parameters

Reference results:

$$P_f^{(1)}(z_1) = \mathbf{Prob}(g(x_1,x_2) - z_1 \le 0))$$

$$\left.\frac{\partial P_f^{(1)}}{\partial z_1}\right|_{z_1} = \phi(-1+z_1)$$

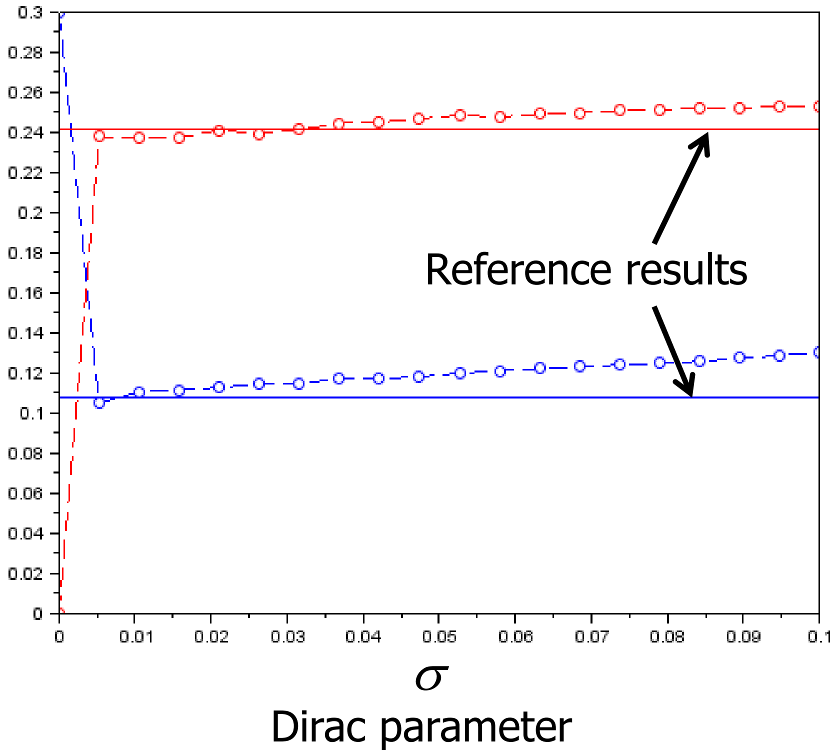
$$\left.\frac{\partial P_f^{(1)}}{\partial z_1}\right|_{z_1=0} = \phi(-1) = 0,24197$$

$$P_f^{(2)}(z_2) = \mathbf{Prob}(g(x_1,x_2) - z_2 \le 0))$$

$$\left.\frac{\partial P_f^{(2)}}{\partial z_2}\right|_{z_2} = 2\phi(-2+2z_2)$$

$$\left.\frac{\partial P_f^{(2)}}{\partial z_2}\right|_{z_2=0} = 2\phi(-2) = 0,10798$$

Sensitivity estimation using Monte Carlo and comparison with reference values



5. Optimization under uncertainties – main methods

Robust design

Numerical effort ++ (→ ROBUSTNESS)

Numerical effort ++ (↓ RELIABILITY)

RELIABILITY \ ROBUSTNESS	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	Optimal design	Robust design
Constraint function with X, P deterministic	Admissible design	Optimal and admissible design	Robust and admissible design
Constraint function with X, P uncertain	Reliable design	Optimal and reliable design	Robust and reliable design

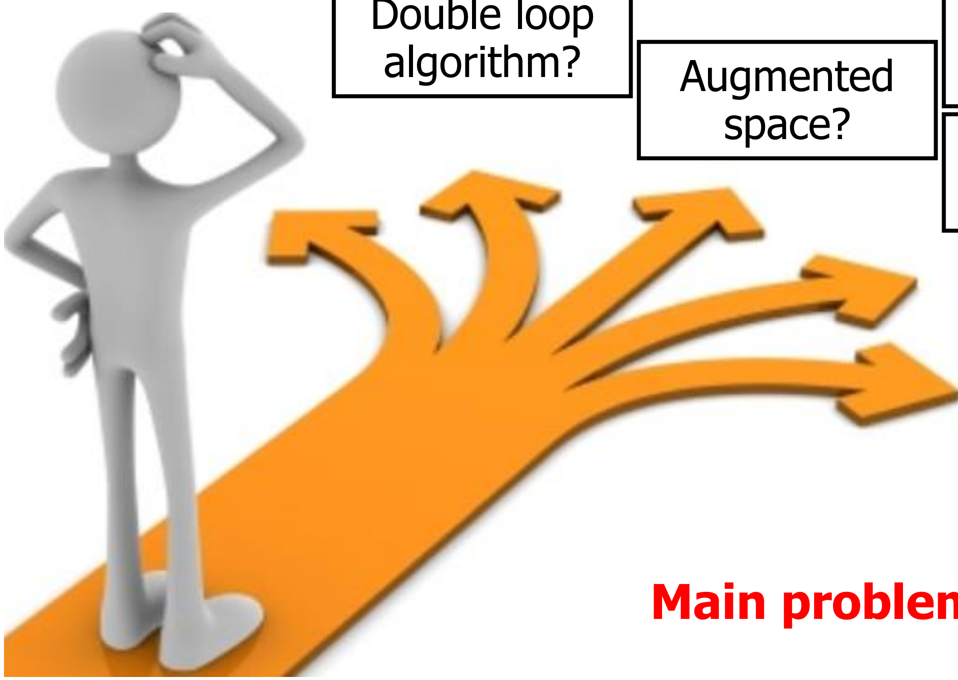
Numerical effort ++++

Focus on RBDO : a hard problem to solve ...

$$\textbf{Find} \quad \tilde{\mathbf{T}}_{OptRel} = \underset{\mathbf{T}}{\textbf{Argmin}}\ f\left(\mathbf{T}\right)$$

$$\textbf{under} \quad \textbf{Prob}\left(g\left(\mathbf{X}\left(\mathbf{T},\omega\right)\right) \leq 0\right) \leq \Pr_{Target}$$

$$\Leftrightarrow P_f\left(\mathbf{T}\right) \leq \Pr_{Target}$$



Double loop algorithm?

Augmented space?

Single loop algorithm?

Hybrid space?

Meta-model?

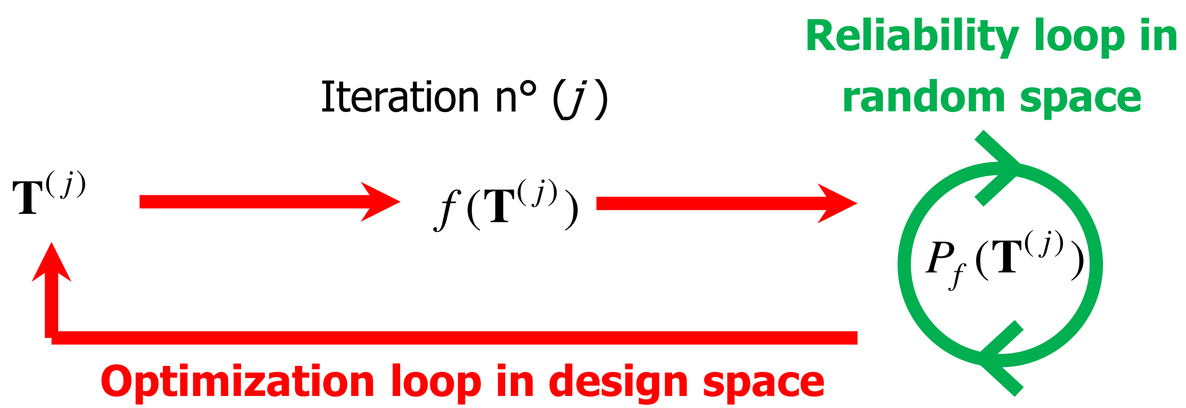
Sequential formulation?

Main problem : computation time!!

A time consuming dooble loop problem

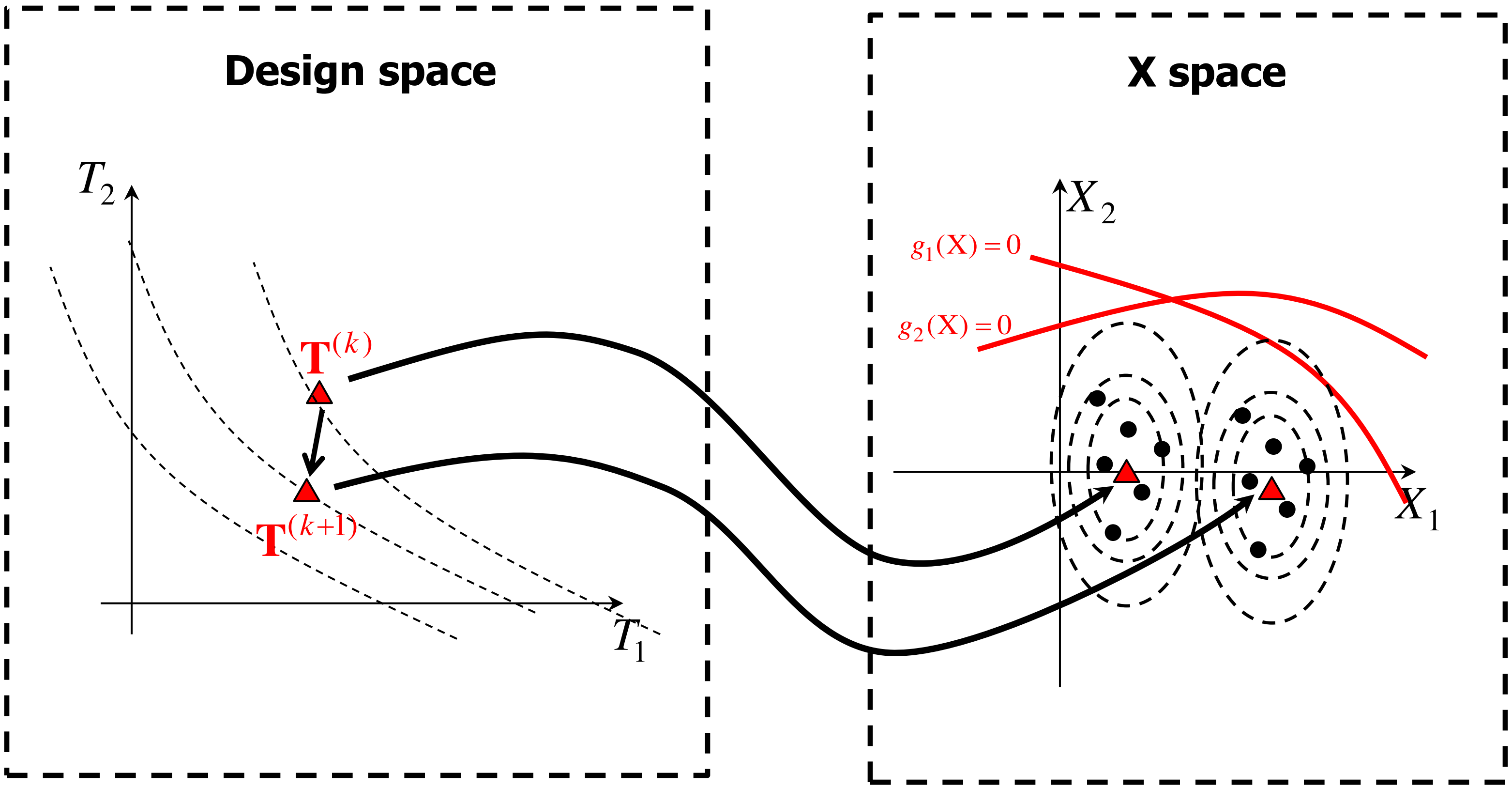
First order optimality conditions:

$$\begin{bmatrix} \nabla^2 f(\mathbf{T}^{(k)}) & \nabla P_f(\mathbf{T}^{(k)}) \\ \nabla P_f(\mathbf{T}^{(k)})^t & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{T}^{(k+1)} - \mathbf{T}^{(k)} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -\nabla f(\mathbf{T}^{(k)}) \\ \mathrm{Pr}_{\mathbf{target}} - P_f(\mathbf{T}^{(k)}) \end{Bmatrix}$$



DOUBLE LOOP VERY TIME CONSUMING

A time consuming dooble loop problem



Problem of RBDO

Aim : propose economical methods for resolution

- ✓ Double level methods based on RIA (Reliability Index Analysis) or PMA (Performance Measure Approach)
- ✓ Single level methods
- ✓ Sequential methods
- ✓ Meta-model - based approaches

5. Optimization under uncertainties – main methods

5.1 Double level methods based on RIA (Reliability Index Analysis) or PMA (Performance Measure Approach)

5.2 Single level methods

5.3 Sequential methods

5.4 Meta-model - based approaches

Double level method based on RIA

$$\text{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\text{Argmin}}\ f(\mathbf{T}) \quad \text{under} \quad P_f(\mathbf{T}) \leq \Pr_{Target}$$

↓ Reliability Index Approach

$$\text{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\text{Argmin}}\ f(\mathbf{T}) \quad \text{under} \quad \beta(\mathbf{T}) \geq \beta_{Target}$$

$$\beta_{Target} = -\Phi^{-1}(\Pr_{Target})$$

Reliability index easier to compute than failure probability but need the double loop and the sensitivity estimation at each step.

Double level method based on PMA

[Tu et al. 1999]

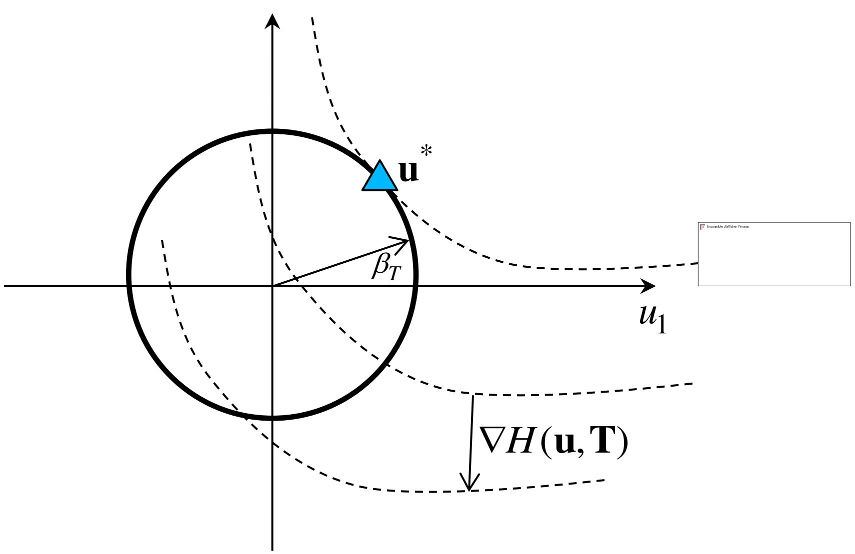
$$\textbf{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\textbf{Argmin}}\, f(\mathbf{T})$$

$$\textbf{under} \quad \beta(\mathbf{T}) \geq \beta_{Target}$$

Verify $\beta(\mathbf{T}) \geq \beta_{Target}$ $\longleftrightarrow$ $g(\mathbf{T}) = H(\mathbf{u}^*, \mathbf{T}) \geq 0$

$$\textbf{with} \quad \mathbf{u}^* = \underset{\mathbf{u}}{\textbf{Argmin}}\, H(\mathbf{u}, \mathbf{T})$$

$$\textbf{under} \quad \|\mathbf{u}\| = \beta_{Target}$$



Can be easier to compute than the RIA but still a double loop.

5. Optimization under uncertainties – main methods

Single loop algorithm

Aim : increase the CV speed avoiding a second loop / level

→ Replacing the reliability assessment by a deterministic constraint

Single loop algorithm – KKT optimality conditions based approach

[Madsen et. al, 1992]

KKT optimality condition based approach

FORM Lagrangian $L(\mathbf{u}^*,\lambda) = \|\mathbf{u}^*\| + \lambda H(\mathbf{u}^*)$

Optimality conditions:

$$\begin{cases} H(\mathbf{u}^*) = 0 \\ \nabla_u(\|\mathbf{u}^*\|) + \lambda \nabla_u H(\mathbf{u}^*) = 0 \end{cases} \rightarrow \quad \frac{\mathbf{u}^*}{\|\mathbf{u}^*\|} + \lambda \nabla_u H(\mathbf{u}^*) = 0$$

$$\rightarrow \quad \mathbf{u}^* = -\lambda \|\mathbf{u}^*\| \nabla_u H(\mathbf{u}^*)$$

$$\rightarrow \quad \|\mathbf{u}^*\| = |\lambda| \|\mathbf{u}^*\| \|\nabla_u H(\mathbf{u}^*)\|$$

$$\rightarrow \quad \lambda = +\frac{1}{\|\nabla_u H(\mathbf{u}^*)\|}$$

Single loop algorithm – KKT optimality conditions based approach

[Madsen et. al, 1992]

$$\begin{cases} H(\mathbf{u}^*) = 0 \\ \mathbf{u}^* = -\nabla_u H(\mathbf{u}^*) \dfrac{\|\mathbf{u}^*\|}{\|\nabla_u H(\mathbf{u}^*)\|} \end{cases} \rightarrow \quad \|\nabla_u H(\mathbf{u}^*)\| \mathbf{u}^* + \nabla_u H(\mathbf{u}^*) \|\mathbf{u}^*\| = 0$$

$$\rightarrow \quad \|\nabla_u H(\mathbf{u}^*)\| \mathbf{u}^{*t}.\mathbf{u}^* + \mathbf{u}^{*t}.\nabla_u H(\mathbf{u}^*) \|\mathbf{u}^*\| = 0$$

$$\rightarrow \quad \|\nabla_u H(\mathbf{u}^*)\| + \mathbf{u}^{*t}.\nabla_u H(\mathbf{u}^*) = 0$$

Find $\mathbf{T}^* = \underset{\mathbf{T},\mathbf{u}^*}{\mathbf{Argmin}}\, f(\mathbf{T})$

under $\Phi(-\|\mathbf{u}^*\|) \leq P_f^t$

$H(\mathbf{u}^*) = 0$

$\|\nabla_u H(\mathbf{u}^*)\| + \mathbf{u}^{*t}.\nabla H(\mathbf{u}^*) = 0$

$\mathbf{u}^*$: contain $\mathbf{T}$ / isoprobabilistic transformation

$$\mathbf{u} = \Im(\mathbf{X}, \mathbf{T})$$

Simultaneous search of $\mathbf{T}$ and of u*

Optimality condition
=> Hessian of H.

Single loop algorithm – Approximate Moment Approach (AMA)

$$\textbf{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\textbf{Argmin}}\ f(\mathbf{T})$$
$$\textbf{under} \quad P_f(\mathbf{T}) \leq \Pr_{Target}$$

$$P_f(\mathbf{T}) \approx \Phi\left(-\frac{m_g(\mathbf{T})}{\sigma_g(\mathbf{T})}\right)$$

Strong assumption

$$\rightarrow \quad \textbf{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\textbf{Argmin}}\ f(\mathbf{T})$$
$$\textbf{under} \quad \Phi\left(-\frac{m_g(\mathbf{T})}{\sigma_g(\mathbf{T})}\right) \leq \Pr_{Target}$$

$$m_g(\mathbf{T}) \approx g(\mathbf{T})$$

$$\sigma_g^2(\mathbf{T}) \approx \sum_{j=1}^{n} \left.\frac{\partial g}{\partial X_j}\right|_{\mathbf{T}}^2 \sigma_j^2$$

Single loop algorithm – Approximate Moment Approach (AMA)

$$\text{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\text{Argmin}}\, f(\mathbf{T})$$
$$\text{under} \quad \Phi\left(-\frac{m_g(\mathbf{T})}{\sigma_g(\mathbf{T})}\right) \leq \Pr_{Target}$$

$$\rightarrow \quad \text{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\text{Argmin}}\, f(\mathbf{T})$$
$$m_g(\mathbf{T}) + \Phi^{-1}(\Pr_{Target})\sigma_g \geq 0$$

"Safety margin"

$$\rightarrow \quad \text{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\text{Argmin}}\, f(\mathbf{T})$$
$$g(\mathbf{T}) + \Phi^{-1}(\Pr_{Target})\sqrt{\sum_{j=1}^{n} \left.\frac{\partial g}{\partial X_j}\right|_{\mathbf{T}}^2 \sigma_j^2} \geq 0$$

5. Optimization under uncertainties – main methods

Sequential approaches

Aim of sequential approaches: find the RBDO solution solving a succession of deterministic optimization problems.

Sequential approaches – Safety Factor Approach (SFA)

[Wu Y.T., et al, 1998]

$$\textbf{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\textbf{Argmin}}\, f(\mathbf{T})$$

$$\textbf{under} \quad \textbf{Prob}\left(g(\mathbf{X}(\mathbf{T},\omega)) \le 0\right) \le \Pr_{Target}$$

$$g(\mathbf{X}^*(\mathbf{T})) > 0$$

Choice of $\mathbf{X}^*(\mathbf{T})$?

Iterative scheme:

(i) For a given value of $\mathbf{T}^{(k)}$, find the deterministic margin $s^{(k)}$ such that:

$$\textbf{Prob}\left(g(\mathbf{X}(\mathbf{T}^{(k)},\omega)) + s^{(k)} \le 0\right) = \Pr_{Target}$$

(ii) Compute the coordinate of the MPFP P* in the physical space: $\mathbf{X}^*(\mathbf{T}^{(k)})$

and use it in the optimization problem, $s^{(k)}$ is the deterministic margin i.e. a **Global Safety factor** needing a costly iterative setup since $\mathbf{X}^*$ is a function of $\mathbf{T}^{(k)}$.

(iii) The solution is $\mathbf{T}^{(k+1)}$, go back to (i) if convergence is not reached

Sequential approaches – SAP (Sequential approximate programming)

[Cheng et al, 2006]

$$\textbf{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\textbf{Argmin}}\, f(\mathbf{T})$$
$$\textbf{under} \quad \beta(\mathbf{T}) \geq \beta_{Target}$$

For a given value of $\mathbf{T}^{(k)}$, solve the following problem:

$$\textbf{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\textbf{Argmin}}\, f(\mathbf{T})$$
$$\textbf{under} \quad \beta^{(k)}(\mathbf{T}) \geq \beta_{Target}$$

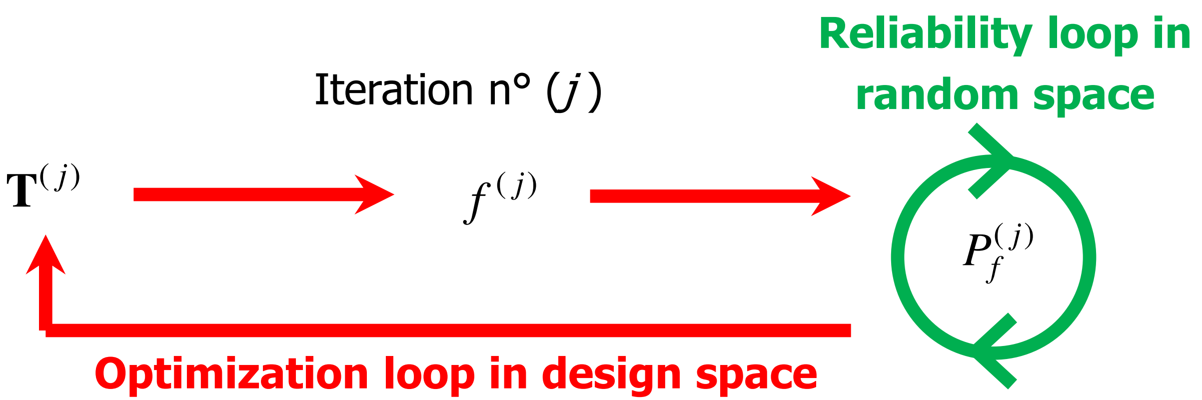
With: $$\beta^{(k)}(\mathbf{T}) \approx \beta(\mathbf{T}^{(k-1)}) + \nabla_T \beta(\mathbf{T}^{(k-1)})(\mathbf{T}^{(k)} - \mathbf{T}^{(k-1)})$$

Evaluate by finite differences

Iterative scheme still convergence…

A kind of meta-model on reliability index!!

Sequential approaches – calibration of safety factors – idea !!



DOUBLE LOOP VERY TIME CONSUMING

Idea : the aim of partial safety factors calibration is to provide a determinitic rule that is calibrated for a given target reliability level (reliability index)

Giving engineering values $\mathbf{T}$, the rule can be written : $g(\mathbf{T},\gamma) > 0$

Sequential approaches – Safety Factor Approach (SFA) – idea !!

Two separate problems

Partial safety factors calibration preliminary loop

↓

Partial safety factors for a target reliability level $\boldsymbol{\gamma}^*$ ←

A sub-optimization problem

$$\boldsymbol{\gamma}^* = \mathbf{Argmin}\, W(\gamma) = \sum_{j=1}^{L} \left(\beta_j(\gamma) - \beta_t\right)^2$$

L specific design situations

Not so easy to solve but can be solved using the MPFP

↓

$$\textbf{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\mathbf{Argmin}}\, f(\mathbf{T})$$

$$\textbf{under} \quad g(\mathbf{T}, \boldsymbol{\gamma}^*) \geq 0$$

A kind of deterministic meta-model!!

5. Optimization under uncertainties – main methods

5.1 Double level methods based on RIA (Reliability Index Analysis) or PMA (Performance Measure Approach)

5.2 Single level methods

5.3 Sequential methods

5.4 Meta-model - based approaches

Preliminary remark

Objective function needs **APPROXIMATION**

Find $x^* \in \mathbb{R}^n$ such that : $x^* = \mathbf{Argmin}\ f(x)$ $f : \mathbb{R}^n \to \mathbb{R}$

under $g_j(x) \leq 0$ $g_j : \mathbb{R}^n \to \mathbb{R}$

$j = 1, \ldots, m$

Constraint function can only need **CLASSIFICATION**

Metamodel-based approach

$$\textbf{Find} \quad \mathbf{T}^* = \underset{\mathbf{T}}{\textbf{Argmin}}\ f(\mathbf{T})$$

$$\textbf{under} \quad \textbf{Prob}\left(g(\mathbf{X}(\mathbf{T},\omega)) \leq 0\right) \leq \Pr_{Target}$$

May be time consuming in industrial applications

→ Use meta-models to replace the time consuming constraint by an approximation with no computation time.

→ Two possibilities:

$$g(\mathbf{X}(\mathbf{T},\omega)) \longrightarrow \tilde{g}(\mathbf{X}(\mathbf{T},\omega))$$

Meta-model on the performance function

$$\textbf{Prob}\left(g(\mathbf{X}(\mathbf{T},\omega)) \leq 0\right) - \Pr_{Target} \longrightarrow \tilde{P}_f(\mathbf{T})$$

Meta-model on the constraint function

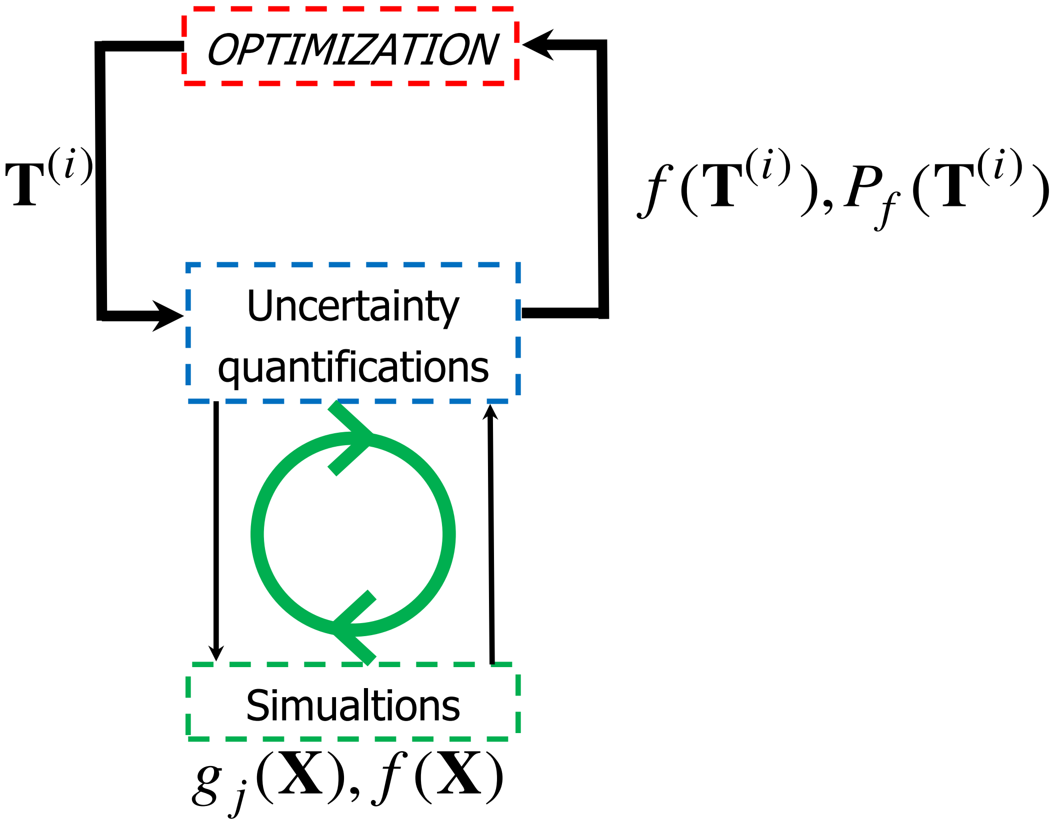
Metamodel-based approach

[Eldred, 2002]

Four (Three) meta-model - based approaches:

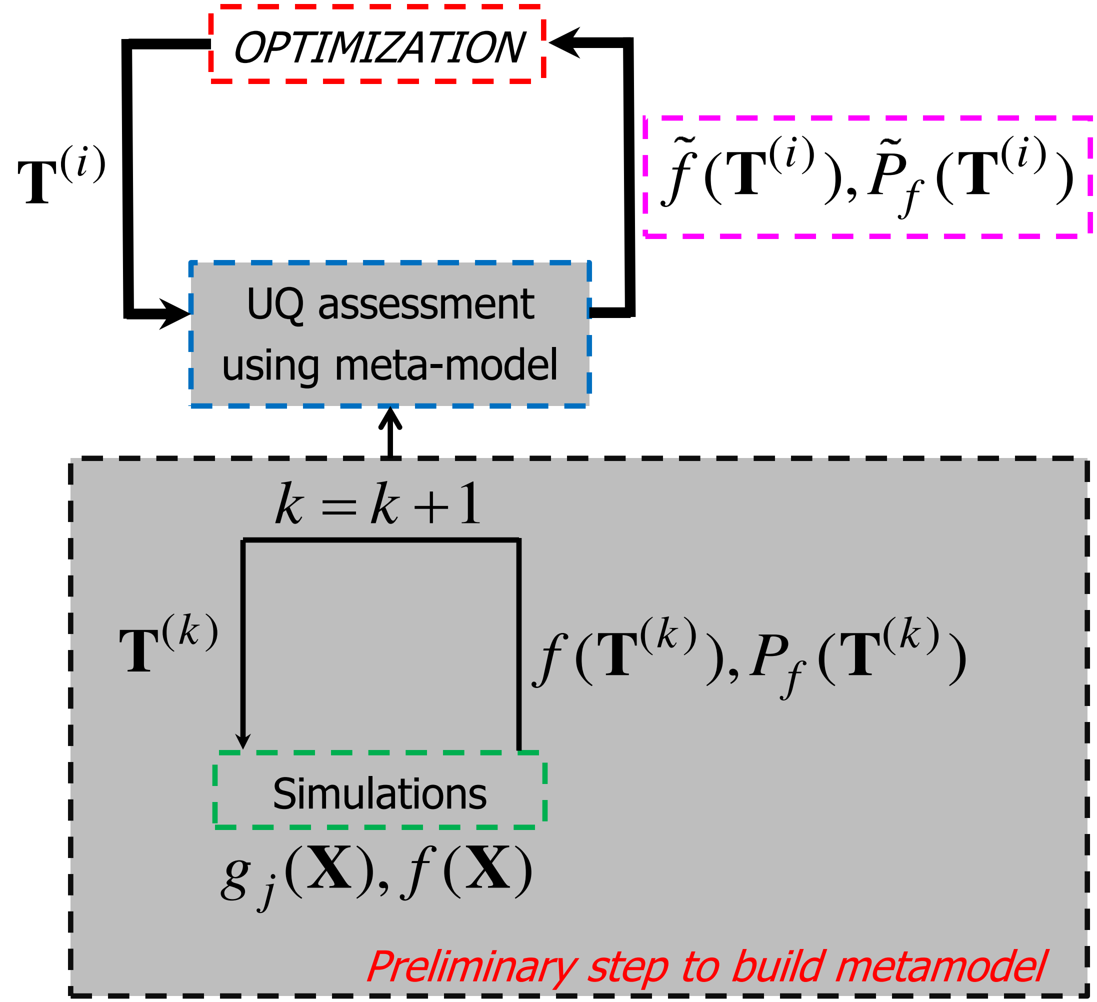
✓ **Formulation #1 (nested)**

→ Double loop approach



Metamodel-based approach

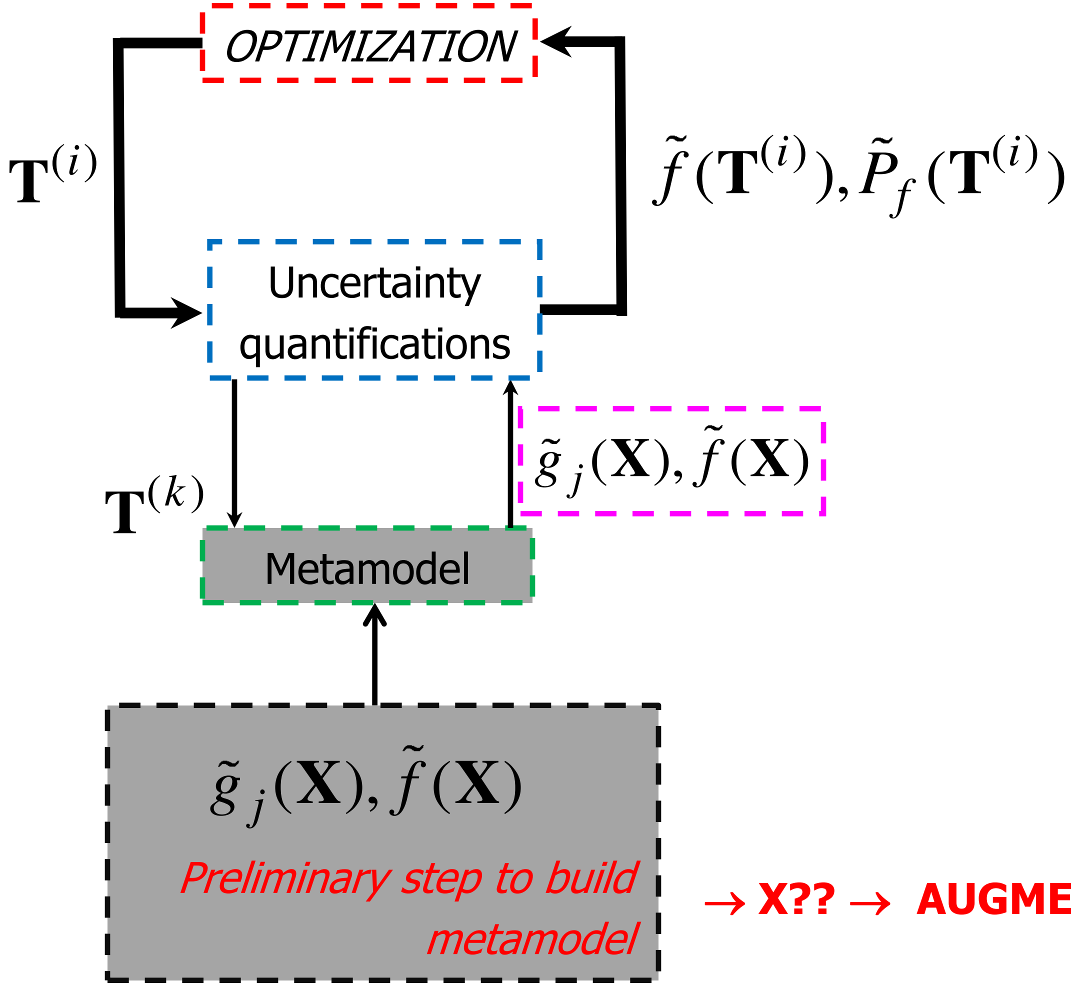
✓ **Formulation #2 (Layered / Nested)**



Metamodel-based approach

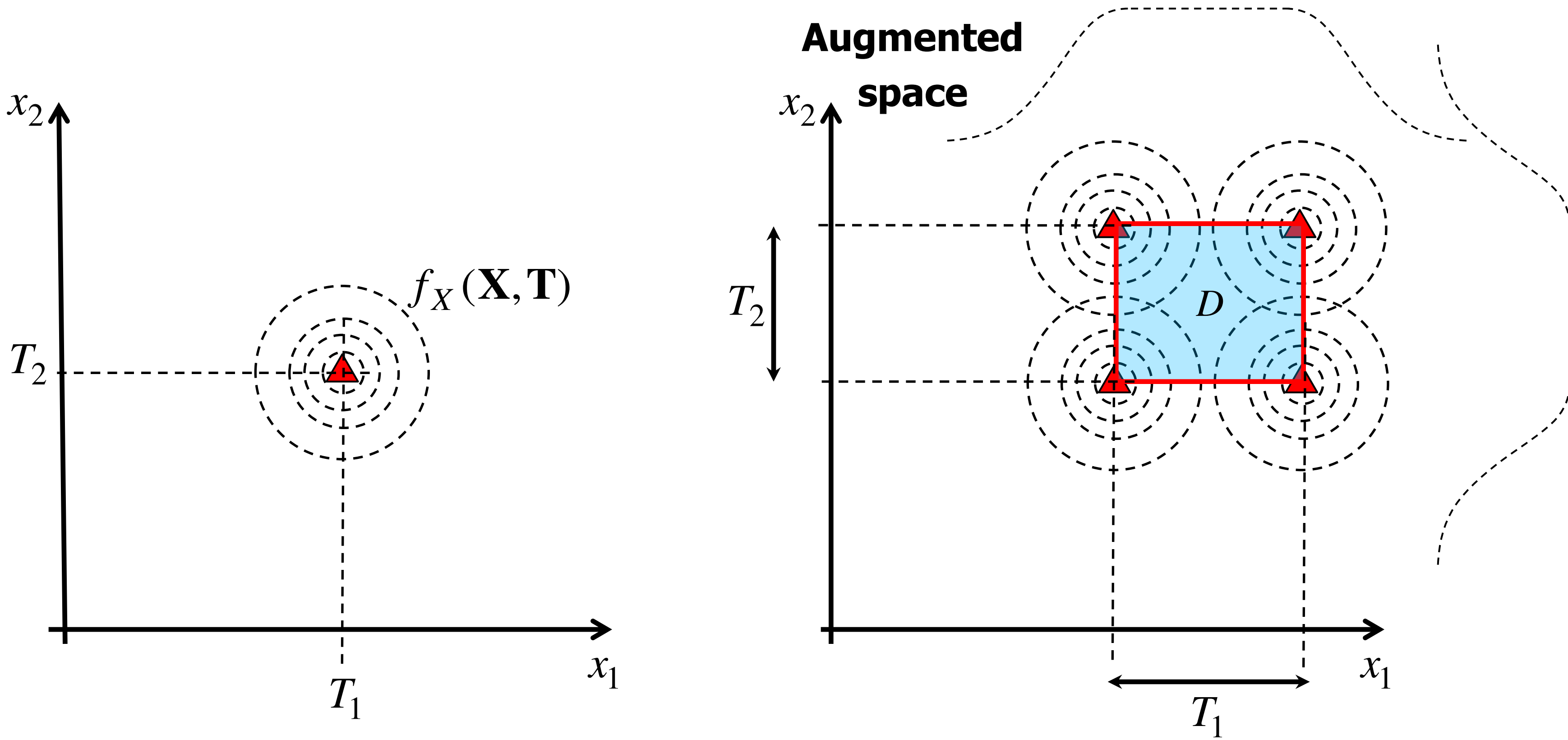
[Eldred, 2002]

✓ **Formulation #3 (Nested / Layered)**



→ X?? → AUGMENTED SPACE

Augmented space



Augmented pdf:

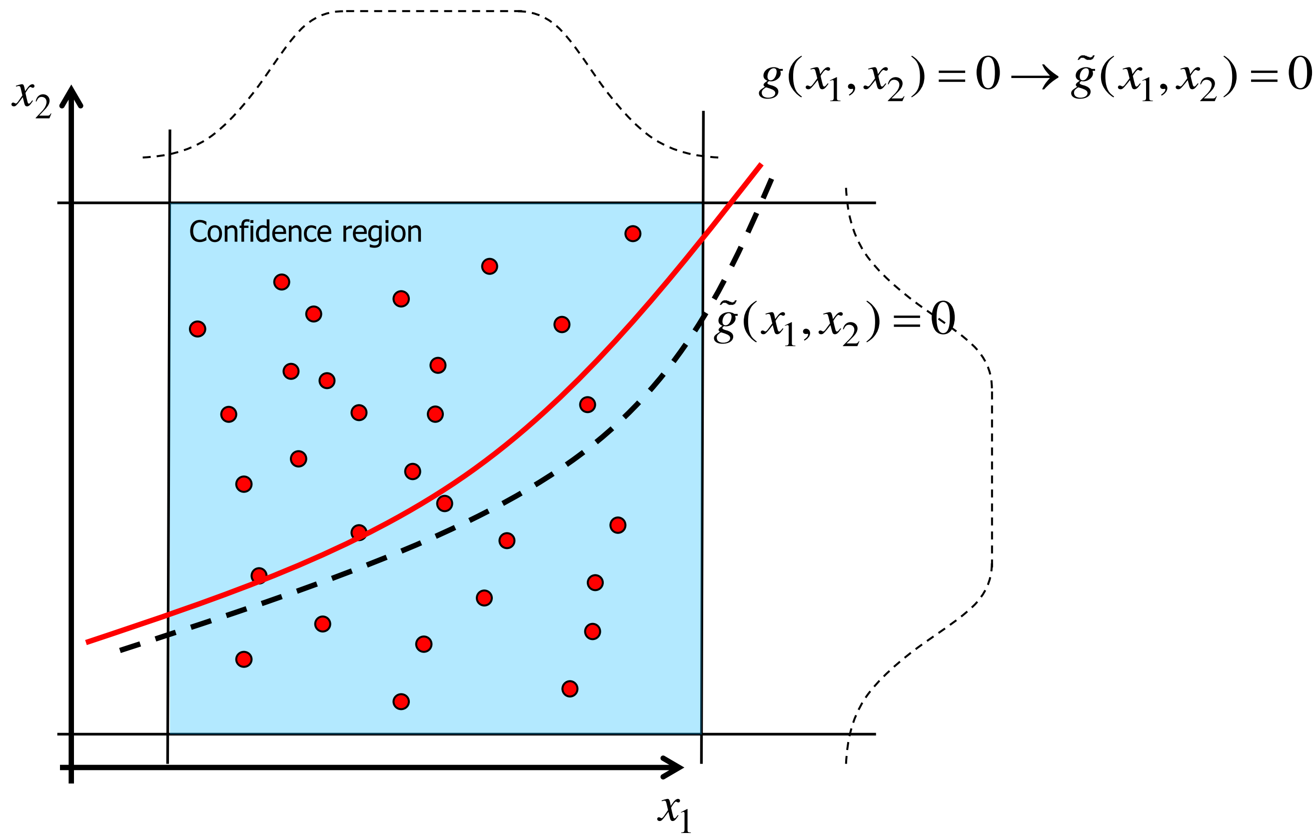
$$h_{\mathbf{X}}(\mathbf{X}) = \int_D f_{\mathbf{X}}(\mathbf{X},\mathbf{T})\pi(\mathbf{T})\mathbf{dT}$$

Instrumental density

Metamodel-based approach – iterative scheme

[Dubourg, 2011]

Augmented space

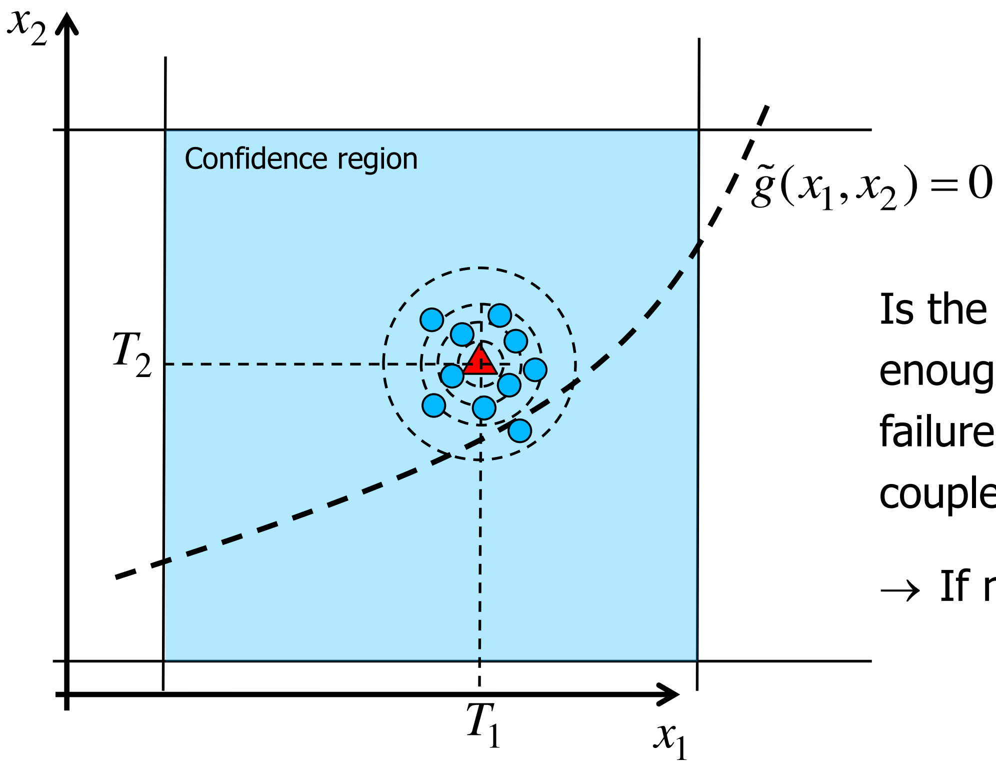


Confidence region: $R_{h_0} = \left\{ X \in \mathbb{R}^n \mid h(x) > h_0 \right\}$

Metamodel-based approach – iterative scheme

[Dubourg, 2011]

Augmented space



Is the surrogate accurate enough to compute the failure probability for the couple (T_1, T_2) ?

→ If no : local refinement

6. Conclusions

Come back to initial objectives

Optimization under uncertainties

- ✓ **Objective #1** – What are the industrial issues?
- ✓ **Objective #2** – What are the differences between optimization with and without uncertainty?
- ✓ **Objective #3** – What are the different kind of problems and mathematical formulations?
- ✓ **Objective #4** – What are the main difficulties and tools needed to optimize a system under uncertainties?
- ✓ **Objective #5** – What are the recent methods with advantages and drawbacks?

To keep in mind

- ✓ Deterministic optimization is quite difficult to solve, Optimization under uncertainties is still more complex.
- ✓ The problem formulation is **CRUCIAL**
- ✓ The link between optimization variables and random variable laws is **CRUCIAL.**
- ✓ **Be Careful :**
  - ✓ In optimization: the admissible space is defined as $g(\mathrm{X}) \leq 0$
  - ✓ In reliability: the admissible space (safe) is defined as $g(\mathrm{X}) > 0$
- ✓ Many methods and tools exists to decrease computation time
  - ✓ Score functions
  - ✓ Meta-models
  - ✓ …
- ✓ Most of recent methods are based on surrogate models (SVM / Kriging)

7. Bibliography

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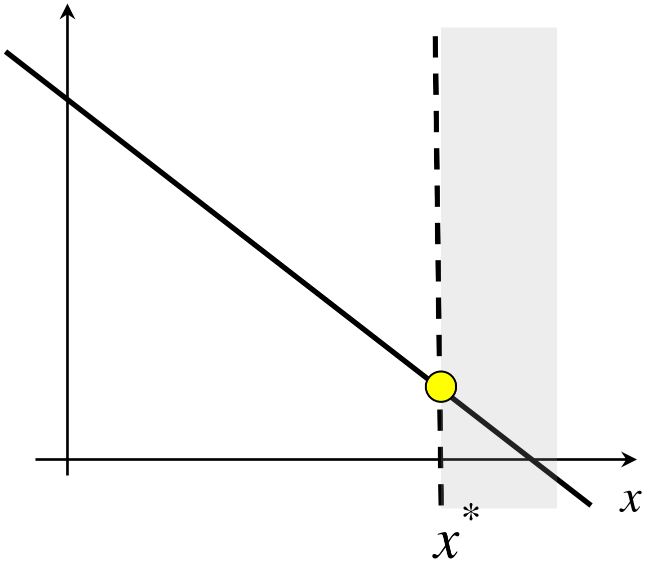
8. Exercise #0 – 1D trivial exercice

Problem formulation

Deterministic problem to solve:

$$x^* = \underset{x}{\textbf{Argmin}} \quad f(x) = 5 - x$$
$$\textbf{under} \quad g(x) = x - 4 \leq 0$$



Question #1: find the solution writing the first order optimality conditions

$$\begin{cases} f'(x) + \lambda g'(x) = 0 \\ \lambda g(x) = 0 \\ \lambda \geq 0 \end{cases} \longrightarrow$$

$\lambda = 0$ No solution

$g(x) = 0$ $\quad x^* = 4$ Solution

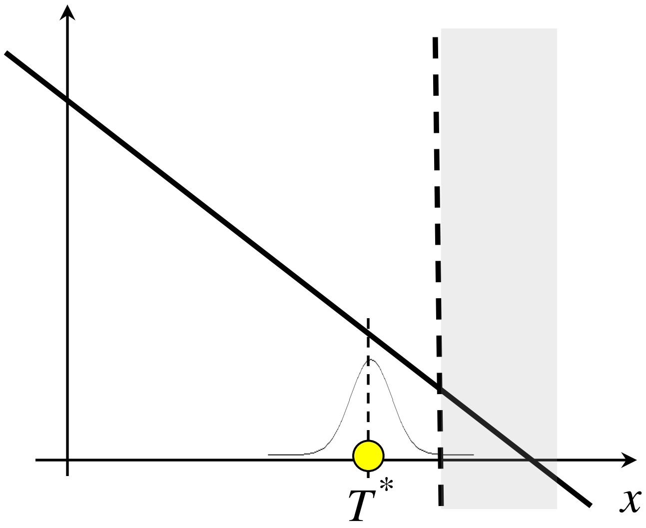
$\lambda^* = 4$

Problem formulation

Optimization under uncertainties

x is a random variable $x(\omega)$ following a Gaussian distribution

$$N(T, \sigma = 0.2)$$



Question #2: formulate and solve a RBDO problem considering uncertainty only in the constraint function.

$$T^* = \underset{T}{\textbf{Argmin}} \quad f(T) = 5 - T$$
$$\textbf{under} \quad g(T) = \textbf{Prob}(-x(T,\omega) + 4 \leq 0) \leq 0.001$$

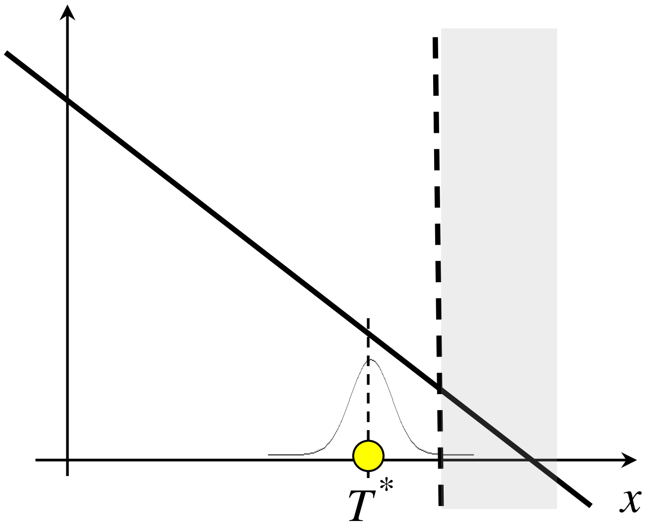
$$\textbf{Prob}(-x(T,\omega) + 4 \leq 0) = \Phi\left(-\frac{-T + 4}{\sigma}\right)$$

$$T - 4 - \sigma\Phi^{-1}(0.001) \leq 0$$
$$T - 3.382 \leq 0$$
$$T^* = 3.382$$

Problem formulation

Question #3: formulate and solve a robust and reliable problem considering uncertainty in both constraint and objective function.

$$T^* = \underset{T}{\textbf{Argmin}} \quad \Psi(T) = m_f(T) + \sigma_f(T)$$
$$\textbf{under} \quad g(T) = \textbf{Prob}(-x(T,\omega) + 4 \leq 0) \leq 0.001$$



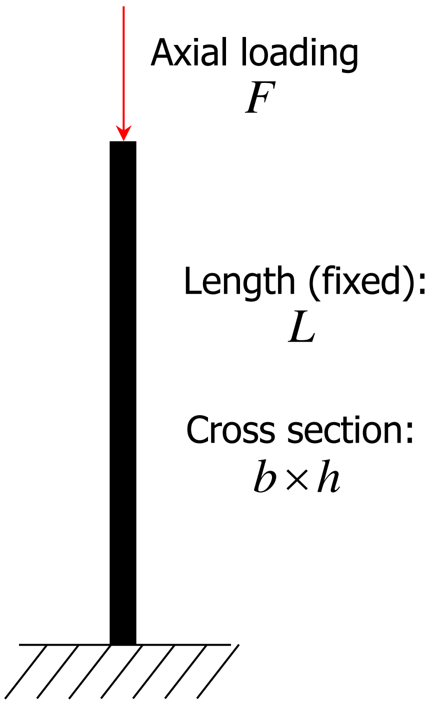
$$\Psi(T) = m_f(T) + \sigma_f(T)$$
$$= 5 - T + \sigma$$

$T^* = 3.382$ same solution as RBDO

9. Exercise #1 – Simple beam under axial loading

Problem definition

Aim : minimize the structural weight under buckling constraints



Deterministic formulation:

$$b^*, h^* = \textbf{Argmin} \quad f(b,h) = b \times h$$
$$\textbf{under} \quad g_1(b,h) = F - \frac{\pi^2 E b h^3}{12 L^2} \le 0$$
$$g_2(b,h) = h - b \le 0$$

RBDO formulation (what are the target values?):

$$T_b, T_h = \textbf{Argmin} \quad f(T_b, T_h) = T_b \times T_h$$
$$\textbf{under} \quad \textbf{Prob}\left[\frac{\pi^2 E b(\omega) h(\omega)^3}{12 L(\omega)^2} - F(\omega) \le 0 \right] - P_f^c \le 0$$
$$g_2(b,h) = T_h - T_b \le 0$$

2 optimization variables

4 random variables :

- ✓ Gaussian
- ✓ Constant standard deviation

Questions

Optimization without uncertainties

1. Optimization without uncertainties: write the optimality conditions.

Optimization under uncertainties

1. Give the Cornell Reliability index from the two first statistical moments
2. Based on the Cornell index, write the optimality conditions of the RBDO problem

Resolution without uncertainty

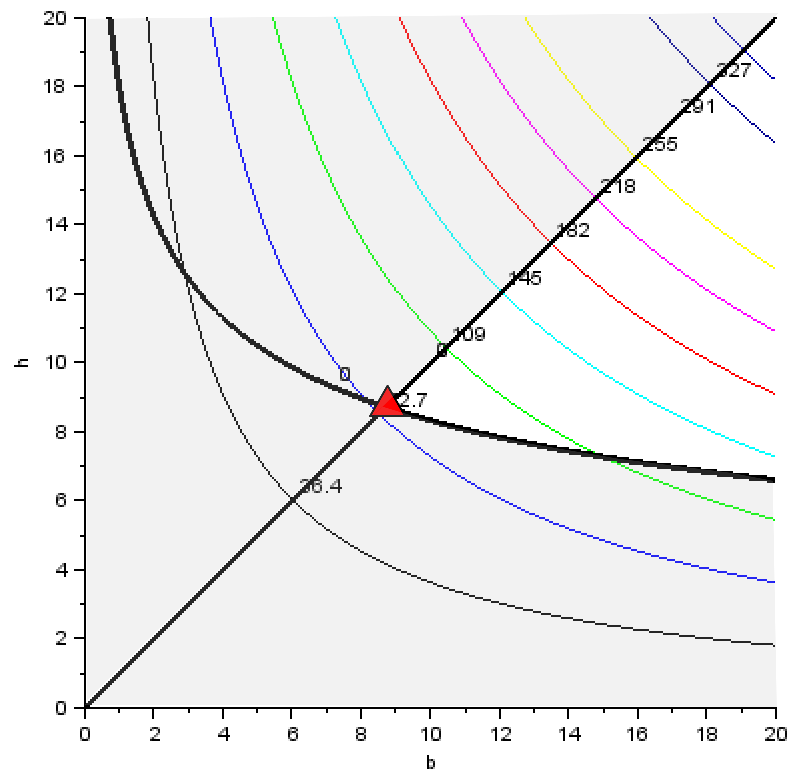
Resolution of the deterministic formulation:

$$\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0 \qquad \lambda_1, \lambda_2 \geq 0$$

$$\begin{cases} h - \lambda_1 \dfrac{\pi^2 E h^3}{12 L^2} - \lambda_2 = 0 \\ b - \lambda_1 \dfrac{\pi^2 E b h^2}{4 L^2} + \lambda_2 = 0 \end{cases}$$

$$\frac{\pi^2 E b h^3}{12 L^2} - F = 0$$

$$h - b = 0$$



Solution :

$$b^* = 8{,}72 mm$$

$$h^* = 8{,}72 mm$$

$$l_1^* = 0{,}04$$

$$l_2^* = 4{,}36$$

Resolution under uncertainties

Resolution of the RBDO formulation:

$$T_b, T_h = \textbf{Argmin} \quad f(T_b, T_h) = T_b \times T_h$$

$$\textbf{under} \quad \textbf{Prob}\left[\frac{\pi^2 E b(\omega) h(\omega)^3}{12 L(\omega)^2} - F(\omega) \le 0\right] - P_f^c \le 0$$

$$g_2(b,h) = T_h - T_b \le 0$$

? P_f function de $T_b \times T_h$

Approximation using the two first moments:

$$m_g(\mathbf{T}) \approx g(\mathbf{T}) = \frac{\pi^2 E T_b T_h^3}{12 T_L^2} - T_F$$

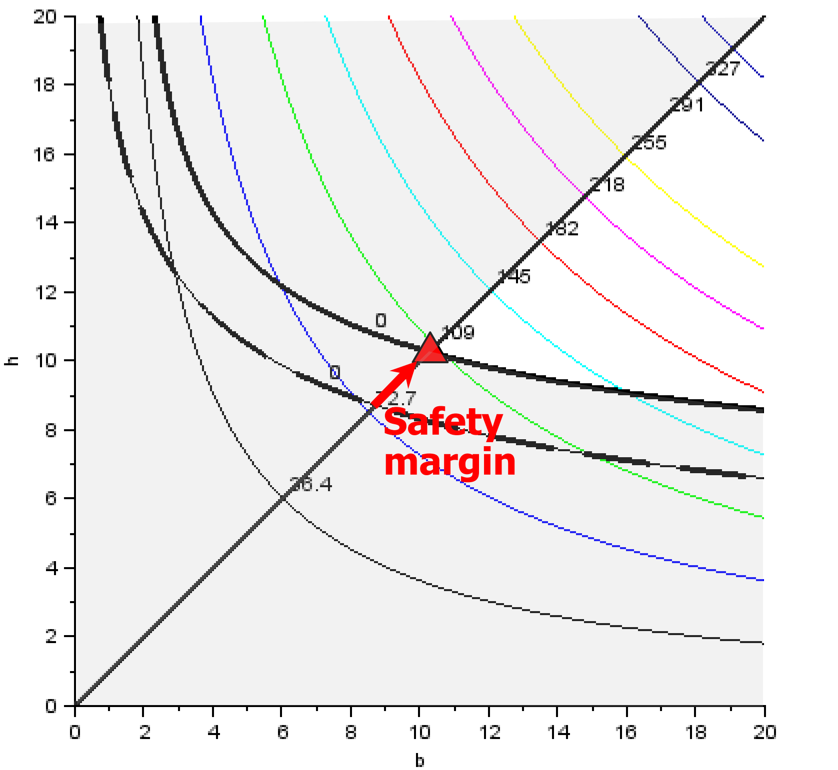
$$\sigma_g^2(\mathbf{T}) \approx \sum_{j=1}^{n} \left.\frac{\partial g}{\partial X_j}\right|_{\mathbf{T}}^{2} \sigma_j^2$$

$$P_f(\mathbf{T}) \approx \Phi\left(-\frac{m_g(\mathbf{T})}{\sigma_g(\mathbf{T})}\right) = \Phi\left(-\beta_c(\mathbf{T})\right)$$

Simple illustration – Simple beam under axial loading

$$\sigma_g^2(\mathbf{T}) \approx \sum_{j=1}^{n} \left.\frac{\partial g}{\partial X_j}\right|_{\mathbf{T}}^{2} \sigma_j^2$$

$$= \left(\frac{\pi^2 E T_h^3}{12 T_L^2}\right)^2 \sigma_b^2 + \left(\frac{\pi^2 E T_b T_h^2}{4 T_L^2}\right)^2 \sigma_h^2 + \left(\frac{\pi^2 E T_b T_h^3}{6 T_L^3}\right)^2 \sigma_L^2 + \sigma_F^2$$



Resolution of the deterministic formulation:

$$\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0 \quad \lambda_1, \lambda_2 \geq 0$$

$$\begin{cases} h - \lambda_1 \left.\dfrac{\partial \beta(T)}{\partial T_b}\right|_T - \lambda_2 = 0 \\ b - \lambda_1 \left.\dfrac{\partial \beta(T)}{\partial T_h}\right|_T + \lambda_2 = 0 \end{cases}$$

$$\beta_t - \beta(\mathbf{T}) = 0$$

$$T_h - T_b = 0$$

Solution :

$$b^* = 10,26mm$$

$$h^* = 10,26mm$$

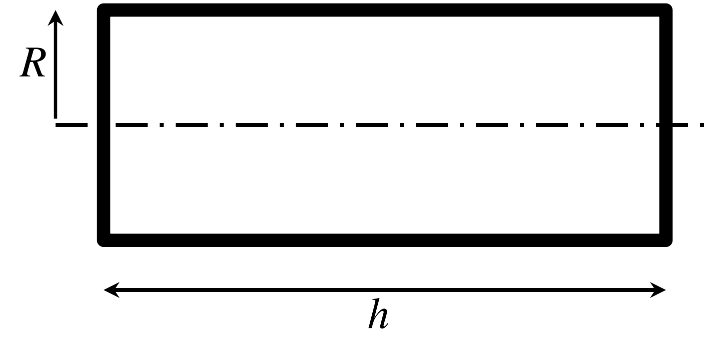
$$\lambda_1^* = 12,63$$

$$\lambda_2^* = 5,64$$

10. Exercise #2 – Application to a container

Problem definition

We want to design (find value of target values and tolerances) of a container that must contain at least 33cm3 of a liquid and using, for economical reasons, the minimum quantity of material.



$$R \to T_R \pm t_R / 2$$
$$h \to T_h \pm t_h / 2$$

T_R, T_h, t_R, t_h are the design variables.

$T_R \in [1;4] \quad T_h \in [1;10]$

Objective function (To be minimized): $f(R,h) = 2\pi Rh + 2\pi R^2$

Constraint function: $g(R,h) = \pi R^2 h - 33 \geq 0$

#1 without uncertainties

#1: we don't care about uncertainties, "the tolerances will be defined in a meeting with the production department" – Give a solution to the following problems in red.

ROBUSTNESS

RELIABILITY

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	determinitic optimization without constraints	Optimization of the robutness
Constraint function with X, P deterministic	Sizing / dimensionning	determinitic optimization under constraints	Optimization of the robutness under determinitic constraint
Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robutness under uncertain constraint

#1 without uncertainties

#1: we don't care about uncertainties, "the tolerances will be defined in a meeting with the production department"

ROBUSTNESS

RELIABILITY

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	determinitic optimization without constraints	Optimization of the robustness
Constraint function with X, P deterministic	Sizing / dimensionning	determinitic optimization under constraints	Optimization of the robustness under determinitic constraint
Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robustness under uncertain constraint

Each value such that $\pi T_R^2 T_h - 33 \geq 0$ is satisfying $\rightarrow$ $T_R = 1cm$, $T_h = 10{,}51cm$

#1 without uncertainties

#1: we don't care about uncertainties, "the tolerances will be defined in a meeting with the production department"

	ROBUSTNESS		
	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
RELIABILITY — No constraint function	X	deterministic optimization without constraints	Optimization of the robustness
Constraint function with X, P deterministic	Sizing / dimensionning	deterministic optimization under constraints	Optimization of the robustness under deterministic constraint
Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robustness under uncertain constraint

The structure has no function → $$T_R = 0cm^3 \quad T_h = 0cm^3$$ is the less expansive.

#1 without uncertainties

#1: we don't care about uncertainties, "the tolerances will be defined in a meeting with the production department"

	ROBUSTNESS: No objective function	ROBUSTNESS: Objective function with X, P deterministic	ROBUSTNESS: Objective function with X, P uncertain
RELIABILITY: No constraint function	X	determinitic optimization without constraints	Optimization of the robustness
RELIABILITY: Constraint function with X, P deterministic	Sizing / dimensionning	determinitic optimization under constraints	Optimization of the robustness under determinitic constraint
RELIABILITY: Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robustness under uncertain constraint

Find T_R, T_h such as: $$T_R^*, T_h^* = \textbf{Argmin} \quad f(T_R, T_h) = 2\pi T_R T_h + 2\pi T_R^2$$

$$\textbf{under} \quad \pi T_R^2 T_h - 33 \geq 0$$

#1 without uncertainties

$$T_R^*, T_h^* = \textbf{Argmin} \quad f(T_R, T_h) = 2\pi T_R T_h + 2\pi T_R^2$$

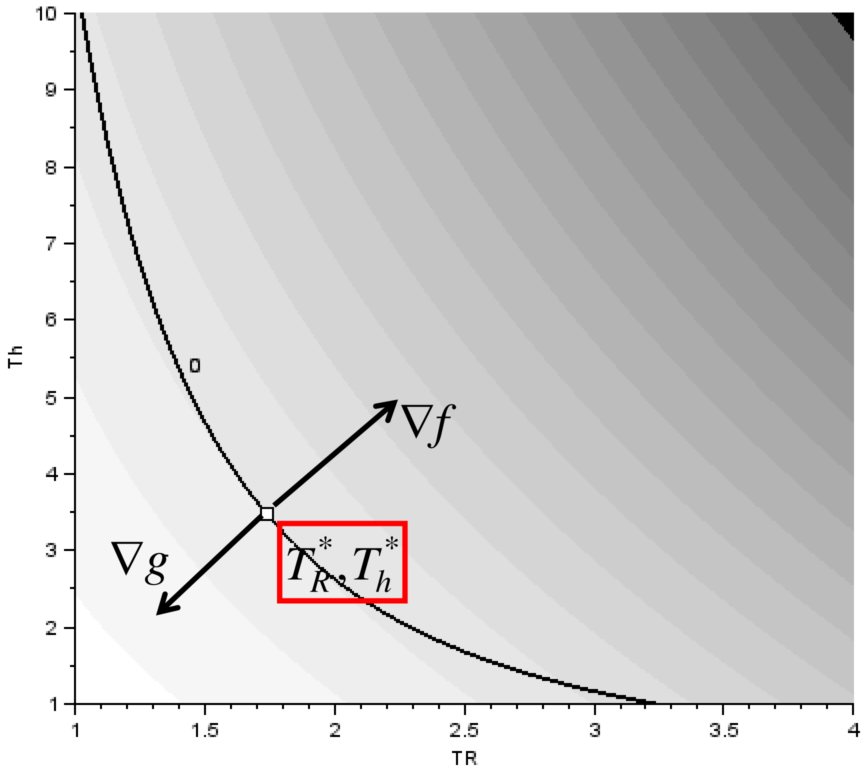
$$\textbf{under} \quad g(T_R, T_h) = -\pi T_R^2 T_h + 33 \leq 0$$

First order optimality conditions:

$$\nabla f(T_R, T_h) + \lambda \nabla g(T_R, T_h) = 0$$
$$g(T_R, T_h) = 0$$
$$\lambda \geq 0$$

$$\begin{cases} 2\pi T_h + 4\pi T_R - 2\lambda\pi T_R T_h = 0 \\ 2\pi T_R - \lambda\pi T_R^2 = 0 \\ -\pi T_R^2 T_h + 33 = 0 \end{cases}$$

$$\begin{cases} T_R^* = 1,74 \\ T_h^* = 3,48 \\ \lambda^* = 1,15 \end{cases}$$



#2 with uncertainties

#2: we care about uncertainties, "the tolerances are discussed with the production department before the optimization" – Give a solution to the following problems in red.

ROBUSTNESS

RELIABILITY

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	determinitic optimization without constraints	Optimization of the robustness
Constraint function with X, P deterministic	Sizing / dimensionning	determinitic optimization under constraints	Optimization of the robustness under determinitic constraint
Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robustness under uncertain constraint

$t_R = 0,6$ → $R \to N(T_R; 0,2)$

$t_h \to 0,6$ → $h \to N(T_h; 0,2)$

Constraint on failure probability : $P_f \leq 0.1$

Robustness function: $\Psi(T_R, T_h) = \bar{f}$

(To be minimized)

#2 with uncertainties

ROBUSTNESS (columns) / RELIABILITY (rows)

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	determinitic optimization without constraints	Optimization of the robutness
Constraint function with X, P deterministic	Sizing / dimensionning	determinitic optimization under constraints	Optimization of the robutness under determinitic constraint
Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robutness under uncertain constraint

Each value such that $\mathbf{Prob}\left(\pi R^2 h - 33 \leq 0\right) \leq 0.1$ is satisfying $\rightarrow$

$$T_R = 2cm$$
$$T_h = 4cm$$

$$P_f = 0,06$$

#2 with uncertainties

RELIABILITY \ ROBUSTNESS	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	determinitic optimization without constraints	Optimization of the robutness
Constraint function with X, P deterministic	Sizing / dimensionning	determinitic optimization under constraints	Optimization of the robutness under determinitic constraint
Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robutness under uncertain constraint

The structure has no function → $T_R = 0cm^3$, $T_h = 0cm^3$ is the less expansive.

$$\Psi(T_R, T_h) = \overline{f} = 0$$

#2 with uncertainties

ROBUSTNESS

RELIABILITY

	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	determinitic optimization without constraints	Optimization of the robutness
Constraint function with X, P deterministic	Sizing / dimensionning	determinitic optimization under constraints	Optimization of the robutness under determinitic constraint
Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robutness under uncertain constraint

Find T_R, T_h such as: $T_R^*, T_h^* = \textbf{Argmin} \quad f(T_R, T_h) = 2\pi T_R T_h + 2\pi T_R^2$

$$\textbf{under} \quad \textbf{Prob}\left(\pi R^2 h - 33 \leq 0\right) \leq 0.1$$

#2 with uncertainties

$$T_R^*, T_h^* = \textbf{Argmin} \quad f(T_R, T_h) = 2\pi T_R T_h + 2\pi T_R^2$$

$$\textbf{under} \quad g(T_R, T_h) = \textbf{Prob}\left(\pi T_R^2 T_h - 33 \le 0\right) - 0.1 \le 0$$

First order optimality conditions:

$$\nabla f(T_R, T_h) + \lambda \nabla g(T_R, T_h) = 0$$
$$g(T_R, T_h) = 0$$
$$\lambda \ge 0$$

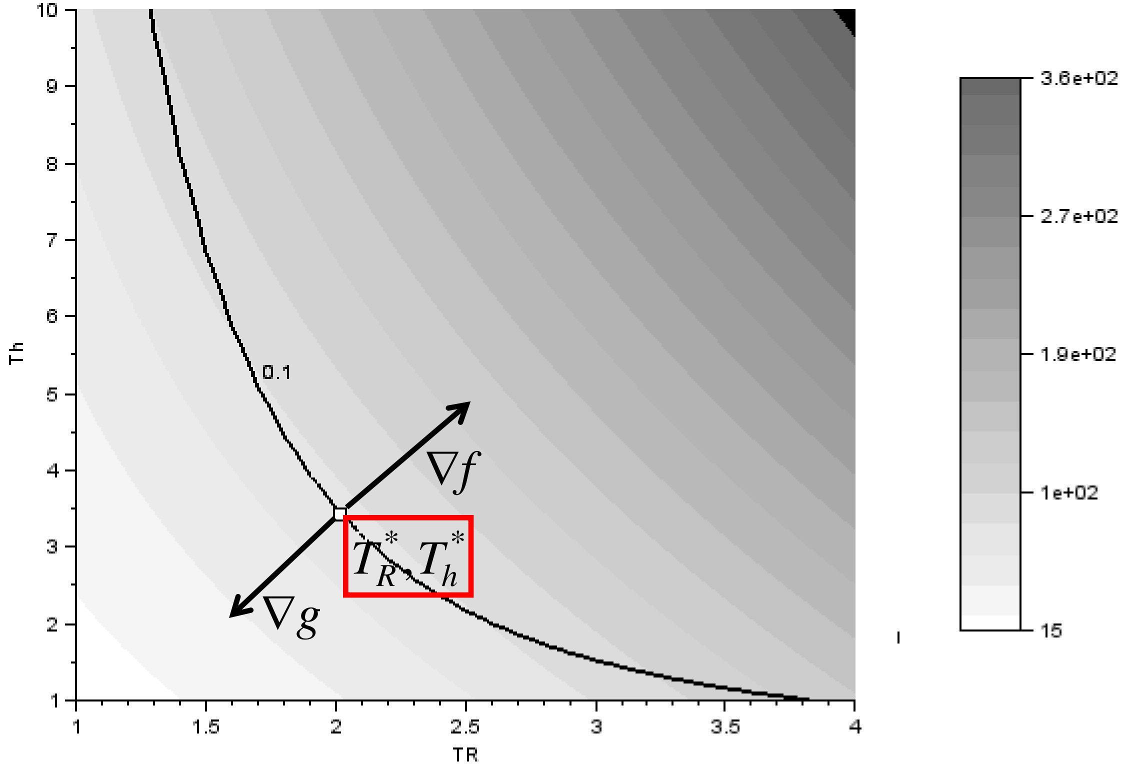
$$\begin{cases} 2\pi T_h + 4\pi T_R + \lambda \dfrac{\partial P_f(T_R, T_h)}{\partial T_R} = 0 \\ 2\pi T_R + \lambda \dfrac{\partial P_f(T_R, T_h)}{\partial T_h} = 0 \\ P_f(T_R, T_h) - 0.1 = 0 \end{cases}$$

$$F(\mathbf{X}) = 0 \Leftrightarrow \mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} - J_F(\mathbf{X}^{(k)}) F(\mathbf{X}^{(k)})$$

$$J_F(T_R, T_h, \lambda) = \begin{bmatrix} 4\pi + \lambda \dfrac{\partial^2 P_f(T_R, T_h)}{\partial^2 T_R} & 2\pi + \lambda \dfrac{\partial^2 P_f(T_R, T_h)}{\partial^2 T_R T_h} & \dfrac{\partial P_f(T_R, T_h)}{\partial T_R} \\ 2\pi + \lambda \dfrac{\partial^2 P_f(T_R, T_h)}{\partial^2 T_R T_h} & \lambda \dfrac{\partial^2 P_f(T_R, T_h)}{\partial^2 T_h} & \dfrac{\partial P_f(T_R, T_h)}{\partial T_h} \\ \dfrac{\partial P_f(T_R, T_h)}{\partial T_R} & \dfrac{\partial P_f(T_R, T_h)}{\partial T_h} & 0 \end{bmatrix}$$

#2 with uncertainties

$$\begin{cases} T_R^* = 2{,}02 \\ T_h^* = 3{,}44 \\ \lambda^* = 55.5 \end{cases}$$



#2 with uncertainties

RELIABILITY \ ROBUSTNESS	No objective function	Objective function with X, P deterministic	Objective function with X, P uncertain
No constraint function	X	determinitic optimization without constraints	Optimization of the robutness
Constraint function with X, P deterministic	Sizing / dimensionning	determinitic optimization under constraints	Optimization of the robutness under determinitic constraint
Constraint function with X, P uncertain	Reliability	Reliability-based design Optimization (RBDO)	Optimization of the robutness under uncertain constraint

Find T_R, T_h such as:

$$T_R^*, T_h^* = \textbf{Argmin} \quad f(T_R, T_h) = \bar{f}\left(T_R, T_h\right)$$

$$\textbf{under} \quad \textbf{Prob}\left(\pi R^2 h - 33 \leq 0\right) \leq 0.1$$

#2 with uncertainties

$$T_R^*, T_h^* = \textbf{Argmin} \quad f(T_R, T_h) = \overline{f}\left(T_R, T_h\right)$$
$$\textbf{under} \quad \textbf{Prob}\left(\pi R^2 h - 33 \leq 0\right) \leq 0.1$$

First order optimality conditions:

$$\nabla f(T_R, T_h) + \lambda \nabla g(T_R, T_h) = 0$$
$$g(T_R, T_h) = 0$$
$$\lambda \geq 0$$

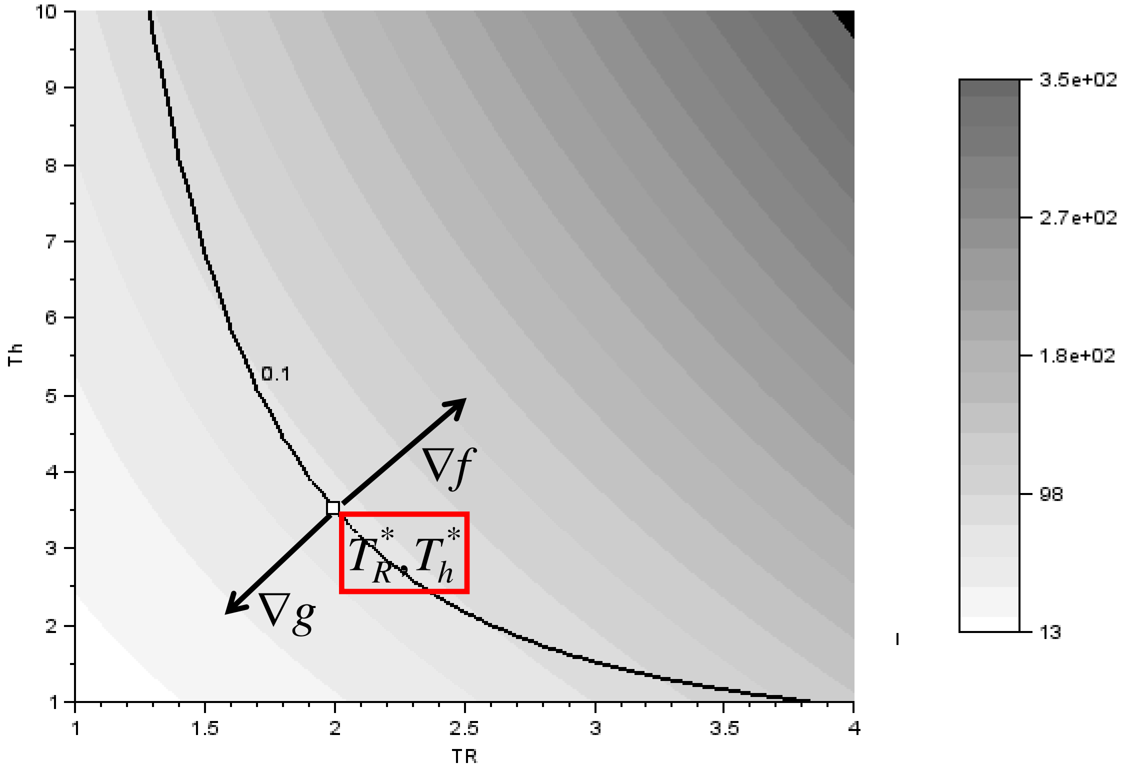
$$\begin{cases} \dfrac{\partial \overline{f}}{\partial T_R} + \lambda \dfrac{\partial P_f(T_R, T_h)}{\partial T_R} = 0 \\ \dfrac{\partial \overline{f}}{\partial T_h} + \lambda \dfrac{\partial P_f(T_R, T_h)}{\partial T_h} = 0 \\ P_f(T_R, T_h) - 0.1 = 0 \end{cases}$$

$$F(\mathbf{X}) = 0 \Leftrightarrow \mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} - J_F(\mathbf{X}^{(k)}) F(\mathbf{X}^{(k)})$$

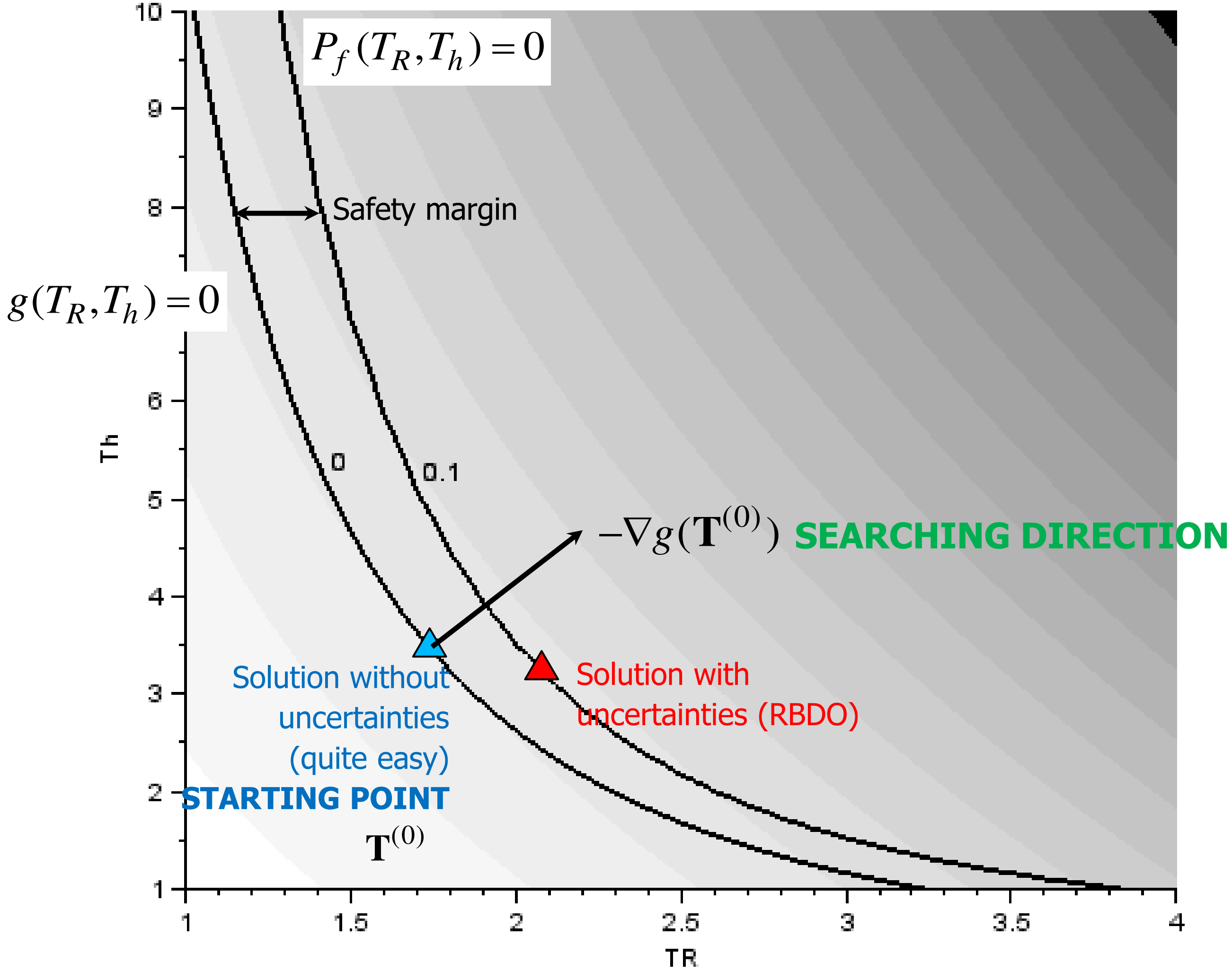
$$J_F(T_R, T_h, \lambda) = \begin{bmatrix} \dfrac{\partial^2 \overline{f}(T_R, T_h)}{\partial^2 T_R} + \lambda \dfrac{\partial^2 P_f(T_R, T_h)}{\partial^2 T_R} & \dfrac{\partial^2 \overline{f}(T_R, T_h)}{\partial^2 T_R T_h} + \lambda \dfrac{\partial^2 P_f(T_R, T_h)}{\partial^2 T_R T_h} & \dfrac{\partial P_f(T_R, T_h)}{\partial T_R} \\ \dfrac{\partial^2 \overline{f}(T_R, T_h)}{\partial^2 T_R T_h} + \lambda \dfrac{\partial^2 P_f(T_R, T_h)}{\partial^2 T_R T_h} & \dfrac{\partial^2 \overline{f}(T_R, T_h)}{\partial^2 T_h} + \lambda \dfrac{\partial^2 P_f(T_R, T_h)}{\partial^2 T_h} & \dfrac{\partial P_f(T_R, T_h)}{\partial T_h} \\ \dfrac{\partial P_f(T_R, T_h)}{\partial T_R} & \dfrac{\partial P_f(T_R, T_h)}{\partial T_h} & 0 \end{bmatrix}$$

#2 with uncertainties

$$\begin{cases} T_R^* = 1{,}99 \\ T_h^* = 3{,}51 \\ \lambda^* = 55.2 \end{cases}$$



Idea


$P_f(T_R,T_h)=0$
Safety margin
$g(T_R,T_h)=0$
0
0.1
$-\nabla g(\mathbf{T}^{(0)})$ SEARCHING DIRECTION
Solution without uncertainties (quite easy)
STARTING POINT
$\mathbf{T}^{(0)}$
Solution with uncertainties (RBDO)
Th
TR
10
9
8
6
5
4
3
2
1
1
1.5
2
2.5
3
3.5
4

Idea

RBDO easier problem to solve:

$$\textbf{Find } \alpha^* \quad \textbf{such that} \quad \textbf{Prob}\left(g\left(\mathbf{X}(\mathbf{T}^{(0)} - \alpha^* \nabla g(\mathbf{T}^{(0)}), \omega)\right) \leq 0\right) = \Pr_{\textbf{target}}$$

Solution approximated by: $\mathbf{T}^{(0)} - \alpha^* \nabla g(\mathbf{T}^{(0)})$