Uncertainty Aggregation
Variability, Statistical Uncertainty, and Model Uncertainty

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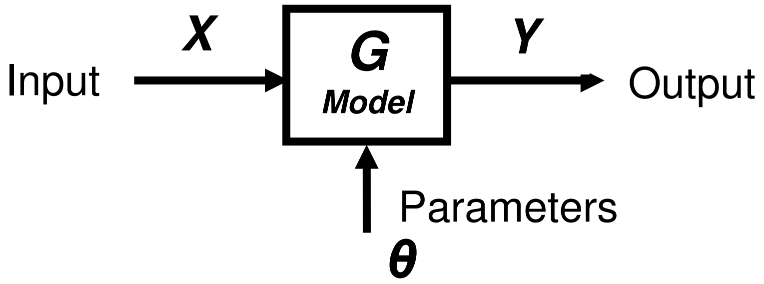
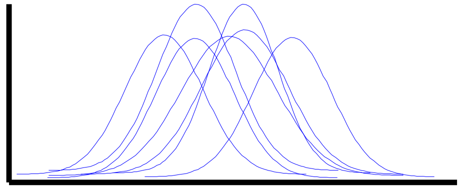
Sources of Uncertainty in Model Prediction

- Natural Variability (Aleatory)

  Variation across
  - Samples → Random variables
  - Time → Random processes
  - Space → Random fields

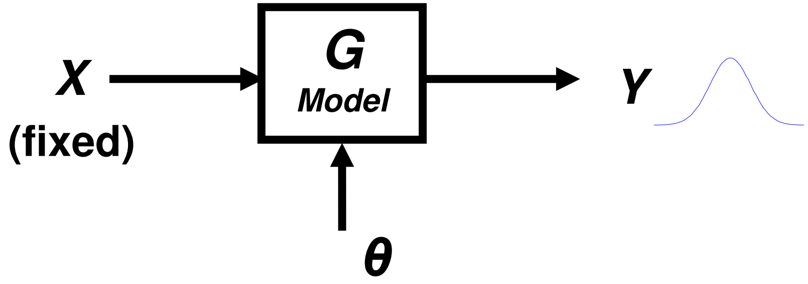


- Input uncertainty (Epistemic)
  - Sparse data
  - Imprecise and qualitative data
  - Measurement errors
  - Processing errors



Multiple PDFs of input X

- Model uncertainty (Epistemic)
  - Model parameters
  - Solution approximation
  - Model form

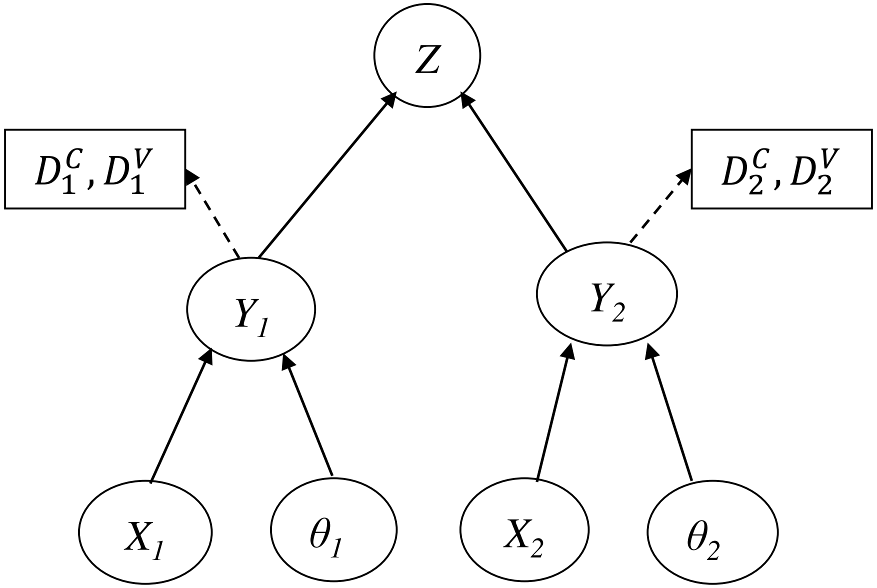


Uncertainty aggregation

- Information at multiple levels
  - Inputs
  - Parameters
  - Model errors
  - Outputs

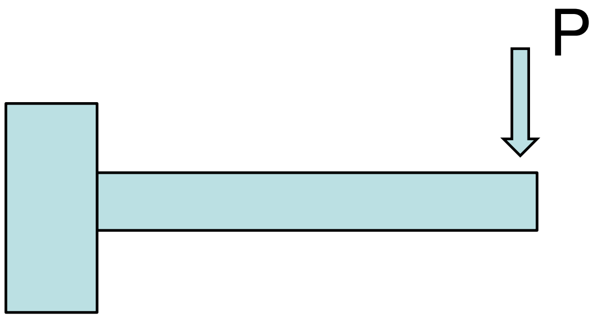
- Heterogeneous information
  - Multiple types of sources, formats
    - models, tests, experts, field data
  - Multiple physics, scales, resolutions
  - Different levels of fidelity and cost

- How to fuse **ALL** available information to quantify uncertainty in system-level prediction?

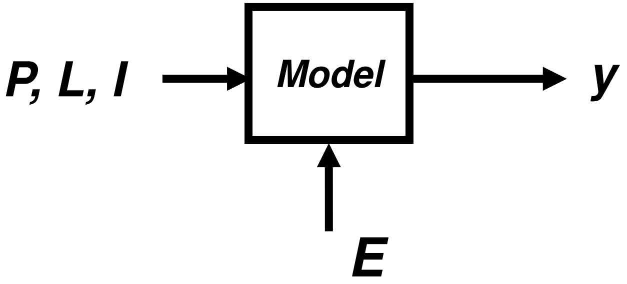


A simple example

- Model for vertical deflection at free end (Euler-Bernoulli) $y = \dfrac{PL^3}{3EI}$
- Assume L and I have only aleatory variability
- $P \rightarrow$ random variable (aleatory), but we may not know its distribution type D and parameters θ_p, thus $P \sim D(\theta_p)$ could have both aleatory and epistemic uncertainty
- E → model parameter (could be only epistemic, only aleatory, or both)
- Model error → infer from tests
- Other issues
  - Boundary condition → degree of fixity → infer from tests
  - Spatial variability of E → random field
  - Temporal variability of P → random process
- Random field and random process parameters need to be inferred from data → could have both types of uncertainty



Cantilever beam



Treatment of Epistemic Uncertainty

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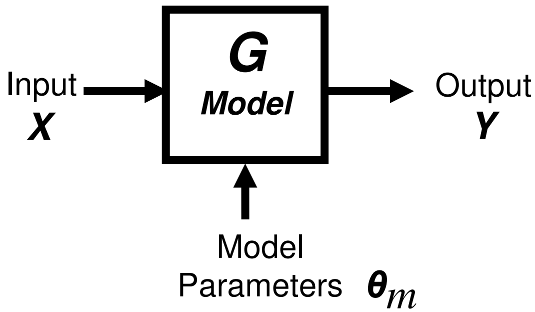

$$Y = G(X, \theta_m) + \varepsilon_M$$
$$\varepsilon_M = \varepsilon_{mf} + \varepsilon_{num}$$

- Statistical Uncertainty
  - Distribution type D and parameters $\boldsymbol{\theta_p}$ of $X \sim D(\boldsymbol{\theta_p})$
- Model Uncertainty
  - System model parameters $\boldsymbol{\theta}_m$ ← **Calibration**
    - uncertainty represented by probability distributions (Bayesian)
  - Model form of G
    - Model form error ε_{mf} (quantified using validation data) ← **Validation**
  - Numerical solution error ε_{num} in G
    - Discretization error (quantified using convergence study) ← **Verification**
    - Surrogate model error (by-product of surrogate model building)
- **Bayesian Approach** → All uncertainty/error terms represented through probability distributions

BACKGROUND TOPICS

Quantities varying over space and time

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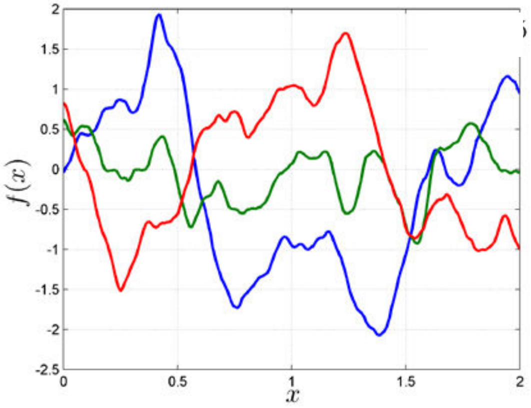
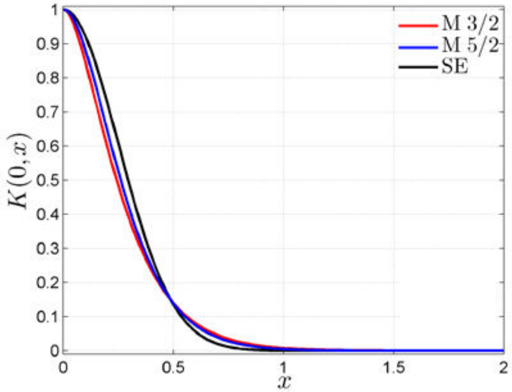
- Quantities expressed as random processes/fields
  - e.g., Loading at one location → random process over time
  - e.g., material properties → random field over space
- Aleatory uncertainty alone
  - Random process/field parameters are deterministic
    - e.g., Gaussian process $w(x) \sim \mathrm{GP}(m(x), C(x,x'))$
    - $m(x) = a + bx + cx^2 \cdots$
    - $C(x,x') = \mathrm{Cov}(w(x), w(x')),$
      - e.g., Squared exponential (SE)

$$C(r,\ell) = \sigma^2 \exp(-r^2/\ell^2)$$

- With epistemic uncertainty
  - Random process/field parameters are uncertain

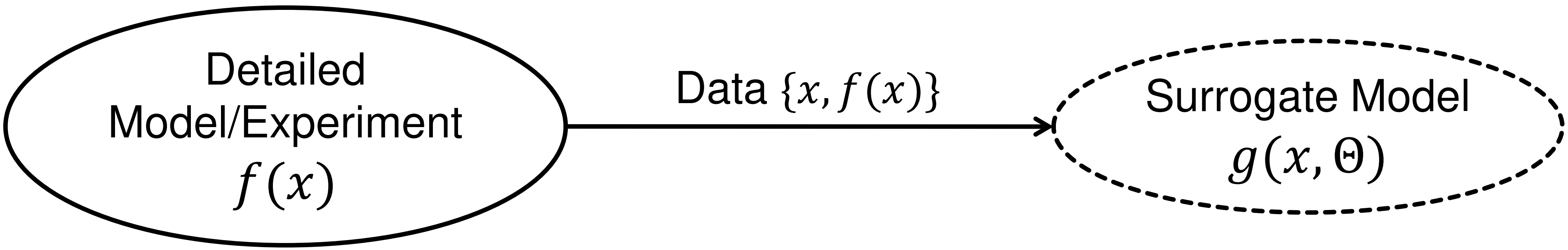


Random process realizations



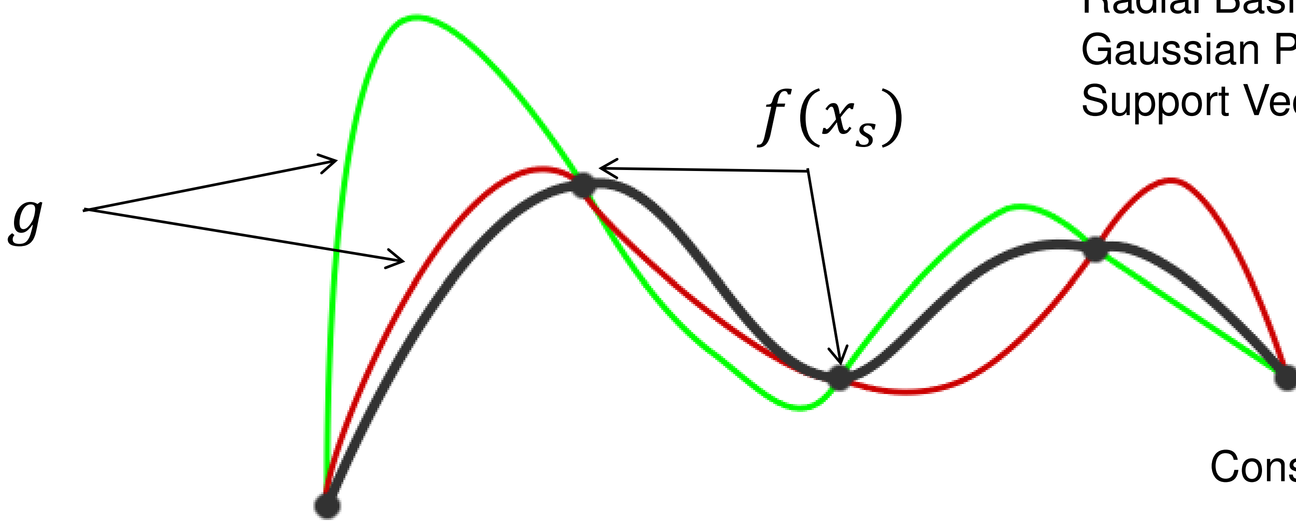
Correlation functions

Surrogate Modeling



- Uncertainty quantification of the output
- Multiple runs of expensive system analysis
- Unknown functional form

- Inexpensive to evaluate at any location
- <u>Examples</u>:
  - Polynomial Chaos Expansions
  - Radial Basis Functions
  - Gaussian Process Models
  - Support Vector Machines



Surrogate model output has uncertainty

Consistent Reconstruction

$$g(x_s) = f(x_s)$$

Global Sensitivity Analysis

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- Y = G(**X**)
- **X** → random variables
- Y → calculated through uncertainty propagation
- Apportion **variance** of Y to inputs **X**
- Analyze sensitivity of output over the entire domain rather than (1) suppressing a variable completely (2) using local derivatives
- Individual effects S_I & Total effects S_T (i.e., in combination with other variables)

$$S_I = \frac{V_{X_i}(E_{X_{\sim i}}(Y \mid X_i))}{V(Y)} \qquad S_T = \frac{E_{X_{\sim i}}(V_{X_i}(Y \mid X_{\sim i}))}{V(Y)}$$

- Single loop sampling approaches exist in literature to calculate S_I and S_T

Parameter estimation: Least Squares

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- Linear regression $Y = X\theta$
  - θ → model parameters (m by 1)
  - "n" ordered input – output observations
    - Each input observation is a vector (1 by m)
    - Each output observation is a scalar
  - Construct X → matrix of inputs (n by m)
  - Construct Y → vector of outputs (n by 1)

$$\hat{\theta} = (X^T X)^{-1}(X^T Y)$$

Coefficient of determination R^2
Proportion of variance explained by the linear regression model

High Value ≠ Accurate Prediction

- Non-linear model $Y = G(X, \theta)$
  - Simple least squares → Minimize $S(\theta) = \sum_{i=1}^{n} (y_i - G(x_i, \theta))^2$
  - α-level confidence intervals on θ using F-statistic $S(\theta) \le S(\hat{\theta})\{1 + \frac{m}{n-m} F^{\alpha}_{m,n-m}\}$
- Advanced methods → weighted, generalized, moving, iterative
  - Unbiased, convergent estimates
  - Applicable to multiple output quantities
- Disadvantages
  - Based on assumption of normally distributed residuals
  - Difficult to include other types of uncertainty in input/output (imprecision, no ordered pairs)

Likelihood function

Likelihood → Probability of observing the data, given specific values of parameters

Example

- Random variable *X;* Available data points x_i (i = 1 to n)
- Suppose we fit a normal distribution to *X*

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

Likelihood function

$$L(\mu,\sigma) = P(\text{observing data } x_i \,|\, (\mu,\sigma)) \propto f_X(x_i)$$

Considering all n data points,

$$L(\mu,\sigma) \propto \prod_{i=1}^{n} f_X(x_i)$$

Maximum likelihood estimate (MLE)

→ Maximize likelihood function and estimate parameters (μ, σ)

Bayesian estimation

- Maximum Likelihood Estimate
  - Maximize $L(\theta)$ → Point estimate of θ
- To account for uncertainty regarding θ
  - Bayesian approach → probability distribution of θ
  - Assume a uniform prior over the domain of θ

$$f(\theta) = \frac{L(\theta)}{\int L(\theta)}$$

- Calculate marginal distributions of individual parameters
- PDF of X → $f_X(x|\theta)$
  - Distribution of θ → $f(\theta)$
  - Each sample of θ → PDF for X
  - Family of PDFs for X

Bayes' Theorem

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- Theorem of Total Probability

$$P(A) = \sum_{i=1}^{n} P(A \mid E_i) P(E_i)$$

(Events E_i are mutually exclusive and collectively exhaustive)

- Bayes theorem

$$P(A \mid B) = \frac{P(B \mid A) P(A)}{P(B)} \longrightarrow P(E_i \mid A) = \frac{P(A \mid E_i) \; P(E_i)}{P(A)}$$

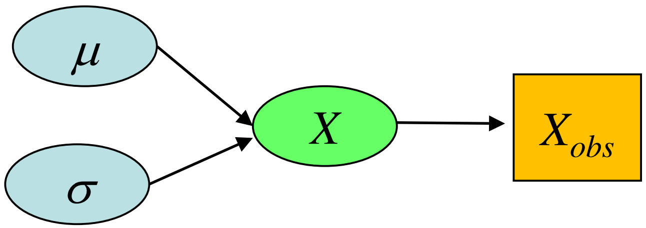
likelihood: $P(A \mid E_i)$; prior: $P(E_i)$; posterior: $P(E_i \mid A)$; normalizing constant: $P(A)$

- In terms of probability densities (continuous variables):
  - θ : parameter to be updated
  - $\mathbf{D}$: experimental data
  - $\Pr(\mathbf{D}|\theta)$: likelihood function of θ
  - $\pi(\theta)$: prior PDF of θ
  - $\pi(\theta|\mathbf{D})$: posterior PDF of θ

$$\pi(\theta \mid \mathbf{D}) = \frac{\Pr(\mathbf{D} \mid \theta) \pi(\theta)}{\int \Pr(\mathbf{D} \mid \theta) \pi(\theta) d\theta}$$

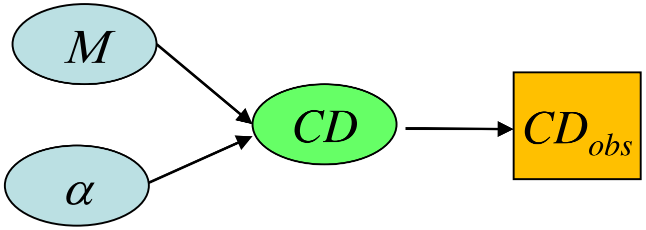
Three examples of Bayesian inference

- Distribution parameters of a random variable → Statistical uncertainty



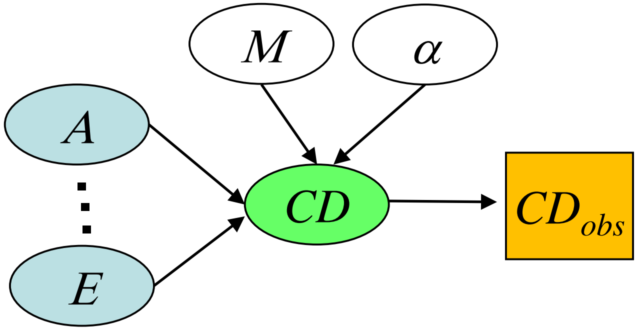
- Distributions of unmeasured test inputs

$$CD = 0.05 - 0.015*M + 4.0e{-}004*\alpha - 7.04e{-}004*M*\alpha + 1.45e{-}003*M^2 + 4.6e{-}004*\alpha^2$$



- Distributions of model coefficients **(Bayesian regression)**

$$CD = 0.05 - A*M + B*\alpha - C*M*\alpha + D*M^2 + E*\alpha^2$$



Construction of posterior distribution

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$$\pi(\theta \mid \mathbf{D}) = \frac{\Pr(\mathbf{D} \mid \theta)\pi(\theta)}{\int \Pr(\mathbf{D} \mid \theta)\pi(\theta)d\theta}$$

- Conjugate distributions → Prior and posterior have same distribution type; only the parameters change.
- Sampling-based methods
  - Markov Chain Monte Carlo methods
    - Metropolis
    - Metropolis-Hastings
    - Gibbs
    - Slice sampling
    - Adaptive improvements
  - Particle filter methods
    - Sequential importance re-sampling (SIR)
    - Rao-Blackwellization

Statistical Uncertainty: Bayesian Inference of Distribution Parameters

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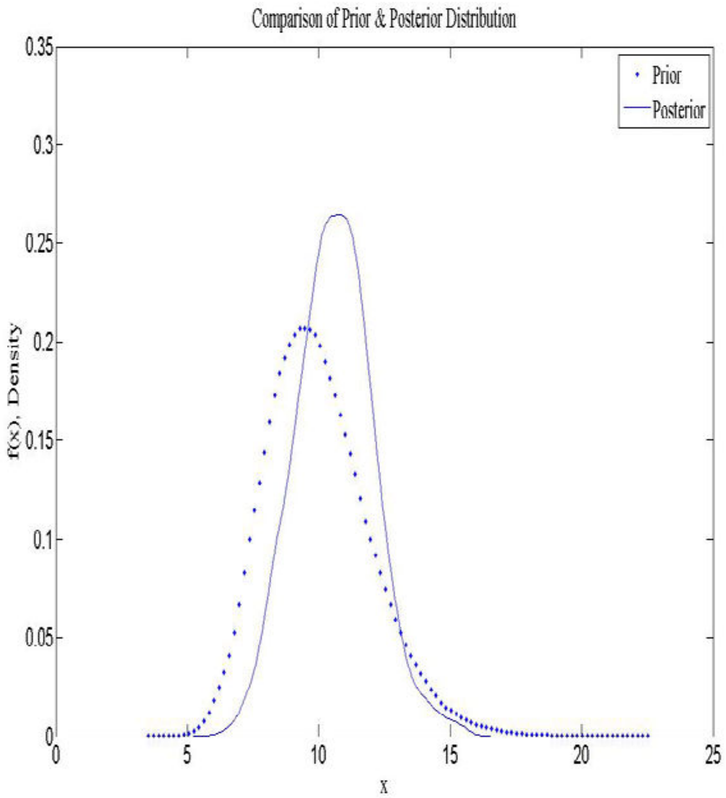
- An RV 'X' has a known 'pdf' type → $X \sim f_X(x|\boldsymbol{\theta})$
- Unknown parameters $\boldsymbol{\theta}$
- Observe instances of 'X' through experiments
- Assume prior distributions for $\boldsymbol{\theta}$ and update them

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i|\theta)$$

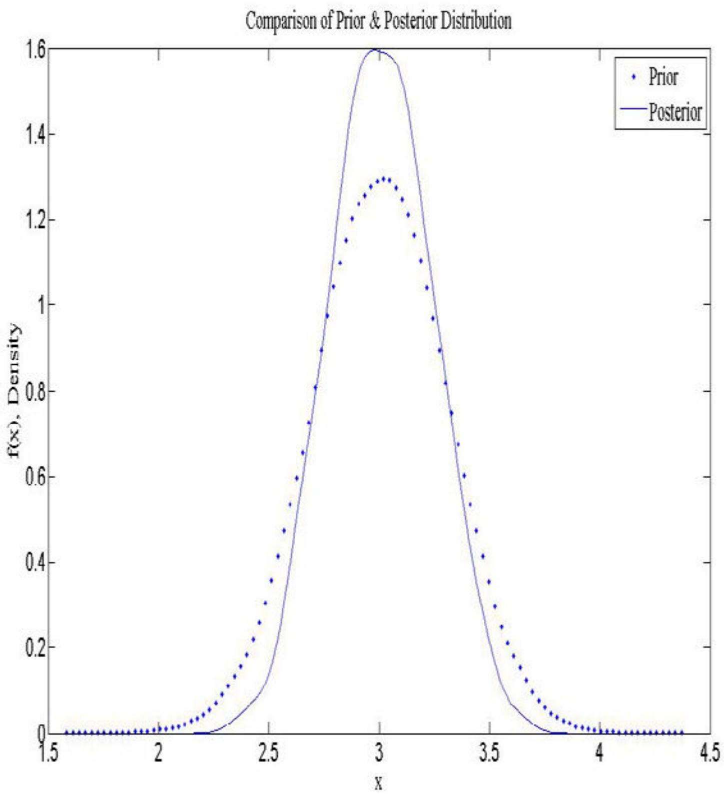
$X \sim N(\mu,\sigma)$ $\theta = \{\mu, \sigma\}$ Observed Values of X = {12, 14}

Parameter No.	Parameter Name	Prior Distribution			Posterior Distribution	
		PDF Type	Mean	Moments	Mean	Variance
1	μ	Log Normal	10	4	10.57	2.19
2	σ	Johnson	3	0.085 -0.006 0.017	3.03	0.05

Comparison of densities



Prior & Posterior of μ



Prior & Posterior of σ

Model Uncertainty: Bayesian Inference of Model Inputs

- Consider a model y = g(**x**) + ε
- Assume prior distributions for **x**
- Observe y through experiments (y_i, i =1 to n)
- Update distributions for ***x***
- ε is assumed to be a normal random variable with zero mean and σ^2 variance, which is calculated from instances of y observed through experiments.

$$L(x) = \prod_{i=1}^{n} \exp\left(-\frac{(y_i - g(x))^2}{2\sigma^2}\right)$$

Model Uncertainty: Bayesian Inference of Model Coefficients

- Consider a model y = b0 + b1*X1 + b2*X2 …… + ε
- The model coefficients **b** are unknown
- Model calibration data are collected (x_i, y_i) (i = 1 to n)
- Prior distributions are assumed for **b** and updated.
- ε is assumed to be a normal random variable with zero mean and σ^2 variance
- This method is applicable irrespective of whether the model is linear or not → $y = g(\mathbf{x},\mathbf{b})$

$$L(b) = \prod_{i=1}^{n} \exp\left(-\frac{(y_i - g(x_i, b))^2}{2\sigma^2}\right)$$

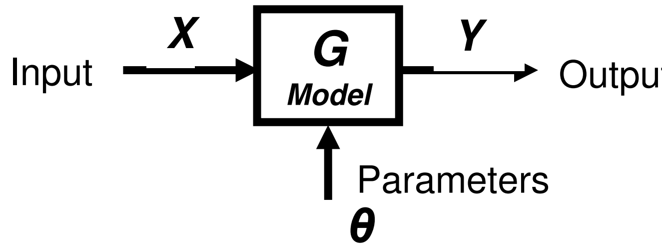
Summary of Background Topics

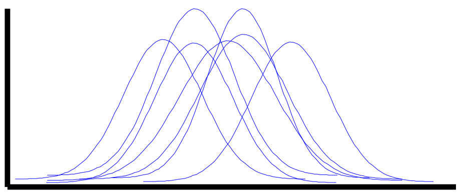
- Aleatory vs. epistemic uncertainty
  - In random variables and random processes/fields
- Surrogate modeling
  - Adds to the uncertainty in prediction
- Sensitivity analysis
  - Variance-based
- Parameter estimation
  - Distribution parameters (statistical uncertainty)
  - Model parameters (model uncertainty)
  - Likelihood, Bayes' theorem, and MCMC

STATISTICAL UNCERTAINTY

Input uncertainty due to data inadequacy

- Sources of data inadequacy
  - Sparsity
  - Imprecision (i.e., interval)
  - Vagueness, ambiguity
  - Missing
  - Erroneous, conflicting
  - Measurement noise
  - Processing errors





Multiple PDFs of input X

- Data inadequacy leads to epistemic uncertainty in the quantification of model inputs and parameters
  - Value of a deterministic variable
  - Value of distribution parameter of a random variable
  - Values of parameters of random process or random field

Topics

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- Parametric approach
  - Family of distributions
  - Model selection
  - Ensemble modeling

- Non-parametric approach
  - Kernel density
  - General approach with point and interval data

- Separating aleatory and epistemic uncertainty

Non-Probabilistic Methods to handle epistemic uncertainty

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- Interval analysis
- Fuzzy sets / possibility theory
- Evidence theory
- Information gap theory

Probabilistic Methods

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- Frequentist → confidence bounds
- P-boxes, imprecise probabilities
- Family of distributions
- Bayesian approach

Statistical uncertainty → Distribution type
→ Distribution parameters

Family of distributions

- Johnson
- Pearson
- Beta
- Gamma

Four-parameter families

Johnson family of distributions

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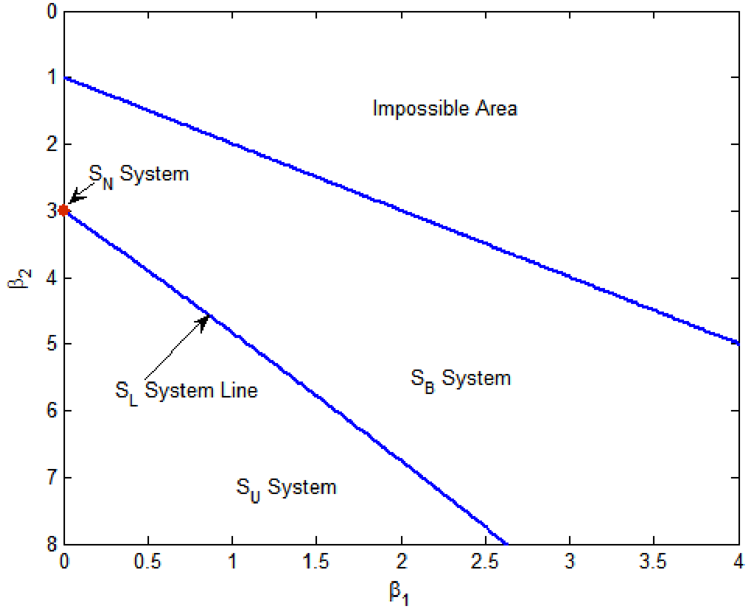
PDF: $$f_x(x) = \frac{\delta}{\lambda\sqrt{2\pi}} g'\left(\frac{x-\xi}{\lambda}\right) \exp\left\{-\frac{1}{2}\left[\gamma + \delta \cdot g\left(\frac{x-\xi}{\lambda}\right)\right]^2\right\}$$

CDF: $$F(x) = \Phi\{\gamma + \delta . g[(x-\xi)/\lambda]\}$$

Inverse CDF: $$Z = \gamma + \delta . g[(x-\xi)/\lambda]$$

Z - standard normal variate

$$g(y) = \ln(y), \text{ for lognormal } (\mathrm{S_L})$$
$$= \ln\left[y + \sqrt{y^2+1}\right], \text{ for unbounded } (\mathrm{S_U})$$
$$= \ln[y/(1-y)], \text{ for bounded } (\mathrm{S_B})$$
$$= y, \text{ for normal } (\mathrm{S_N})$$



$$\beta_1 \equiv m_3^2 / m_2^3 \qquad \beta_2 \equiv m_4 / m_2^2$$

Statistical uncertainty: Parametric approach

Case 1: Known distribution type

- Estimate distribution parameters of *X*
- Assume distribution type is known → $f_X(x|P)$
- Data D → *m* point data (x_i, i = 1 to *m*)

$$L(P) = \text{Prob}\,(D\,|\,P) = \text{Prob}\,(\;x_i\,|\,P)$$

$$= \int_{x_i-\frac{\varepsilon}{2}}^{x_i+\frac{\varepsilon}{2}} f_X(x\,|\,P) \;= \varepsilon\, f_X(x_i\,|\,P) \;\propto\; f_X(x_i\,|\,P)$$

$$L(P) \propto \prod_{i=1}^{m} f_X(x_i\,|\,P)$$

- Consider an interval (a, b) for X →

$$L(P) \propto \text{Prob}\,(D\,|\,P)$$

$$= \text{Prob}\,(x \in [a,b]\,|\,P)$$

$$= \int_a^b f_X(x\,|\,P)dx$$

- Likelihood can include both point data and interval data

$$L(P) \propto \left(\prod_{i=1}^{m} f_X(x_i\,|\,P)\right)\left(\prod_{i=1}^{n}\int_{a_i}^{b_i} f_X(x\,|\,P)dx\right)$$

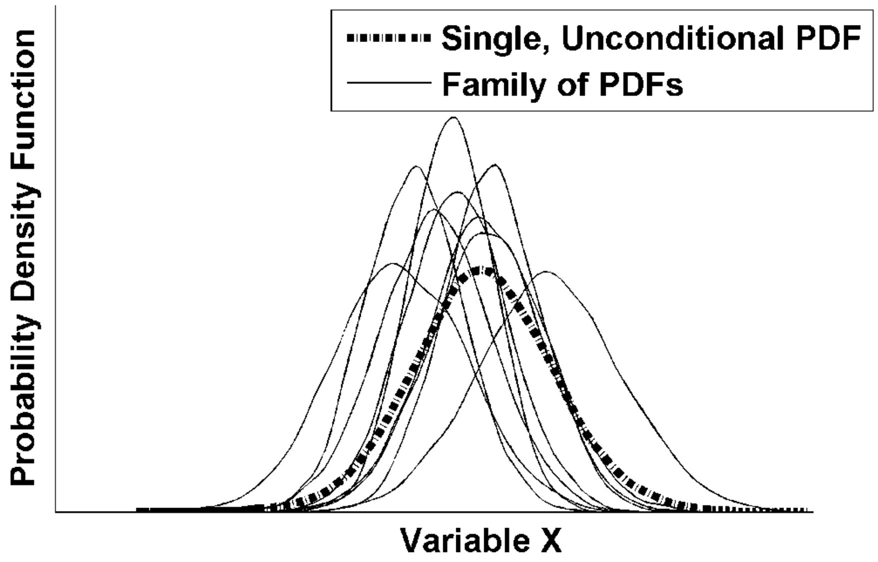
Estimation of Parameters

- Maximum Likelihood Estimate
  - Maximize $L(\boldsymbol{P})$
- To account for uncertainty in $\boldsymbol{P}$
  - Bayesian updating → Joint distribution of $\boldsymbol{P}$
  - Assume a uniform prior over the domain of $\boldsymbol{P}$

$$f(P) = \frac{L(P)}{\int L(P)}$$

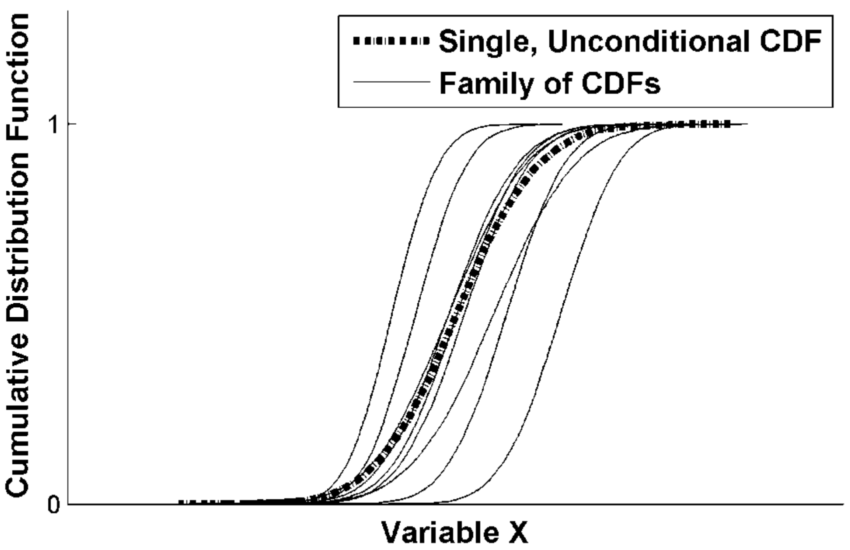
- Calculate marginal distributions of individual parameters
- PDF of X → $f_X(x|P)$ → Two loops of sampling
  - Distribution of P → $f(P)$
  - Each sample of P → PDF for X
  - Family of PDFs for X

Family vs. Single



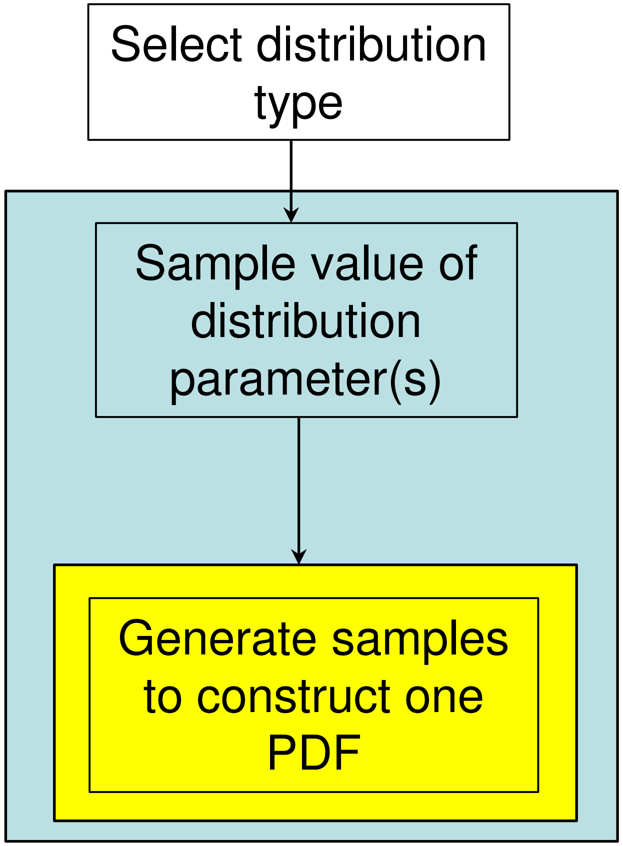
Single distribution spreads over the entire range

Includes both types of uncertainty: aleatory and distribution parameter (epistemic)



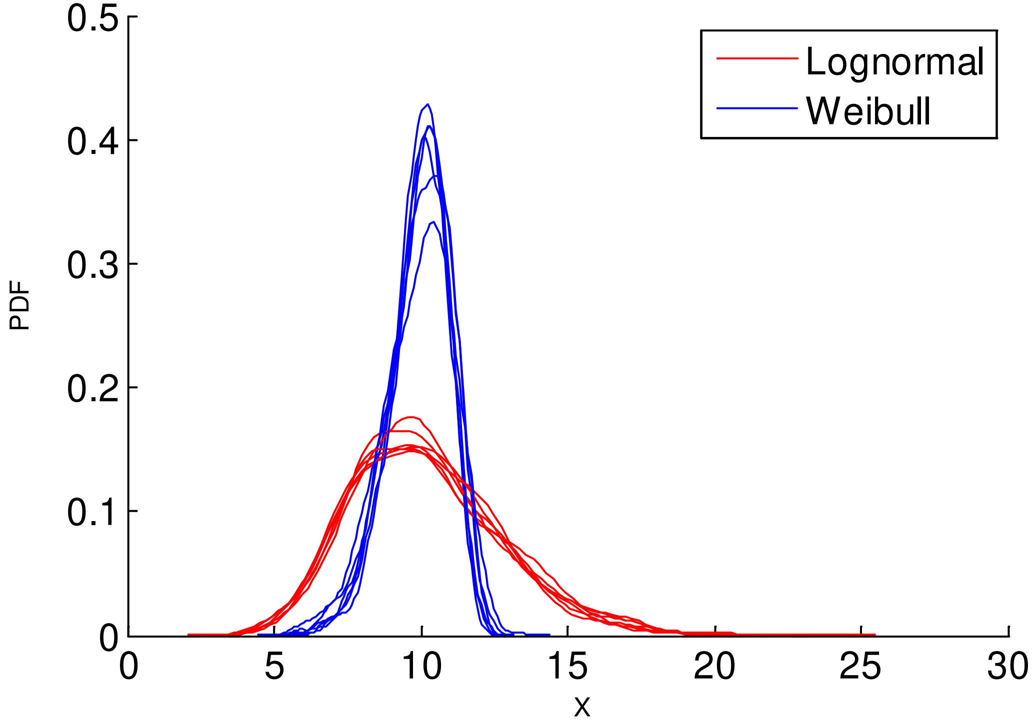
Case 2: Uncertain Distribution Type
Parametric Approach

- Distribution type → T_1 or T_2
  - Uncertainty → $P(T_1)$ and $P(T_2)$
- Given a distribution type, parameters are uncertain
  - Sample value of distribution parameter(s)
- Conditioned on the distribution type, value of parameters
  - Sample random values by inverting CDF
- Can collapse all three loops into a single loop for sampling



Example

- Variable X is either Lognormal or Weibull
- Lognormal
  - $\lambda \sim N(2.3, 0.23)$
  - $\xi \sim N(0.1, 0.01)$
- Weibull
  - $a \sim N(10.5, 1.05)$
  - $b \sim N(11.7, 1.17)$



Quantify distribution type uncertainty

- How to quantify uncertainty in a particular distribution type ?
- Compare two possible distribution types
- Two approaches
  - Bayesian model averaging → Ensemble modeling
  - Bayesian hypothesis testing → Model selection

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Bayesian model averaging

- Suppose f^1 and f^2 are two competing PDF types for X
- The corresponding parameters are ϕ and θ
- BMA assigns weight to each PDF type

$$f_X(x \mid w, \phi, \theta) = w f_X^1(x \mid \phi) + (1-w) f_X^2(x \mid \theta)$$

- Estimate PDFs of w, ϕ and θ simultaneously
  - Construct likelihood → $L(w, \phi, \theta)$ using data (D)
    - Point values → product of pdf's
    - Intervals → product of ranges of cdf values over intervals
  - Bayesian inference → $f(w, \phi, \theta \mid \mathrm{D})$

Uncertainty representation

- Physical variability → Expressed through the PDF

$$f_X(x \mid w, \phi, \theta)$$

- Distribution type uncertainty → w
- Distribution parameter uncertainty → ϕ and θ
- Unconditional distribution ➔ collapsing into single loop

$$f_X(x) = \int\int\int f_X(x \mid w, \phi, \theta) f(w, \phi, \theta \mid D) dw d\phi d\theta$$

Bayesian Hypothesis Testing

Comparing two hypotheses

Bayes' Theorem ⟶ $$\frac{P(H_0 \mid D)}{P(H_1 \mid D)} = \frac{P(D \mid H_0)P(H_0)}{P(D \mid H_1)P(H_1)}$$

Bayes Factor ⟶ $$B = \frac{P(D \mid H_0)}{P(D \mid H_1)}$$

Compute based on $f_Y(y|H_0)$ & $f_Y(y|H_1)$

Confidence in Model

$P(H_0) + P(H_1) = 1$; No prior knowledge ➔ $P(H_0) = P(H_1) = 0.5$

Probability(Model being correct) = $P(H_0|D)$ = B/B+1

Distribution type uncertainty

- Two competing distribution types
  - M_1 with parameter Φ
  - M_2 with parameter θ

$$B = \frac{P(D|M_1)}{P(D|M_2)} = \frac{L(M_1)}{L(M_2)}$$

- Straightforward to calculate
  - $L(M_1, \Phi)$ and $L(M_2, \theta)$
- Necessary to calculate
  - $L(M_1)$ and $L(M_2)$
  - By integrating out Φ and θ

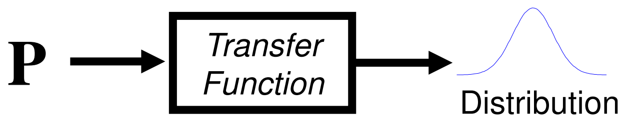
$$P(D|M_1) \propto \int P(D|M_1,\phi) f'(\phi) d\phi$$

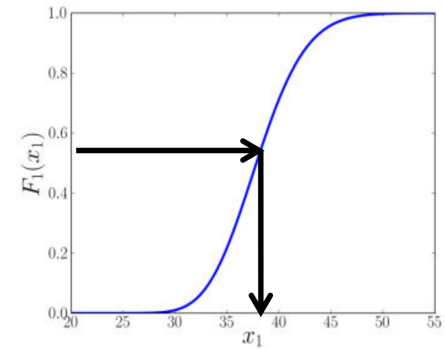
$$P(D|M_2) \propto \int P(D|M_2,\theta) f'(\theta) d\theta$$

- Simultaneously obtain posterior PDFs $f(\phi)$ and $f(\theta)$
- Inherently these PDFs are conditioned on M_1 and M_2 respectively

Aleatory and epistemic uncertainty

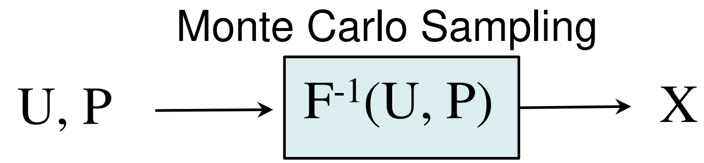
- Distribution parameter uncertainty



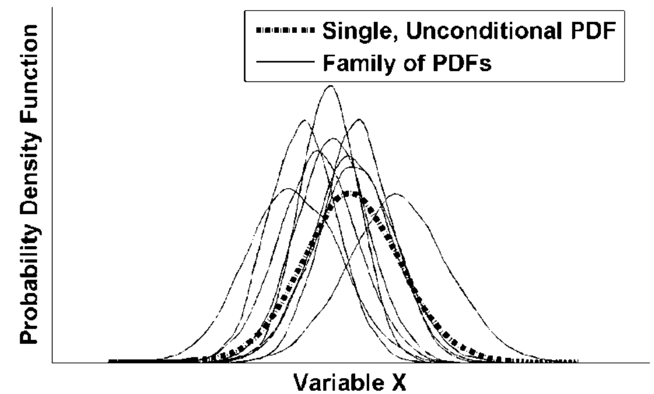


- Introduce auxiliary variable U → CDF of X

$$U = \int_{-\infty}^{X} f_X(x \mid P)\,dx$$



Uncertainty propagation → single loop sampling of aleatory and epistemic uncertainty



Sankararaman & Mahadevan, RESS, 2013

Distribution type uncertainty

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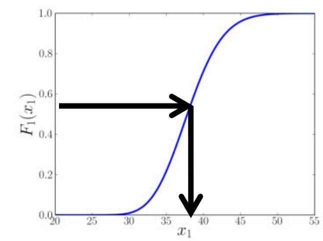
For a given distribution type D

U → auxiliary variable

$$\mu_X, \sigma_X \rightarrow f_X(x), F_X(x)$$

PDF CDF

$$\longrightarrow \quad X = F_X^{-1}(U \mid \mu_X, \sigma_X) = h(U, \mu_X, \sigma_X)$$

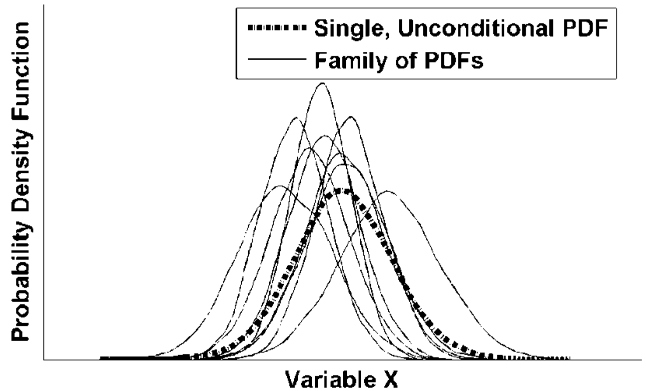


Multiple competing distributions (D_k) each with parameters $\boldsymbol{\theta}_k$

$$X = h(U, D, \theta)$$

Composite distribution

$$f_{X|\boldsymbol{\Theta}}(x|\theta) = \sum_{k=1}^{N} w_k\, f_{X|\Theta_k}(x|\boldsymbol{\theta}_k)$$



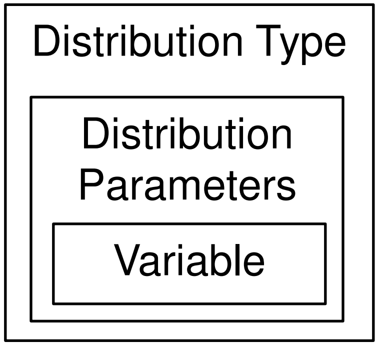
Sankararaman & Mahadevan, MSSP, 2013
Nannapaneni & Mahadevan, RESS, 2016

Sampling the input for uncertainty propagation

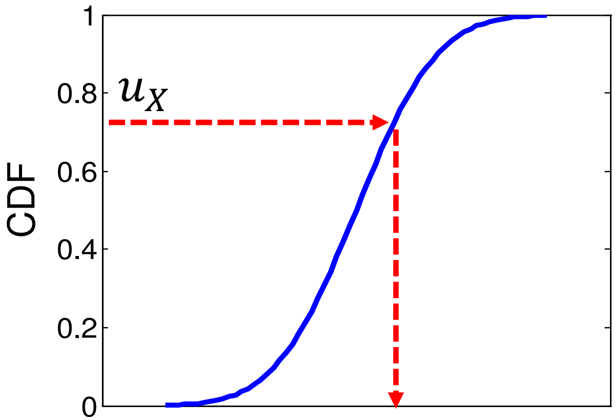
Inputs with aleatory + epistemic uncertainty

- Brute force approach - Nested three-loop sampling

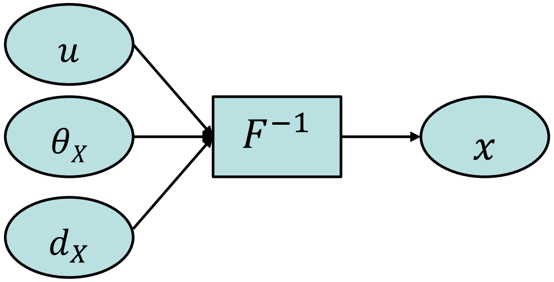
  - Computationally expensive



Auxiliary variable approach → single loop sampling



$$x = F_X^{-1}(u_X | \boldsymbol{\theta}_X, \boldsymbol{d}_X)$$



$$f_{X|\Theta}(x|\theta) = \sum_{k=1}^{N} w_k \, f_{X|\Theta_k}(x|\boldsymbol{\theta_k})$$

Sampling the multivariate input

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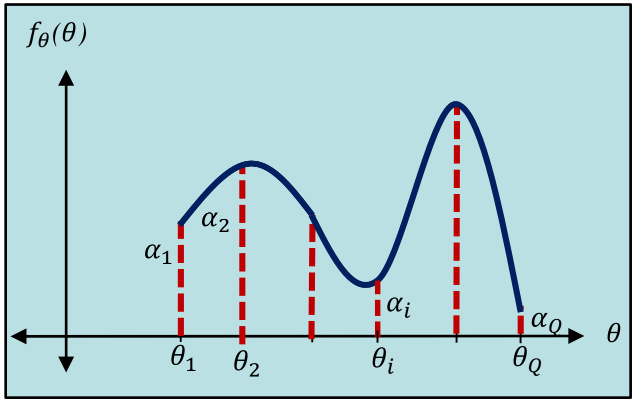
Correlation uncertainty

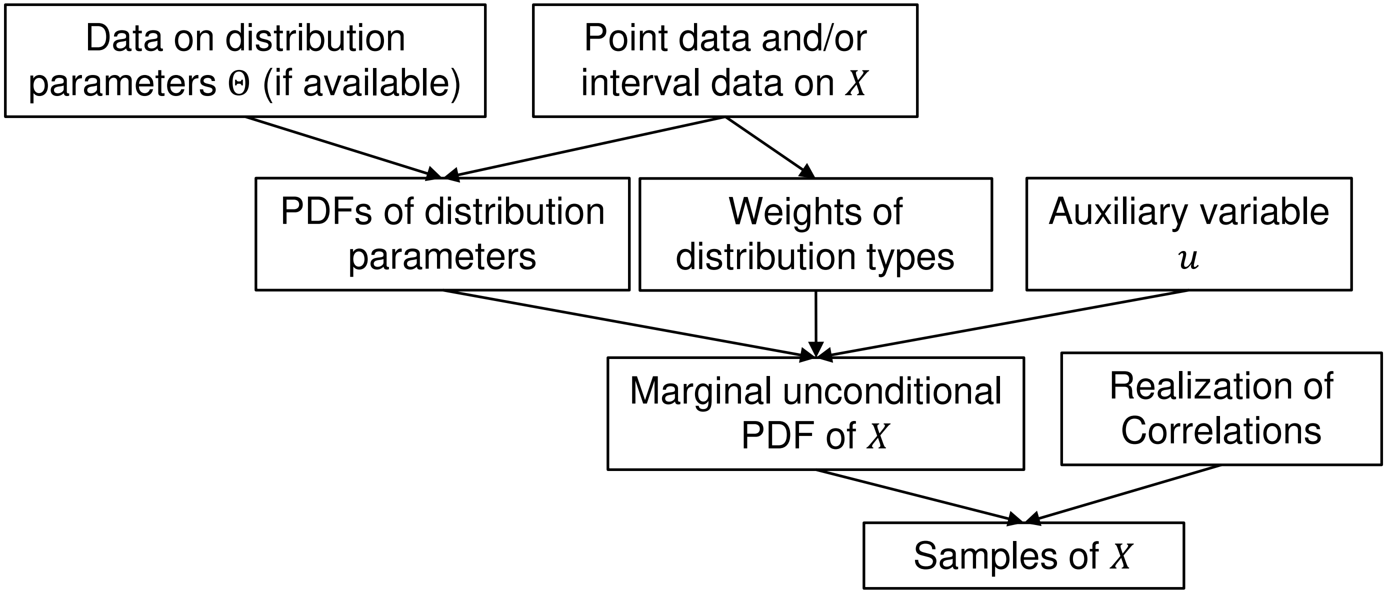
- Data on correlation coefficients
- Construct a non-parametric distribution
- Likelihood



$$L(\alpha) = \prod_{i=1}^{r} f_\Theta(\theta = \theta_p^i|\alpha) \prod_{j=1}^{s} \left[F_\Theta\left(\theta = \theta_b^i|\alpha\right) - F_\Theta(\theta = \theta_a^i|\alpha)\right]$$

- Use MLE to construct PDF of correlation coefficient
- Sample the correlation coefficient from the non-parametric distribution
- Multivariate sampling
  - Transform the correlated non-normal variables to uncorrelated normal variables
  - Sample the uncorrelated normals; then convert to original space

Summary of parametric approach



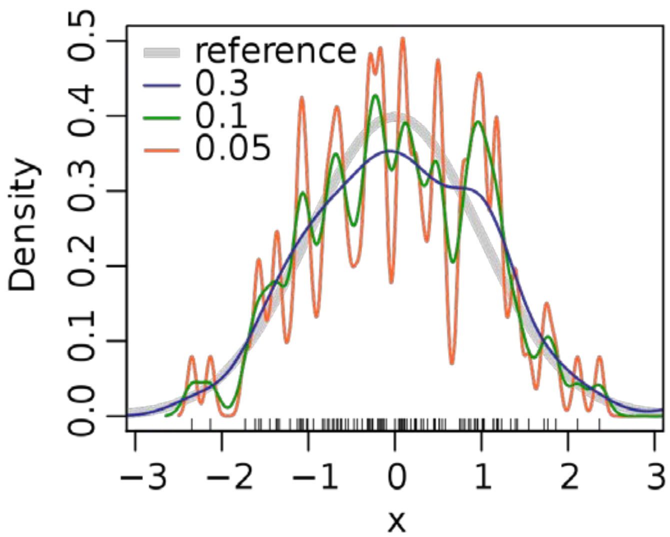
- Fitting parametric probability distributions to sparse and interval data
- Auxiliary variable
  - Distinguish aleatory and epistemic contributions
  - Facilitates sensitivity analysis
  - Supports resource allocation for further data collection

Case 2: Uncertain Distribution Type
Kernel density estimation

- Non-parametric PDF
  - $x_1, x_2, x_3 \ldots x_n$ are i.i.d samples from a PDF to be determined

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

  - K → kernel function → symmetric and must integrate to unity
  - h → smoothing parameter → "bandwidth"



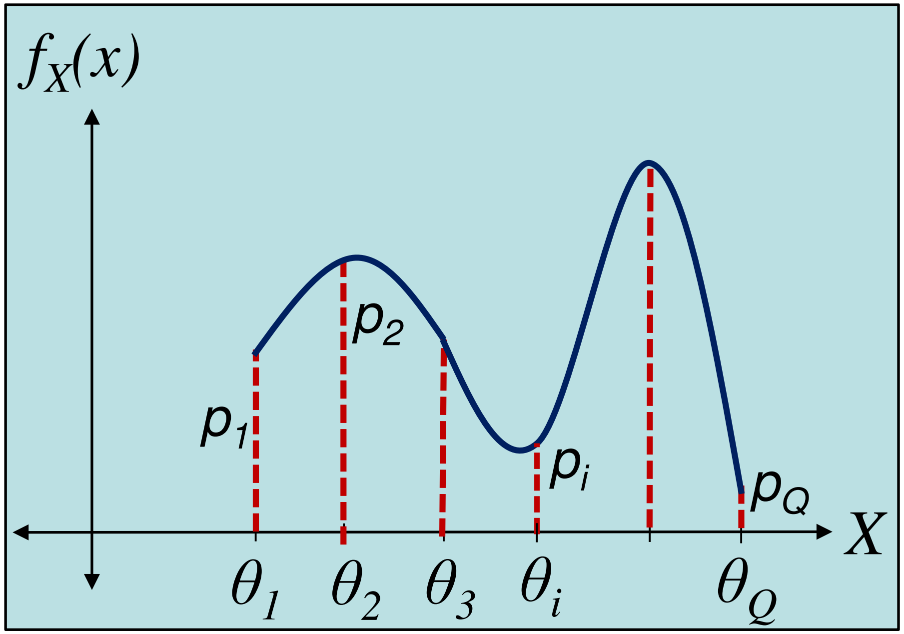
- Larger the h, smoother the PDF
- Optimal h for normal PDF → $h = \left(\frac{4\hat{\sigma}^5}{3n}\right)^{\frac{1}{5}}$
- MATLAB
  - [f, x] = ksdensity (samples)
  - plot (x, f) → PDF
- Multi-variate kernel densities available

Case 2: Uncertain Distribution Type
Likelihood-based Non-Parametric Approach

- Discretize the domain of $X \rightarrow \theta_i$, $i = 1$ to Q
- PDF values at each of these Q points known
  - $f_X(x = \theta_i) = p_i$ for $i = 1$ to Q
- Interpolation technique
  - Evaluate $f(x)$ over the domain
- Construct likelihood

$$L \propto \left(\prod_{i=1}^{n} \int_{a_i}^{b_i} f_X(x)dx \right) \left(\prod_{i=1}^{m} f_X(x_i) \right)$$

- Maximize $L \rightarrow$ Find p_i



Sankararaman & Mahadevan, RESS, 2011

Pros/cons of non-parametric approach

- Flexible framework
  - Integrated treatment of point data and interval data
  - Fusion of multiple types of information
    - Probability distributions
    - Probability distributions of distribution parameters
    - Point data, interval data
- Results in a single distribution
  - Not a family, as in the parametric approach
  - Smaller number of function evaluations for uncertainty propagation
- Cannot distinguish aleatory and epistemic uncertainty

Statistical Uncertainty: Summary

- Epistemic uncertainty regarding parameters of stochastic inputs → represented by probability distributions → family of distributions
- Three options discussed
  - Use 4-parameter distributions (families of distributions)
  - Introduce auxiliary variable to separately capture aleatory uncertainty
  - Use non-parametric distributions
- Above discussion covered sparse and imprecise data

References

1. Huang, S., S. Mahadevan, and R. Rebba, "Collocation-Based Stochastic Finite Element Analysis for Random Field Problems," *Probabilistic Engineering Mechanics*, Vol. 22, No. 2, pp. 194-205, 2007.
2. Mahadevan, S., and A. Haldar, "Practical Random Field Discretization for Stochastic Finite Element Analysis," *Journal of Structural Safety*, Vol. 9, No. 4, pp. 283-302, July 1991.
3. Hombal, V., and Mahadevan, S., "Bias Minimization in Gaussian Process Surrogate Modeling for Uncertainty Quantification," *International Journal for Uncertainty Quantification*, Vol. 1, No. 4, pp. 321-349, 2011.
4. Bichon, B.J., M. S. Eldred, L. P. Swiler, S. Mahadevan, J. M. McFarland, "Efficient Global Reliability Analysis for Nonlinear Implicit Performance Functions," *AIAA Journal*, Vol. 46, No. 10, pp. 2459-2468, 2008.
5. Li, C., and Mahadevan, S., "An Efficient Modularized Sample-Based Method to Estimate the First-Order Sobol Index," *Reliability Engineering & System Safety*, Vol. 153, No. 9, pp 110-121, 2016.
6. Sparkman, D., Garza, J., Millwater, H., and Smarslok, B., "Importance Sampling-based Post-processing Method for Global Sensitivity Analysis," Proceedings, 15th AIAA Non-Deterministic Approaches Conference, San Diego, CA, 2016.
7. Zaman, K., McDonald, M., and S. Mahadevan, "A Probabilistic Approach for Representation of Interval Uncertainty," *Reliability Engineering and System Safety,* Vol. 96, No. 1, pp. 117-130, 2011.
8. Zaman, K., M. McDonald, and S. Mahadevan, "Inclusion of Correlation Effects in Model Prediction under Data Uncertainty," *Probabilistic Engineering Mechanics*, Vol. 34, pp. 58-66, 2013.
9. Zaman, K., M. McDonald, and S. Mahadevan, "Probabilistic Framework for Propagation of Aleatory and Epistemic Uncertainty," *ASME Journal of Mechanical Design*, Vol. 33, No. 2, pp. 021010-1 to 021010-14, 2011.
10. Sankararaman, S., and S. Mahadevan, "Likelihood-Based Representation of Epistemic Uncertainty due to Sparse Point Data and Interval Data," *Reliability Engineering and System Safety*, Vol. 96, No. 7, pp. 814-824, 2011.
11. Sankararaman, S, and Mahadevan, S., "Distribution Type Uncertainty due to Sparse and Imprecise Data," *Mechanical Systems and Signal Processing*, Vol. 37, No. 1, pp. 182-198, 2013.
12. Sankararaman, S., and Mahadevan, S., "Separating the Contributions of Variability and Parameter Uncertainty in Probability Distributions," *Reliability Engineering & System Safety*, Vol. 112, pp. 187-199, 2013.
13. Nannapaneni, S., and Mahadevan, S., "Reliability Analysis under Epistemic Uncertainty," *Reliability Engineering & System Safety*, Vol. 155, No. 11, pp. 9–20, 2016.

MODEL UNCERTAINTY

Activities to address model uncertainty

- Model Verification → Numerical Error
- Model Calibration → Model parameters
- Model Selection → Model form uncertainty
- Model Validation → Model form uncertainty

Model Verification

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Code verification

- Method of manufactured solutions
- Code to code comparisons

Solution verification

ε_{num} → Numerical error
ε_{h} → Discretization error
$\varepsilon_{in\text{-}obs}$ → Input obs error
$\varepsilon_{y\text{-}obs}$ → Output obs error
ε_{su} → Surrogate model error
ε_{uq} → UQ error
ε_{mf} → Model form error

$$
\begin{aligned}
y_{obs} &= y_{pred} + \varepsilon_{pred} - \varepsilon_{y\text{-}obs} \\
&= y_{pred} + \varepsilon_{num} + \varepsilon_{mf} - \varepsilon_{exp} \\
&= g(x, \varepsilon_{h}, \varepsilon_{su}, \varepsilon_{in\text{-}obs}, \varepsilon_{uq}) + \varepsilon_{mf} - \varepsilon_{y\text{-}obs}
\end{aligned}
$$

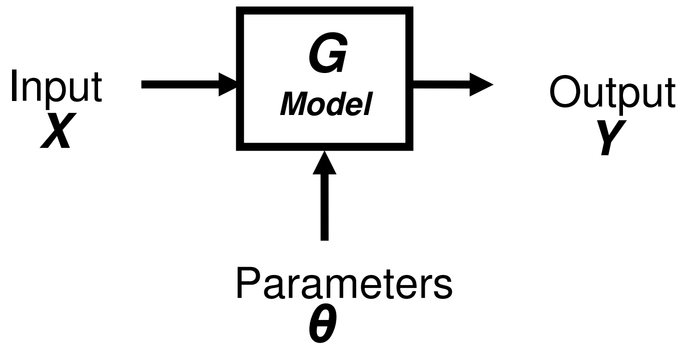
Rebba, Huang & Mahadevan, RESS, 2006
Sankararaman, Ling & Mahadevan, EFM, 2011

Use Bayesian network for systematic aggregation of errors

- Deterministic error (bias) → Correct where it occurs
- Stochastic error → Sample and add to model prediction

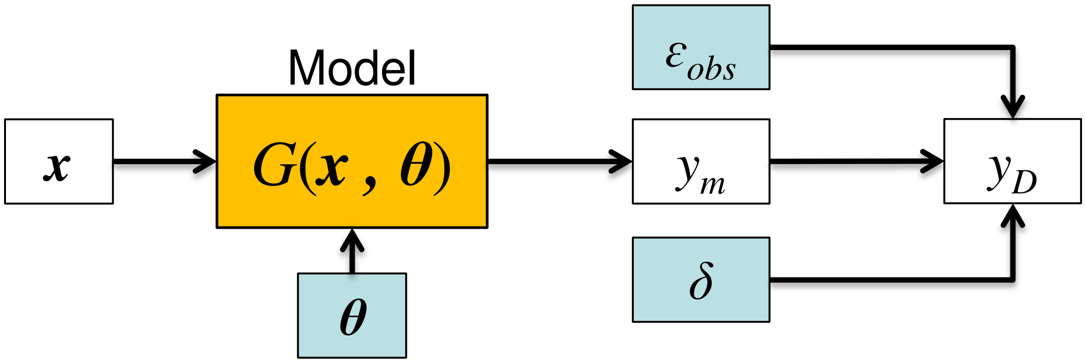
Model Calibration (Parameter estimation)

- 3 techniques
  - Least squares
  - Maximum likelihood
  - Bayesian



- Issues
  - Identifiability, uniqueness
  - Precise or Imprecise data
  - Ordered or un-ordered input-output pairs
  - Data at multiple levels of complexity
  - Dynamic (time-varying) output
  - Spatially varying parameters

Model discrepancy estimation



$$y_D = y_m + \delta + \varepsilon_{obs}$$
$$= G(x;\theta) + \delta(x) + \varepsilon_{obs}$$

Kennedy and O'Hagan, JRS, 2001

- Several formulations possible for model discrepancy:
  1. δ_1 as Constant
  2. δ_2 as i.i.d. Gaussian random variable with fixed mean and variance
  3. δ_3 as independent Gaussian random variable with input dependent mean and variance $\delta_3 \sim N(\mu(x), \sigma(x))$
  4. δ_4 as a stationary Gaussian process
  5. δ_5 as a non-stationary Gaussian process

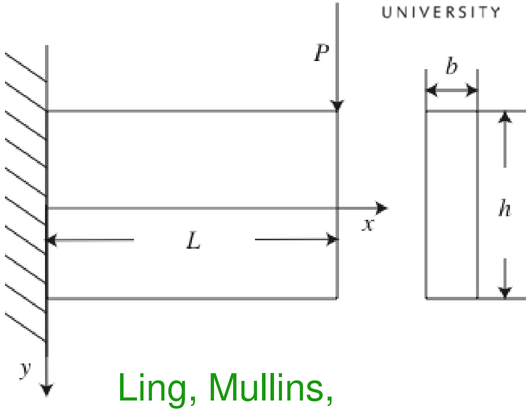
$$\delta \sim N(m(x), k(x, x'))$$

- Result depends on formulation

Ling, Mullins, Mahadevan, JCP, 2014

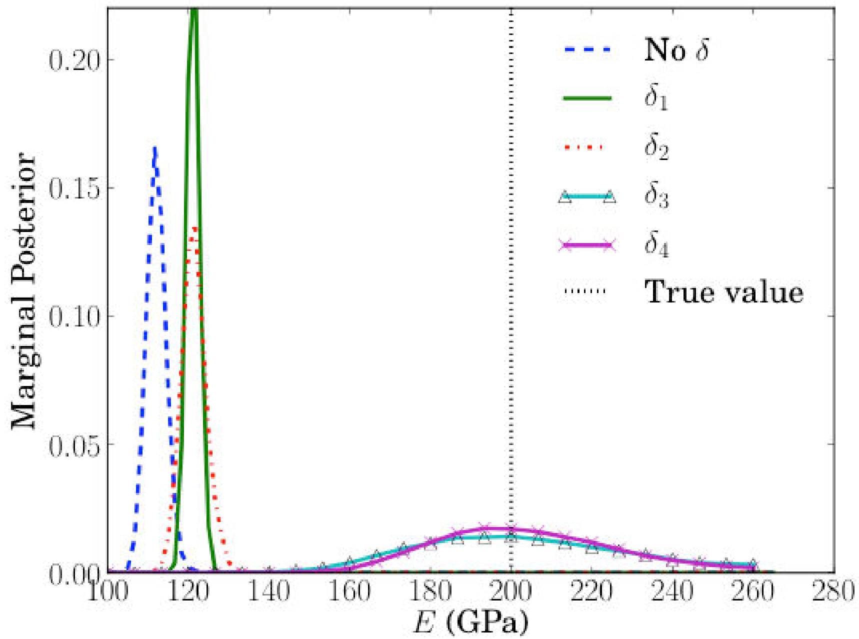
Discrepancy options with KOH

- Calibrate Young's modulus using Euler-Bernoulli beam model
- Synthetic deflection data generated using Timoshenko beam model with $P = 2.5\ \mu N$



Ling, Mullins, Mahadevan, JCP 2014

Calibration



Prediction at $P = 3.5\ \mu N$

Discrepancy	MR	MR
No d	0.5	0.5
IID Gauss	0.66	0.65
Input-dep Gauss	0.55	0.6
Stationary GP	0.36	0.95
Non-stationary GP	0.34	0.93

Input-dependent

Multi-fidelity approach to calibration

(if models of different fidelities are available)

- Need surrogate models in Bayesian calibration
- High-fidelity (HF) model is expensive
- Build surrogate for Low-fidelity (LF) model
- Use HF runs to "correct" the LF surrogate

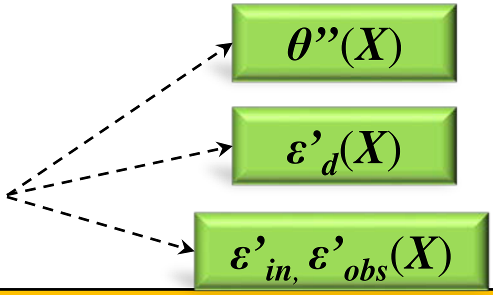
$$Y_{HF} = S_1(X + \varepsilon_{in}, \theta(X)) + \varepsilon_{surr} + \delta_{2,1}(X)$$

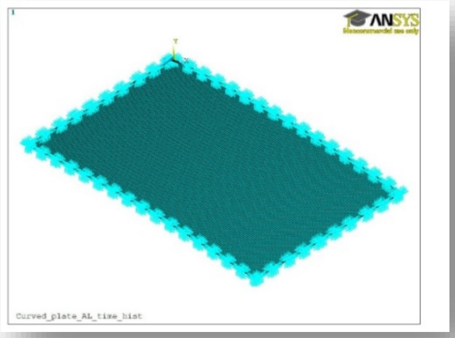
  - Pre-calibration of model parameters → Stronger priors
  - Estimation of HF-LF discrepancy

$$LF_{corr} = S_1(X + \varepsilon_{in}, \theta'(X)) + \varepsilon_{surr} + \delta'_{2,1}(X)$$

- Use experimental data to calibrate model parameters and discrepancy

$$Y_{exp} = S_1(X + \varepsilon_{in}, \theta'(X)) + \varepsilon_{surr} + \delta'_{2,1}(X) + \varepsilon_d(X) + \varepsilon_{obs}(X)$$





Hypersonic panel

Absi & Mahadevan, MSSP, 2015

Absi & Mahadevan, MSSP, 2017

Model Validation

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Quantitative Methods

1. Classical hypothesis testing
2. Bayesian hypothesis testing (equality and interval)
3. Reliability-based method (distance metric)
4. Area metric
5. K-L divergence

(2, 3) Probability measures → Useful in Uncertainty Aggregation

Bayesian hypothesis testing

- Comparison of two hypotheses (H_0 and H_1)
  - H_0 : model agrees with data, H_1 : otherwise
- Validation metric → Bayes factor

$$B = \frac{P(D \mid H_0)}{P(D \mid H_1)}$$

D → obs data

P(model agrees with data)

$$\Pr(H_0|D) = B / B+1$$

Model reliability metric

- Pred → y   Obs → z
- H_0 → $|y - z| \le \delta$
- Compute $P(H_0)$
- $P(H_1) = 1 - P(H_0)$

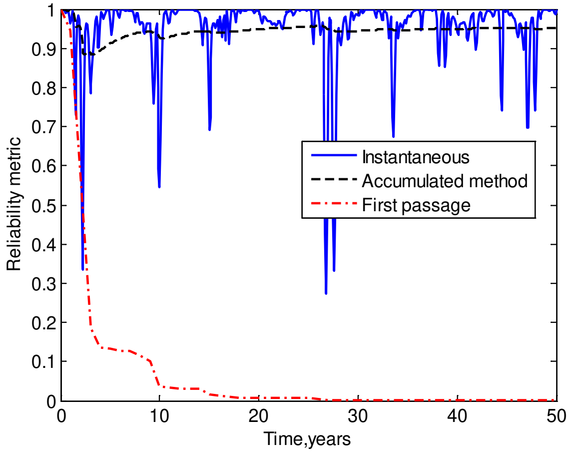
Rebba et al, RESS 2006;
Rebba & Mahadevan, RESS 2008
Mullins et al, RESS, 2016.

Model reliability metric

- Multi-dimensional → Mahalanobis distance

$$M_R = P\left(\sqrt{(\boldsymbol{z}-\boldsymbol{D}_i)^T \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1}(\boldsymbol{z}-\boldsymbol{D}_i)} < \sqrt{\boldsymbol{\lambda}^T \boldsymbol{\Sigma}_{\boldsymbol{z}}^{-1} \boldsymbol{\lambda}}\right)$$

- Input-dependent
  - Expected value
  - Random variable
  - Random field
- Time-dependent (dynamics problems)
  - Use time-dependent reliability methods
    - Instantaneous
    - First-passage
    - Cumulative



Prediction Uncertainty Quantification

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From calibration to prediction

- Same configuration and QOI → can estimate discrepancy
  - Create surrogate for discrepancy or observation
- Different configuration or QOI → KOH discrepancy cannot be propagated
- Embedded discrepancy calibration + propagation $y_D = G\big(x;\ \theta + \delta(x)\big) + \varepsilon_{obs}$
- Combine calibration and validation results
  - Uncertainty aggregation across multiple levels
  - Able to include relevance

Sankararaman & Mahadevan, RESS, 2015

Li & Mahadevan, RESS, 2016

Bayesian state estimation

- Model form error directly quantified using state estimation
- Able to transfer to prediction
  - Estimation of discrepancy at unmeasured locations
  - Estimation of discrepancy for untested, dynamic inputs
  - Translation of model form errors to untested (prediction) configurations

Subramanian & Mahadevan, JCP, MSSP, submitted

Model Uncertainty: Summary

- Several activities to address model uncertainty
  - Calibration
  - Validation
  - Selection
  - Verification (Error quantification)
- Bayesian approach to calibration and validation highlighted
- Approaches to quantify various model errors
- Rigorous approach to error combination (differentiate stochastic and deterministic errors)
- Various error/uncertainty sources can be systematically included in a Bayesian network

UNCERTAINTY AGGREGATION

Error combination: rigorous approach

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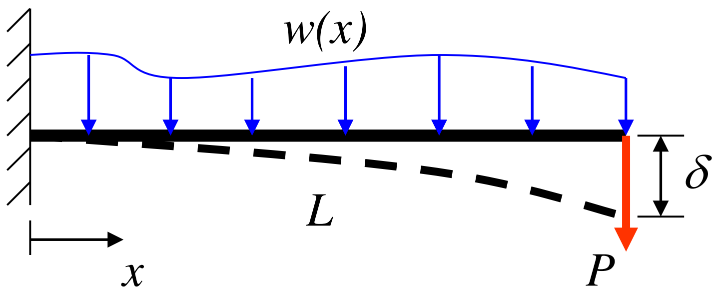
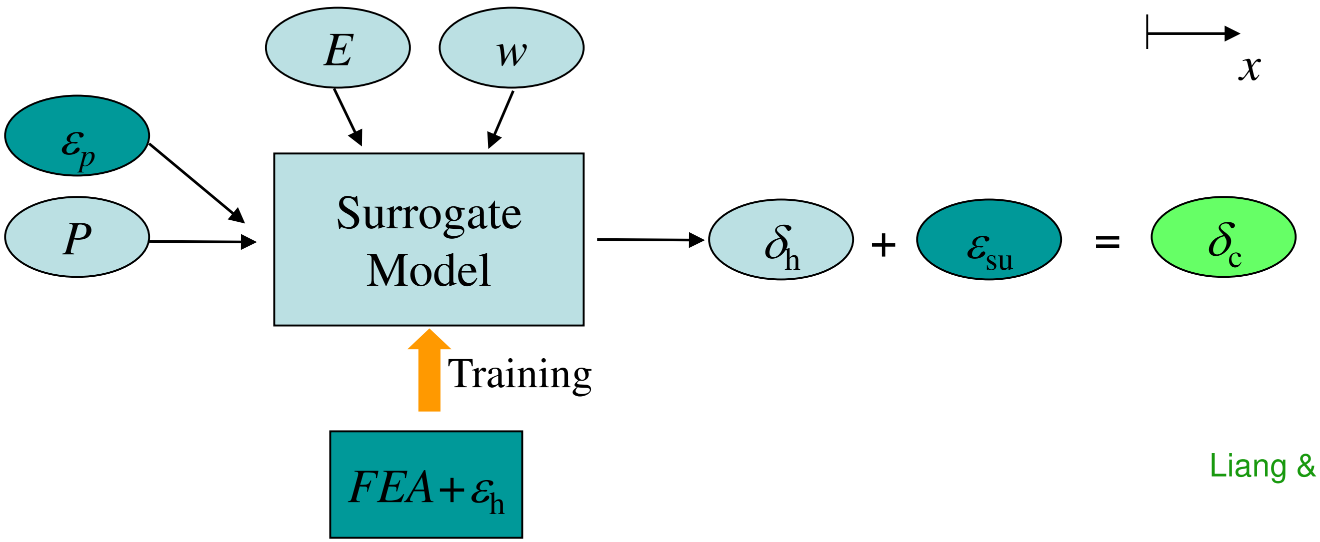
- Current methods use RMS



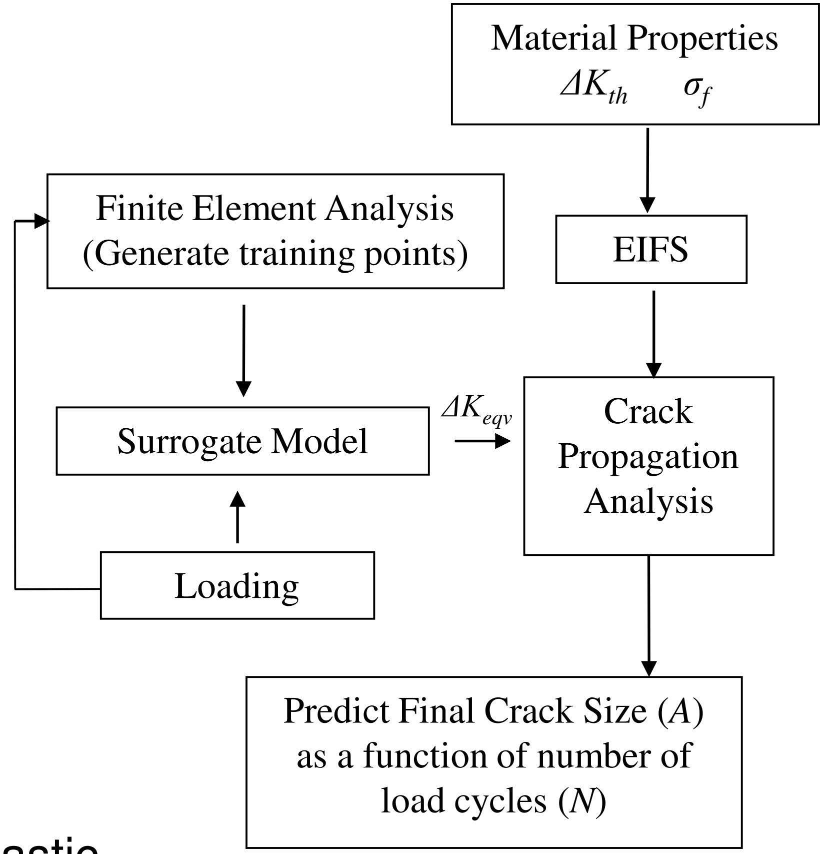


Liang & Mahadevan, IJUQ, 2011

- Correct for deterministic errors; sample stochastic errors
- Surrogate model: e.g., 2nd order polynomial chaos expansion (PCE)
- Corrected model prediction: $\delta_{\mathrm{c}} = \mathrm{PCE}_{\mathrm{h}}(P + \varepsilon_P, E, w) + \varepsilon_{\mathrm{su}}$

Multiple sources of uncertainty in crack growth prediction

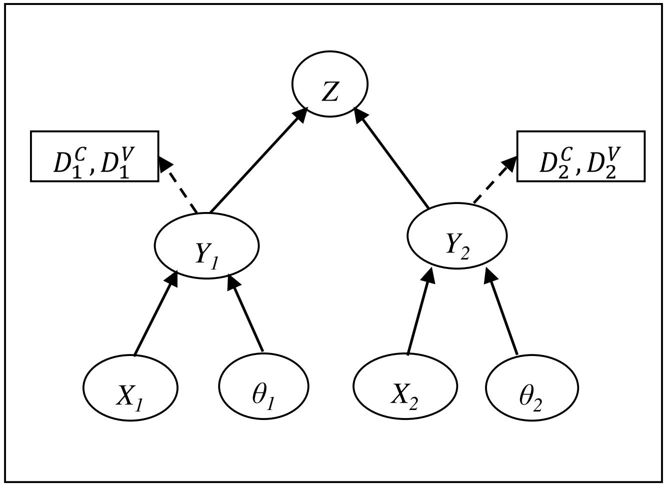
- Physical variability
  - Loading
  - Material Properties
- Data uncertainty
  - Sparse input data
  - Output measurement
- Model uncertainty/errors
  - Finite element discretization error
  - Gaussian process surrogate model
  - Crack growth law
- Complicated interactions
  - Some errors deterministic, some stochastic
  - Combinations could be non-linear, nested, or iterative
  - **Need systematic approach (e.g., Bayesian network) to aggregate uncertainty**



Aggregation of Calibration, Verification and Validation Results

Sankararaman & Mahadevan, RESS, 2015

- Verification → Numerical errors
  - "Correct" the model output
- Calibration data (D^C) → PDF's of θ
- Validation data (D^V) → $P(H_0|D^V)$
- System-level prediction → PDF of Z

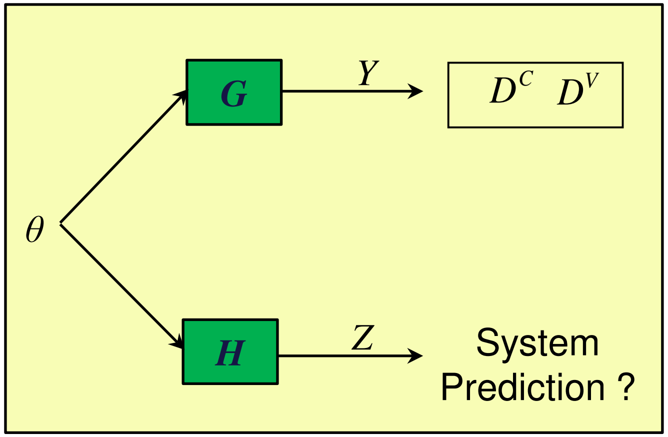


- Sequential → model output

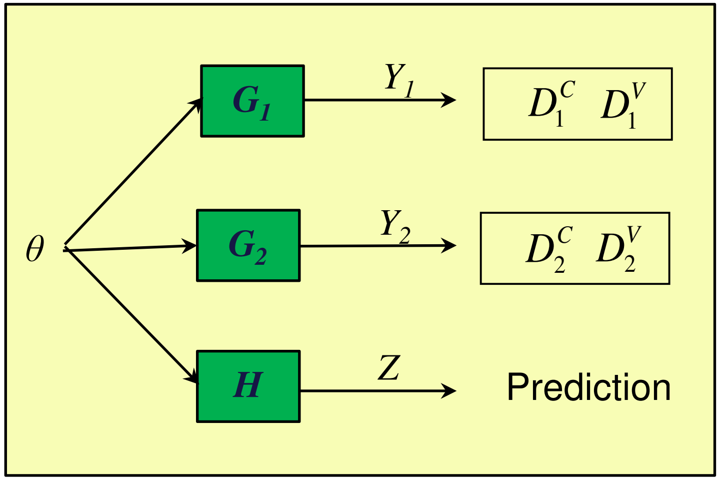
$$\bar{\pi}(y) = \Pr(H_0 \mid D^v)\pi_0(y) + [1 - \Pr(H_0 \mid D^V)]\pi_1(y)$$

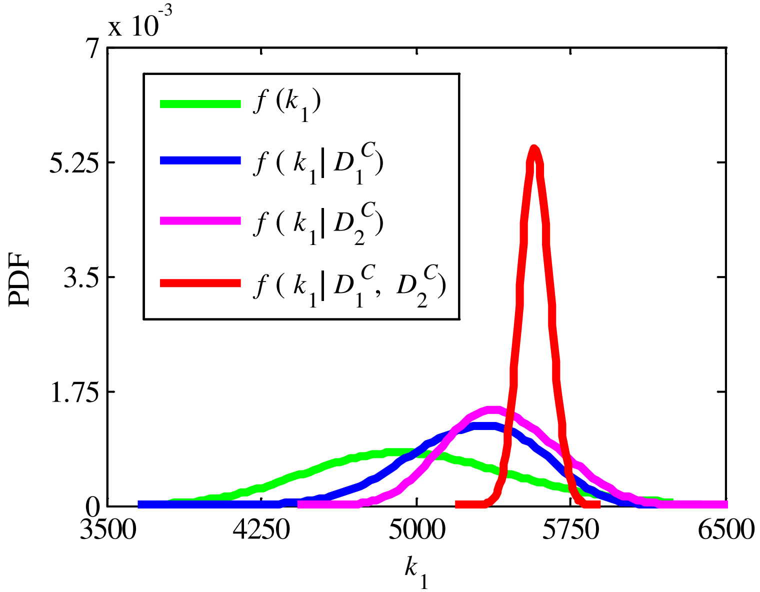
- Non-sequential → model parameter

$$\pi(\theta \mid D^C, D^V) = \pi(\theta \mid D^C, H_0)\Pr(H_0 \mid D^V) + \pi(\theta)[1 - \Pr(H_0 \mid D^V)]$$



Tests at multiple levels of complexity



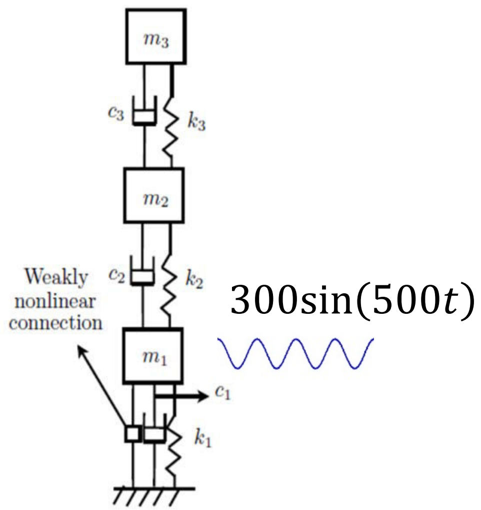


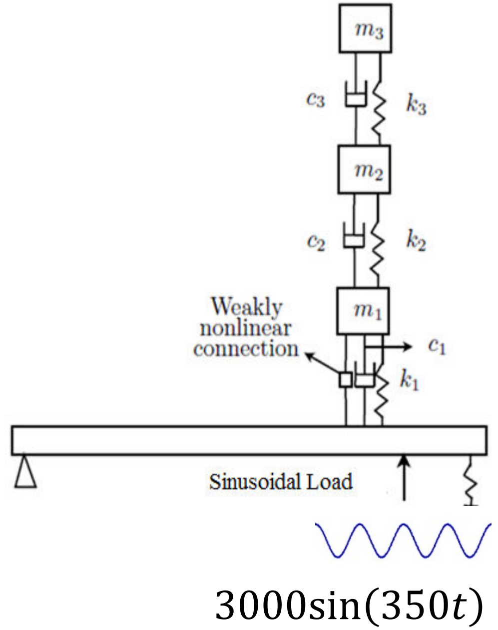
Multi-level integration

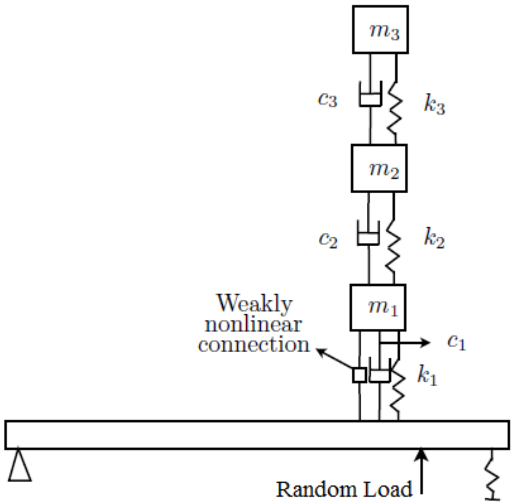
- $f\left(\theta \middle| D_1^{C,V}, D_2^{C,V}\right) = P(G_1)P(G_2)f\left(\theta \middle| D_1^C, D_2^C\right) + P(G_1')P(G_2)f\left(\theta \middle| D_2^C\right) + P(G_1)P(G_2')f\left(\theta \middle| D_1^C\right) + P(G_1')P(G_2')f(\theta)$

Sankararaman & Mahadevan, RESS, 2015

Sandia Dynamics Challenge Problem (2006)







- Level 1
  - Subsystem of 3 mass-spring-damper components
  - Sinusoidal force input on m_1
- Level 2
  - Subsystem mounted on a beam
  - Sinusoidal force input on the beam
- System Level
  - Random load input on the beam
  - Output to predict: Maximum acceleration at m_3

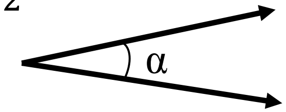
Inclusion of relevance of each level

- At each level
  - Global sensitivity analysis → vector of sensitivity indices
  - Sensitivity vector combines physics + uncertainty
  - Comparison with system-level sensitivity vector quantifies the relevance

- Relevance

Li & Mahadevan, RESS, 2016

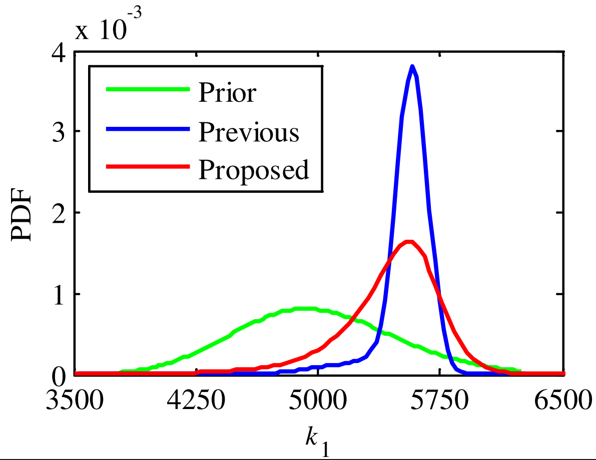
$$S_i = \left(\frac{V_{L_i} \cdot V_S}{\|V_{L_i}\| \|V_S\|} \right)^2$$



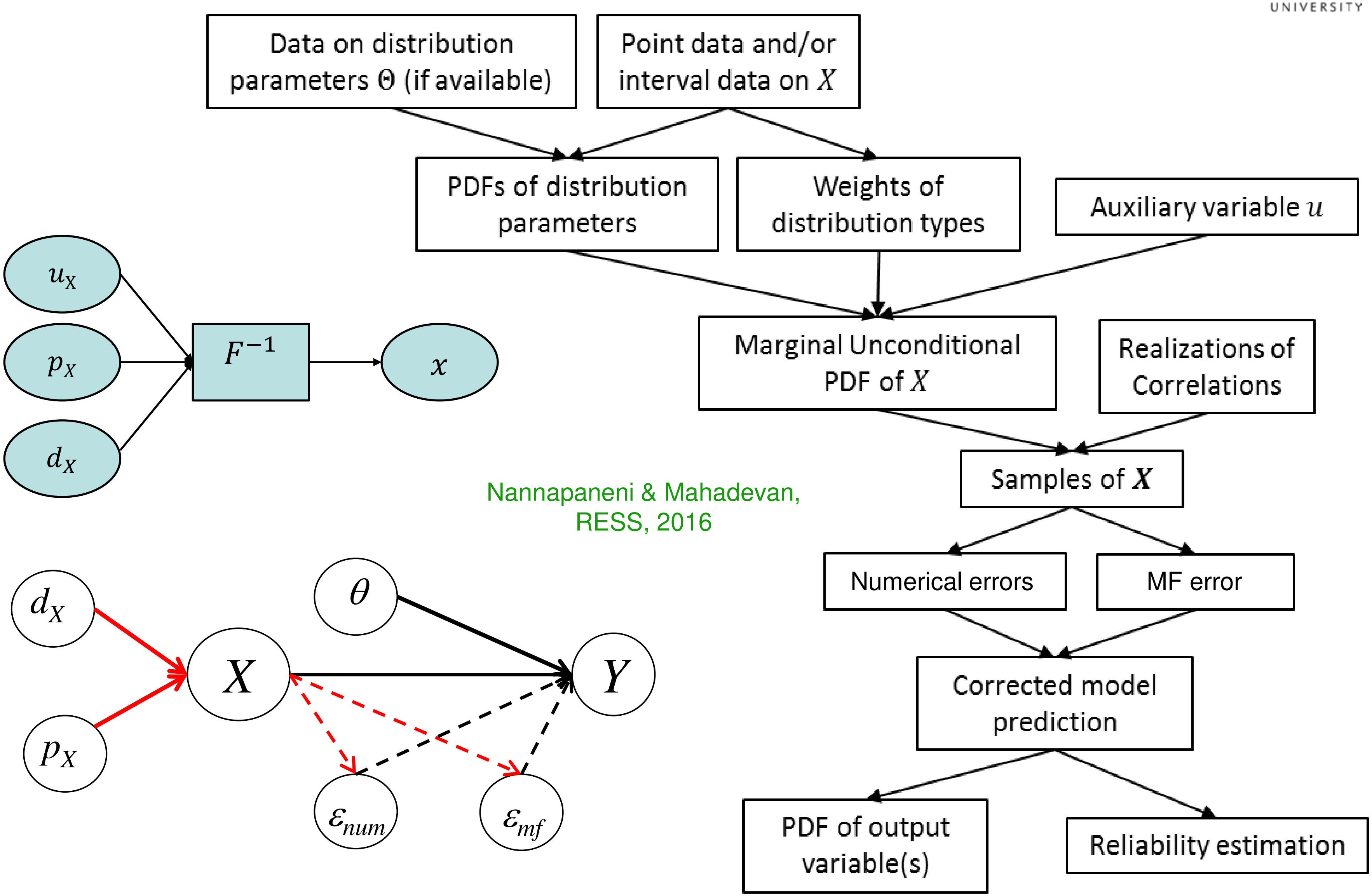
Relevance: $\cos^2(\alpha)$
Non-Relevance: $\sin^2(\alpha)$

- Integration

$$\begin{aligned} & f\left(\theta \middle| D_1^{C,V}, D_2^{C,V}\right) \\ & = P(G_1 G_2 S_1 S_2) f\left(\theta \middle| D_1^C, D_2^C\right) \\ & + P\left(G_1 S_1 \cap (G_2' \cup S_2')\right) f\left(\theta \middle| D_1^C\right) \\ & + P\left(G_2 S_2 \cap (G_1' \cup S_1')\right) f\left(\theta \middle| D_2^C\right) \\ & + P\left((G_1' \cup S_1') \cap (G_2' \cup S_2')\right) f(\theta) \end{aligned}$$



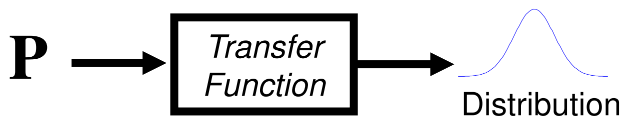
Uncertainty Aggregation Flow



Nannapaneni & Mahadevan, RESS, 2016

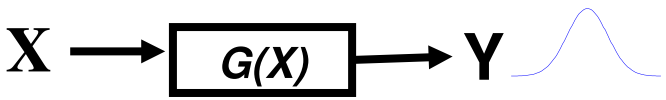
Auxiliary Variable approach

- Data uncertainty



Stochastic mapping

- Model uncertainty

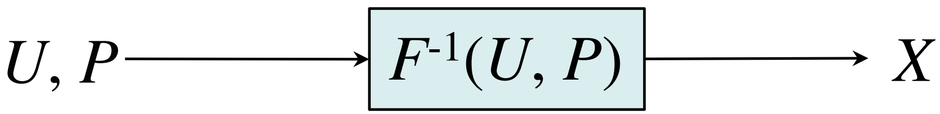


Stochastic mapping

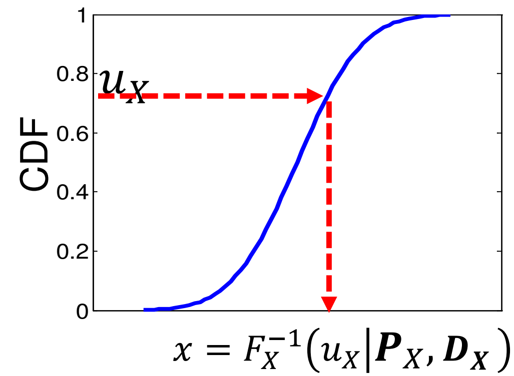
- Introduce auxiliary variable *U* (0, 1)

Sankararaman & Mahadevan, RESS, 2013

$$U = \int_{-\infty}^{X} f_X(x \mid P)\,dx$$



One-to-one mapping



- Prediction
  - Can include aleatory & epistemic sources at same level
- Uncertainty sensitivity analysis
  - Can include aleatory & epistemic sources at same level

Global Sensitivity Analysis

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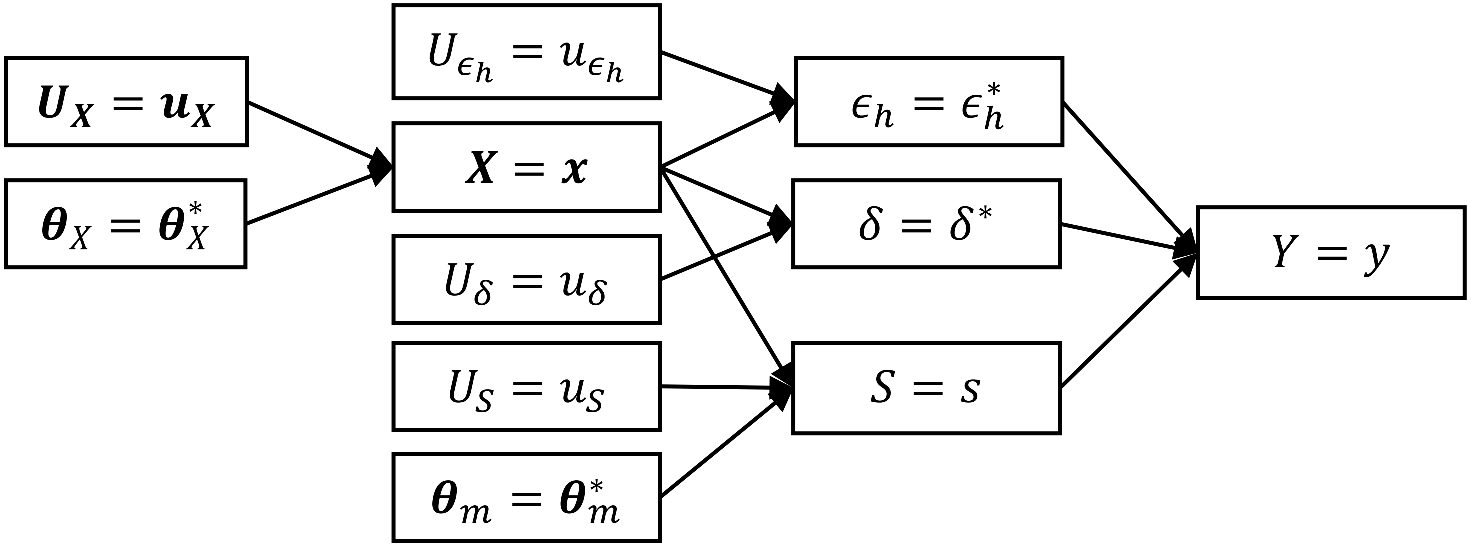


- Deterministic function for GSA:
  - $Y = F(\boldsymbol{\theta}_X, \boldsymbol{U}_X, \boldsymbol{\theta}_m, U_S, U_{\epsilon_h}, U_\delta)$
- Auxiliary variables introduced for
  - Variability in input $\boldsymbol{X}$
  - Model form error $\delta(\boldsymbol{X})$
  - Discretization error $\epsilon_h(\boldsymbol{X})$
  - Surrogate uncertainty in $S(\boldsymbol{\theta}_m, \boldsymbol{X})$

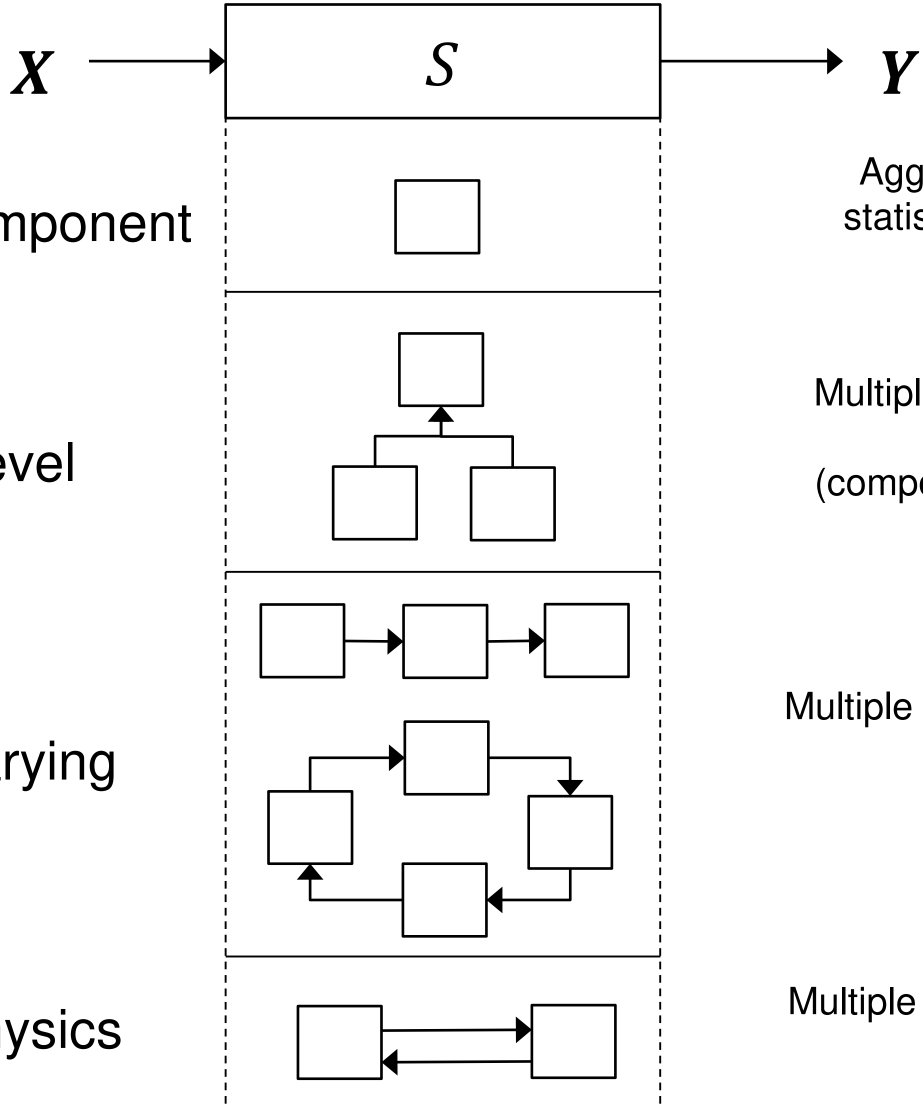
Sobol indices

$$S_i = \frac{V(E(Y|X_i))}{V(Y)}$$

$$S_i^T = 1 - \frac{V(E(Y|X_{-i}))}{V(Y)}$$

Li & Mahadevan, IJF, 2016

Uncertainty aggregation scenarios



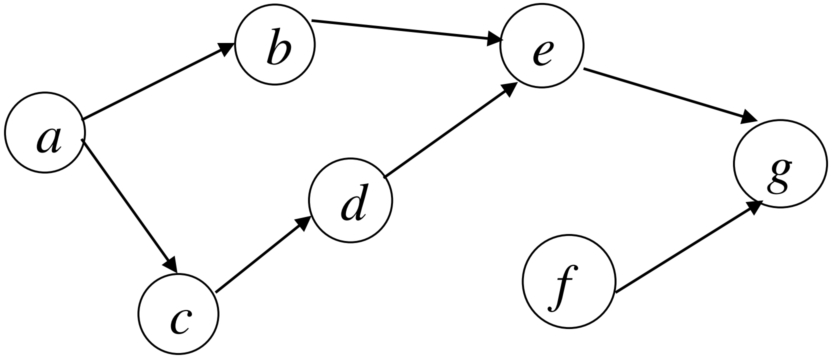
Single-component — Aggregation of input variability, statistical uncertainty, and model uncertainty

Multi-level — Multiple components organized in a hierarchical manner (components, subsystems, system)

Time-varying — Multiple components occurring in a time sequence

Multi-physics — Multiple components with simultaneous interactions

Bayesian network



a, b,….. component nodes (model inputs, outputs, parameters, errors)

g – system-level output

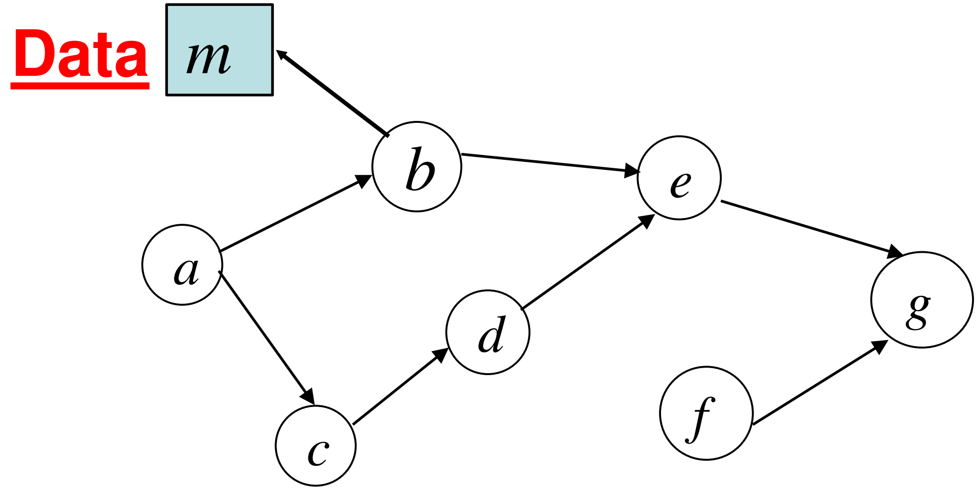
U - set of all nodes { a, b,…, g }

Joint PDF of all nodes

$$\pi(U) = \pi(a) \times \pi(b|\, a) \times \pi(c|\, a) \times \pi(d|\, c) \times \pi(e|\, b, d) \times \pi(f) \times \pi(g|\, e, f)$$

PDF of final output g

$$\pi(g) = \int \pi(U)\, \mathrm{d}a\, \mathrm{d}b \ldots \mathrm{d}f$$



With new observed data m

$$\pi(U, m) = \pi(U) \times \pi(m|\, b)$$

Bayesian network

Construction of BN

- Physics-based
  - Structure based on system knowledge
  - Learn probabilities using models & data
- Data-based
  - Learn both structure and probabilities from data
- Hybrid approach

Uses of BN

- Forward problem: UQ in overall system-level prediction
  - Integrate all available sources of information and results of modeling/testing activities
- Inverse problem: Decision-making at various stages of system life cycle
  - Model development
  - Test planning
  - System design
  - Health monitoring
  - Risk management

Data at multiple levels of complexity

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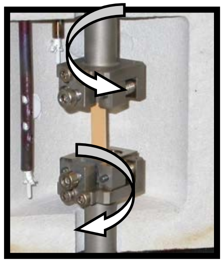
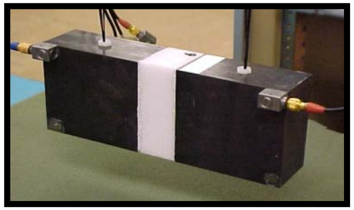
Level 0 | **Level 1** | **Level 2** | **System level**

Foam

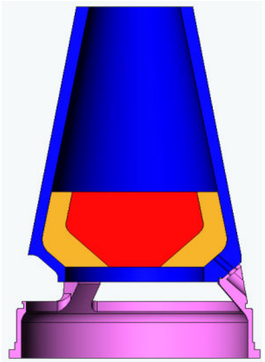


Material characterization



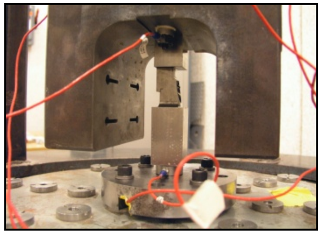
Component level

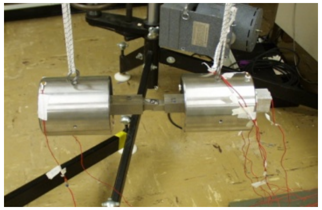
Sub-system level

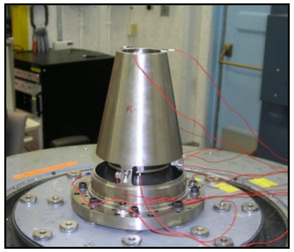


Predict peak acceleration of mass under impact load

Joints







Hardware data and photos courtesy of Sandia National Laboratories

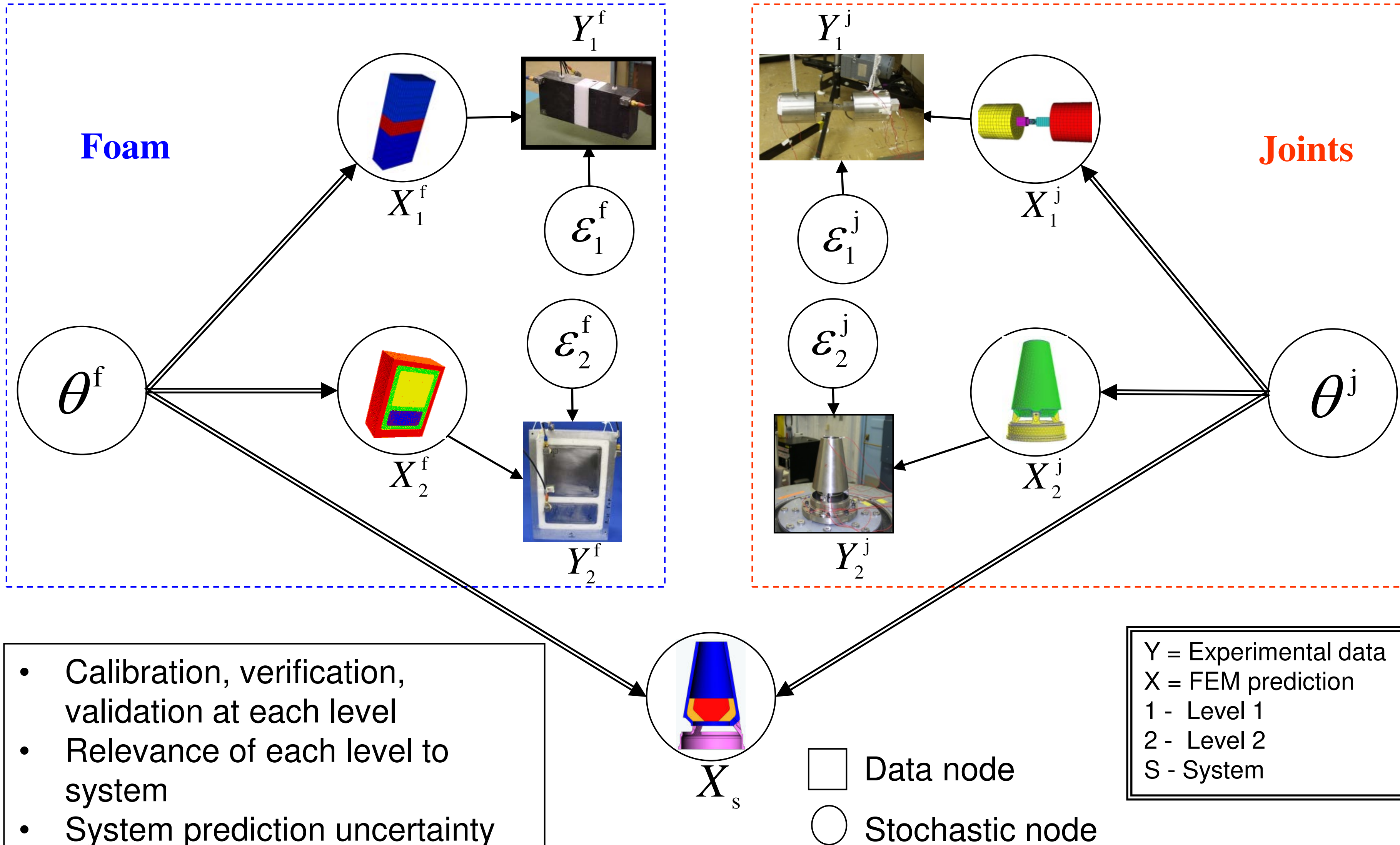
System complexity → Increases

Sources of uncertainty → Increase

Amount of real data → **Decreases**

Urbina et al, RESS, 2011

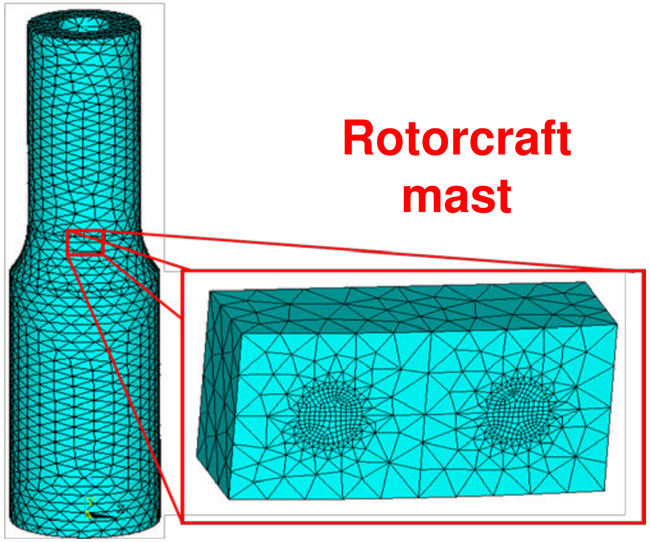
Bayesian Network for Information Fusion (No system-level data)



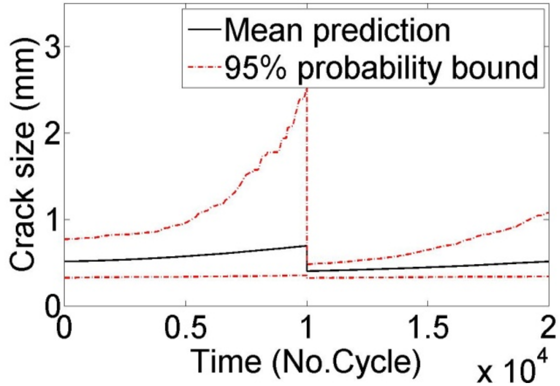
- Calibration, verification, validation at each level
- Relevance of each level to system
- System prediction uncertainty

Y = Experimental data
X = FEM prediction
1 - Level 1
2 - Level 2
S - System

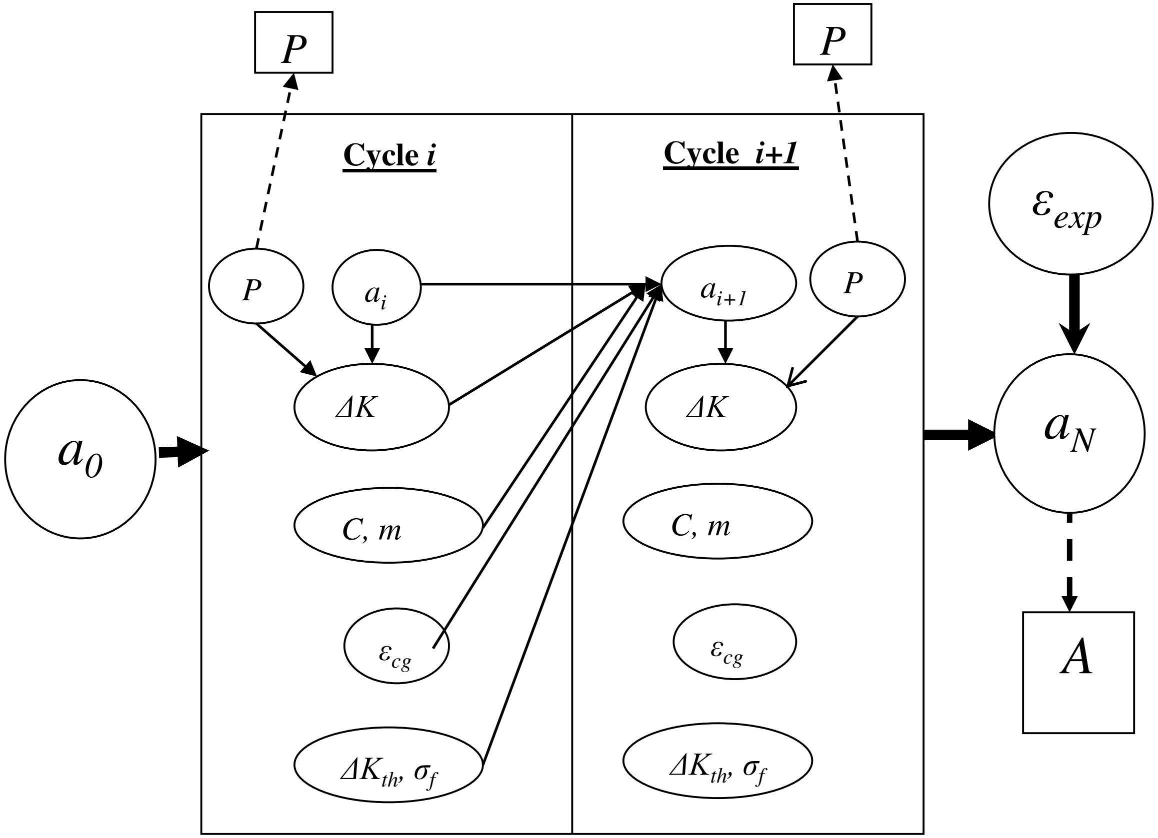
Crack growth prediction
-- Multiple models, time series data



Rotorcraft mast



Overall Crack Growth UQ



Dynamic Bayes Net

Include SHM data
- **Loads monitoring**
- **Inspection**

Sankararaman et al, EFM, 2011
Ling & Mahadevan, MSSP, 2012

Bayesian Network for MEMS UQ

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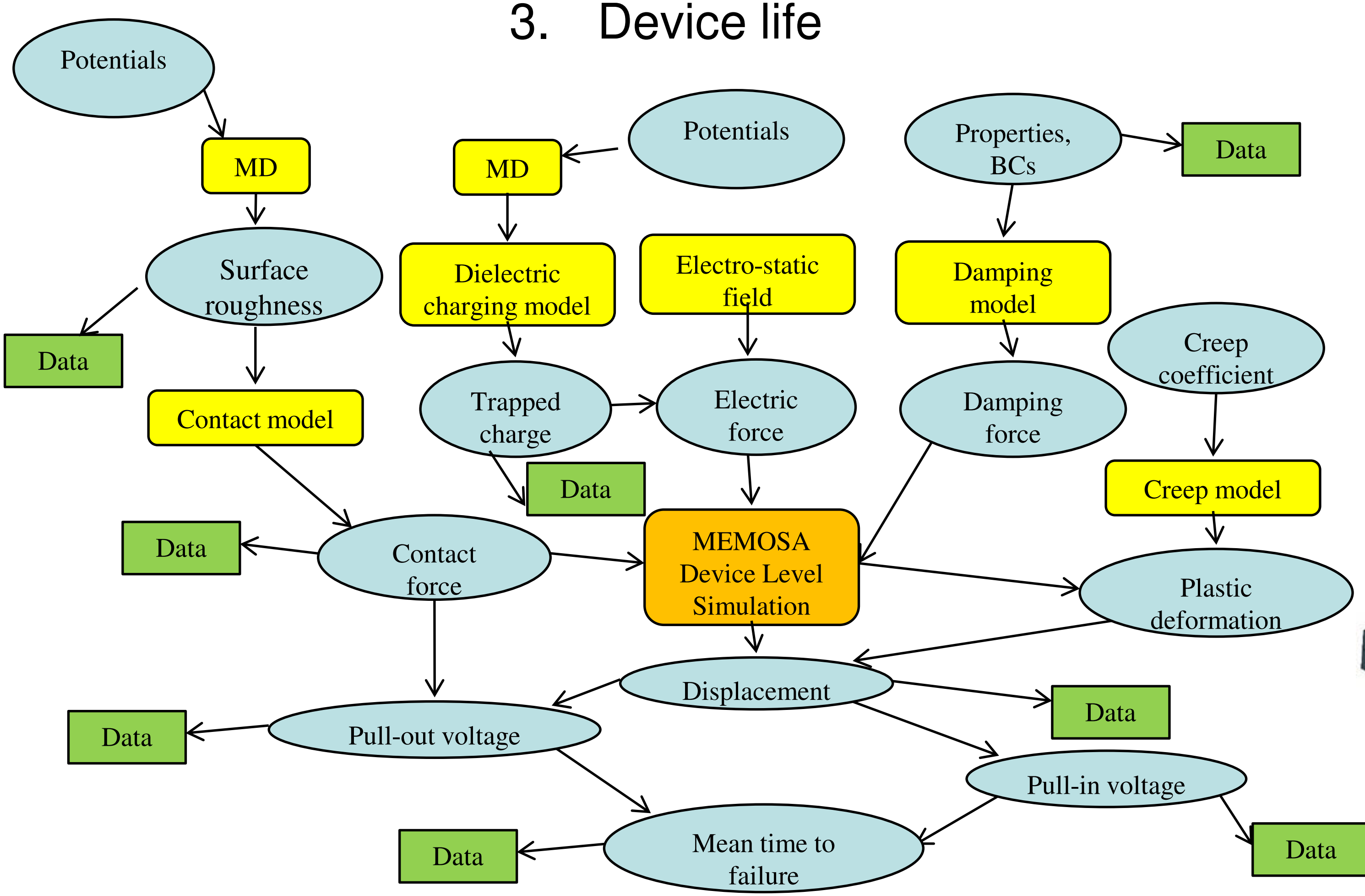
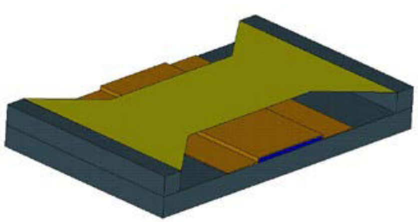
Prediction goals
1. Gap vs. voltage
2. Pull-in and pull-out voltage
3. Device life



Multiple Physics
1. Elasticity
2. Creep
3. Contact
4. Gas damping
5. Electrostatics

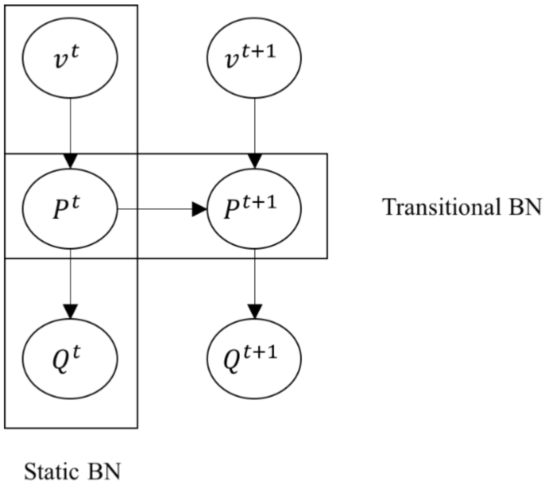


RF MEMS Switch
Purdue PSAAP

Dynamic Bayesian network

Extension of Bayesian network for modeling time-dependent systems

- Uncertainty aggregation over time
- Useful for probabilistic diagnosis and prognosis (SHM)



$$\boldsymbol{P}^{t+1} = G(\boldsymbol{P}^t, \boldsymbol{v}^{t+1})$$

$$\boldsymbol{Q}^t = H(\boldsymbol{P}^t)$$

DBN learning

Two stage learning

- Static BN learning: BN learning techniques (models, data, hybrid)

- Transitional BN learning: Models, Variable selection techniques (data)

DBN Inference

MCMC methods: Expensive

Particle filter methods

- Sequential Importance Sampling (SIS)
- Sequential Importance Resampling (SIR)
- Rao-Blackwellized filter

Analytical approximations

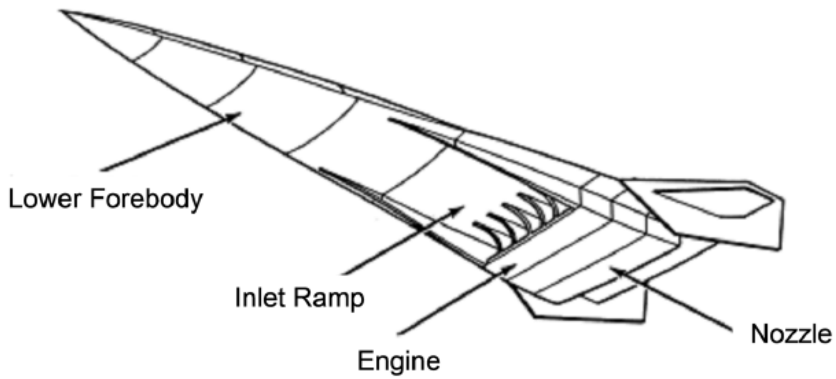
- Gaussian inputs/outputs → Kalman Filter, EKF, UKF

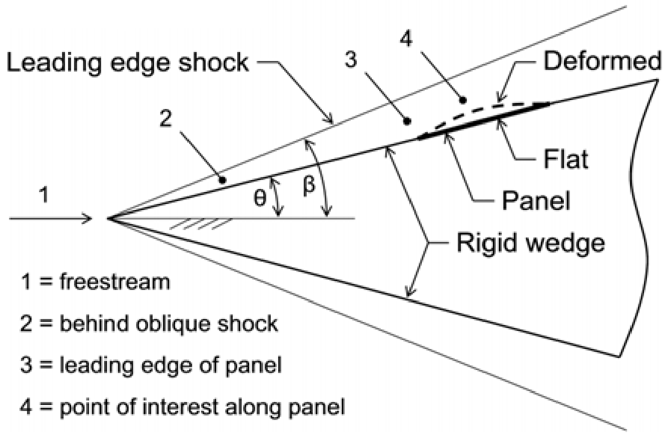
Bartram and Mahadevan, SCHM, 2014

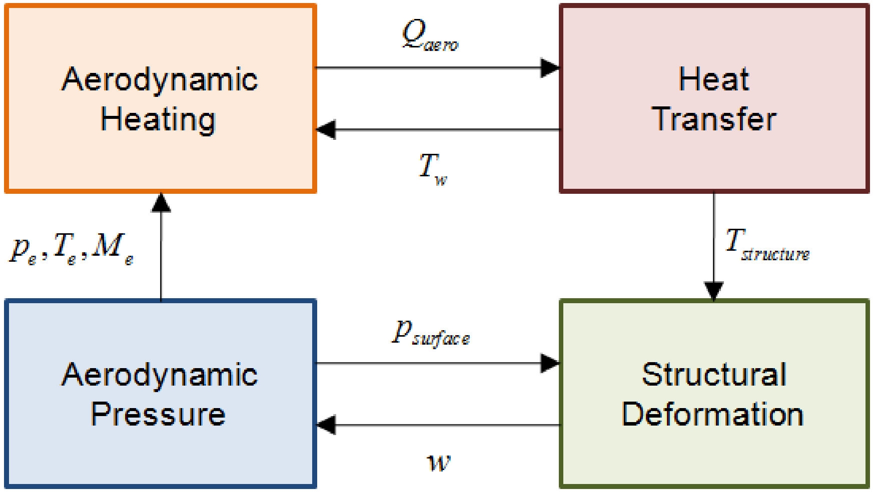
Multi-disciplinary analysis

Hypersonic aircraft panel

– Coupled fluid-thermal-structural analysis







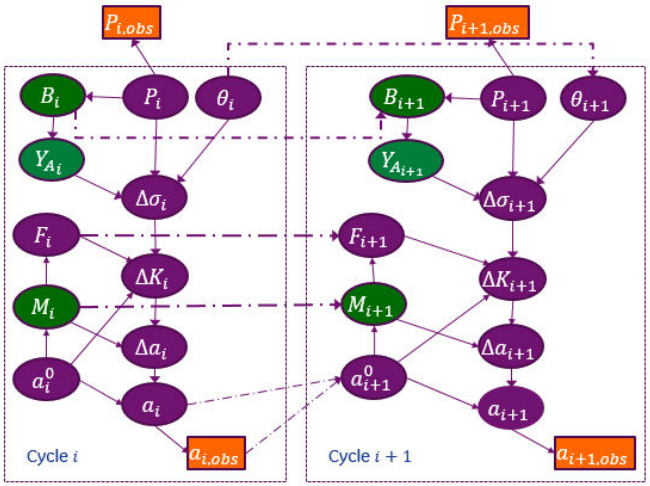
Transient analysis

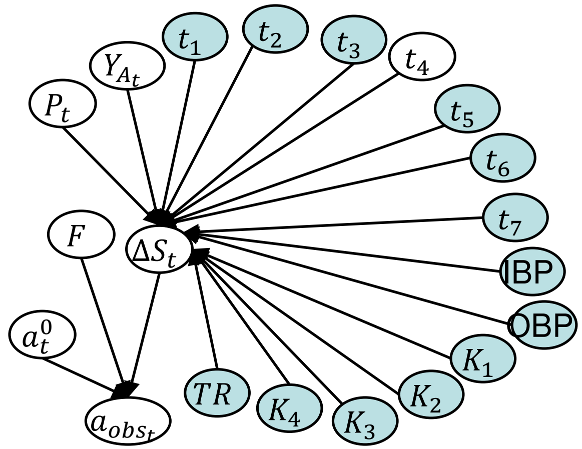
- Model error estimation in different disciplines
- Error accumulates across disciplinary models and over time
- Dynamic Bayesian network (DeCarlo et al, 2014)

Airframe Digital Twin

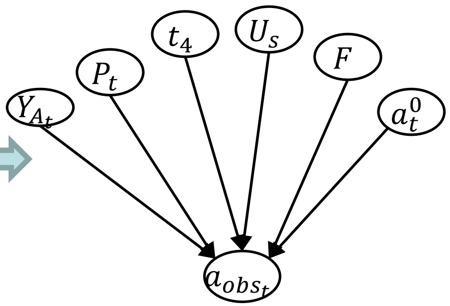
Li et al, AIAA J, 2016
Li & Mahadevan, RESS, 2018

Dynamic Bayesian Network
→ Fusion of multiple models and data sources





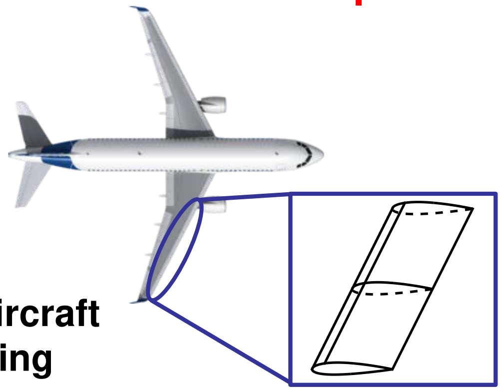
Two-layer BN + UKF

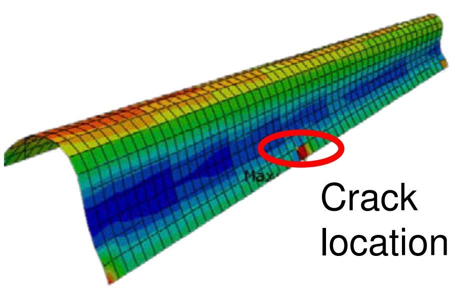


10 hrs → 2 hrs → 1 sec

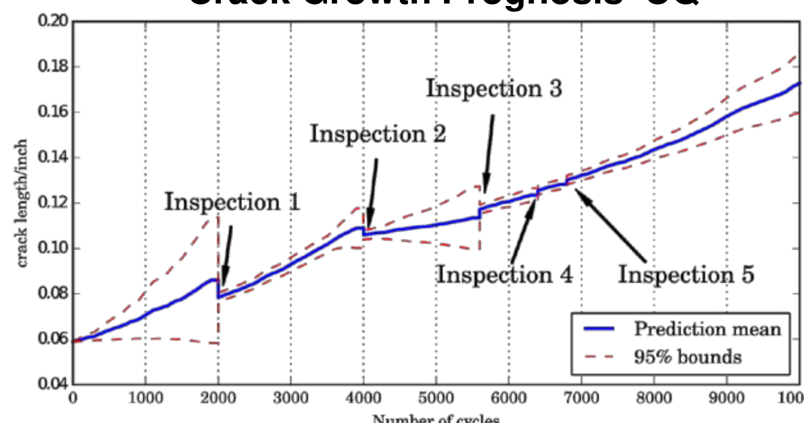
Scalability → GSA → auxiliary variable, stratified sampling
→ Collapse the BN and apply UKF

Aircraft wing





Crack Growth Prognosis UQ



Comprehensive framework for uncertainty aggregation and management

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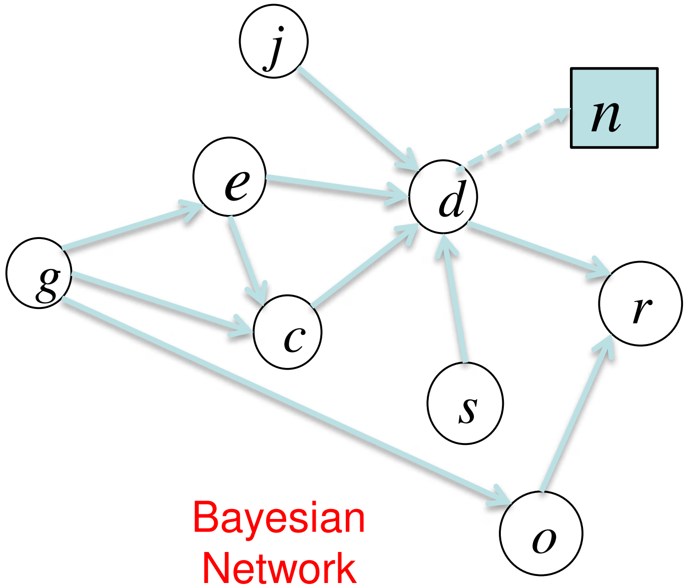
Information fusion

- Heterogeneous data of varying precision and cost
- Models of varying complexity, accuracy, cost
- Include calibration, verification and validation results at multiple levels



Bayesian Network

Facilitates

- Forward problem: Uncertainty aggregation in model prediction
  - Integrate all available sources of information and results of modeling/testing activities
- Inverse problem: Resource allocation for uncertainty reduction
  - Model development, test planning, simulation orchestration, system design, manufacturing, operations, health monitoring, inspection/maintenance/repair

References (1)

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1. Kennedy, M.C.M., and O'Hagan, A., "Bayesian calibration of computer models," Journal of Royal Statistical Society., Series B: Stat. Methodology, Vol. 63 (3), pp. 425–464, 2001.
2. Ling, Y., J. Mullins, and S. Mahadevan, "Selection of Model Discrepancy Priors in Bayesian Calibration," *Journal of Computational Physics*, Vol. 276, No. 11, pp. 665-680, 2014.
3. Sankararaman, S., and S. Mahadevan, "Model Parameter Estimation with Imprecise Input/Output Data," *Inverse Problems in Science and Engineering*, Vol. 20, No. 7, pp. 1017-1041, 2012.
4. Nath, P., Hu, Z., and Mahadevan, S., "Sensor placement for calibration of spatially varying parameters," Journal of Computational Physics, Vol. 343, pp. 150-169, 2017.
5. DeCarlo, E. C., Mahadevan, S., and Smarslok, B. P., "Bayesian Calibration of Coupled Aerothermal Models Using Time-Dependent Data," *16th AIAA Non-Deterministic Approaches Conference at AIAA SciTech*, AIAA 2014-0123, 2014.
6. Absi, G. N., and S. Mahadevan, "Multi-Fidelity Approach to Dynamics Model Calibration," *Mechanical Systems and Signal Processing*, Vol. 68-69, pp. 189-206, 2016.
7. Rebba, R., and S. Mahadevan, "Computational Methods for Model Reliability Assessment," *Reliability Engineering and System Safety*, Vol. 93, No. 8, pp. 1197-1207, 2008.
8. Zhang, R., and S. Mahadevan, "Bayesian Methodology for Reliability Model Acceptance," *Reliability Engineering and System Safety*, Vol. 80, No. 1, pp. 95-103, 2003.
9. Rebba, R., S. Huang, and S. Mahadevan, "Validation and Error Estimation of Computational Models," *Reliability Engineering and System Safety*, Vol. 91, No. 10-11, pp. 1390-1397, 2006.
10. Jiang, X., and S. Mahadevan, "Bayesian Risk-Based Decision Method for Model Validation Under Uncertainty," *Reliability Engineering and System Safety*, Vol. 92, No. 6, pp. 707-718, 2007.
11. Mullins, J., Ling, Y., Mahadevan, S., Sun, L. and Strachan, A., "Separation of aleatory and epistemic uncertainty in probabilistic model validation," *Reliability Engineering & System Safety*, Vol. 147, pp. 49-59, 2016.
12. Li, C. and Mahadevan, S., "Role of calibration, validation, and relevance in multi-level uncertainty integration," *Reliability Engineering & System Safety*, Vol. 148, pp. 32-43, 2016.
13. Mullins, J., Mahadevan, S., Urbina, A., "Optimal test selection for prediction uncertainty reduction," ASME *Journal of Verification, Validation and Uncertainty Quantification*, Vol. 1, No. 4, pp. 041002 (10 pages), 2016.
14. Ao, D., Hu, Z., and Mahadevan, S., :"Dynamics model validation using time-domain metrics," ASME *Journal of Verification, Validation and Uncertainty Quantification,* Vol. 2, No. 1, pp. 011004, 2017.

References (2)

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15. Sankararaman, S., and S. Mahadevan, "Model Validation Under Epistemic Uncertainty," *Reliability Engineering and System Safety*, Vol. 96, No. 9, pp. 1232-1241, 2011.
16. Hombal, V., and Mahadevan, S., "Model Selection Among Physics-Based Models," ASME Journal of Mechanical Design, Vol. 135, No. 2, pp. 021003 (1-15), 2013.
17. Liang, B., and S. Mahadevan, "Error and Uncertainty Quantification and Sensitivity Analysis of Mechanics Computational Models," *International Journal for Uncertainty Quantification*, Vol. 1, No.2, pp. 147-161, 2011.
18. Sankararaman, S., Y. Ling, and Mahadevan, S., "Uncertainty Quantification and Model Validation of Fatigue Crack Growth Prediction," *Engineering Fracture Mechanics*, Vol. 78, No. 7, pp. 1487-1504, 2011.
19. Sankararaman, S., and Mahadevan, S., "Integration of calibration, verification and validation for uncertainty quantification in engineering systems," Reliability Engineering & System Safety, Vol. 138, pp. 194-209, 2015.
20. Li, C. and Mahadevan, S., "Role of calibration, validation, and relevance in multi-level uncertainty integration," Reliability Engineering & System Safety, Vol. 148, pp. 32-43, 2016.
21. Urbina, A., S. Mahadevan, T. Paez,"Quantification of Margins and Uncertainties of Complex systems in the Presence of Aleatory and Epistemic Uncertainty," *Reliability Engineering and System Safety*, Vol. 96, No. 9, pp. 1114-1125, 2011..
22. Nannapaneni, S., and Mahadevan, S., "Reliability Analysis under Epistemic Uncertainty," *Reliability Engineering & System Safety*, Vol. 155, No. 11, pp. 9–20, 2016.
23. Sankararaman, S., and Mahadevan, S., "Separating the Contributions of Variability and Parameter Uncertainty in Probability Distributions," Reliability Engineering & System Safety, Vol. 112, pp. 187-199, 2013.
24. Li C., and Mahadevan, S., "Relative contributions of aleatory and epistemic uncertainty sources in time series prediction," International Journal of Fatigue, Vol. 82, pp. 474-486, 2016.
25. Li, C., and Mahadevan, S., "An Efficient Modularized Sample-Based Method to Estimate the First-Order Sobol Index," *Reliability Engineering & System Safety*, Vol. 153, No. 9, pp 110-121, 2016.
26. Li, C., and Mahadevan, S., "Sensitivity analysis of a Bayesian network," ASME Journal of Risk and Uncertainty in Engineering Systems (Part B: Mechanical Engineering), Vol. 4, No. 1, pp. 011003, 2018.
27. Li, C., Mahadevan, S., Ling, Y., Choze, S., and Wang, L., "Dynamic Bayesian Network for Aircraft Health Monitoring Digital Twin," *AIAA Journal*, Vol. 55, No.3, pp. 930-941, 2017.
28. Li, C., and Mahadevan, S., "Efficient approximate inference in Bayesian networks with continuous variables," Reliability Engineering & System Safety, Vol. 169, pp. 2690280, 2018.