Epistemic Uncertainty Management in Rare Event Probability Estimation

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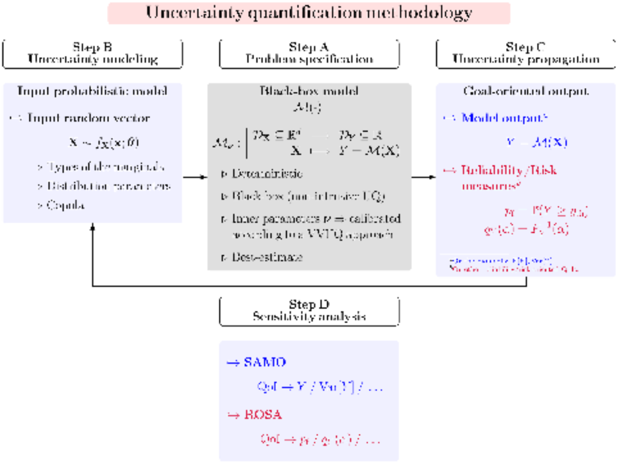
1 – Introduction

Context

- **Numerical simulation**
  - Model, design & predict the behavior of **complex engineering systems**
  - Useful to investigate "rare event" configurations (e.g., for highly-safe systems)
- **Simulators**
  - Static / Spatio-temporal
  - Low / high-fidelity
  - Deterministic / stochastic
  - ...
- **In practice**
  - Substitute to / complementary to (costly or unfeasible) experiments
  - Tainted by several sources of uncertainties
- **There is always a discrepancy between our models and the reality!**

1 – Introduction

UQ Methodology



1 – Introduction

Warning

- **Epistemic Uncertainty Management ≡ EUM**
- **What you will not find in this talk**
  - ✗ A comprehensive literature review about EU
  - ✗ The "True" and "Ultimate" definition of EU (philosophical concerns)
  - ⟹ Check these references instead[1,2,3,4]
- **What you will find in this talk**
  - ✓ A few highlights about the links between EUM and Sensitivity Analysis
  - ✓ Some practical recipes for EUM in the rare event estimation paradigm

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[1] E. Zio and N. Pedroni. *Literature review of methods for representing uncertainty, Cahiers de la Sécurité Industrielle, Report No. 2013-03*. Tech. rep. FONCSI, Toulouse, France, 2013

[2] M. Beer, S. Ferson, and V. Kreinovich. "Imprecise probabilities in engineering analyses". In: *Mechanical Systems and Signal Processing* 37 (2013), pp. 4–29

[3] N. Bousquet. "Expert elicitation and stochastic prior modeling of uncertain inputs – A rationale and some recipes". In: *ETICS 2017*. Porquerolles, France, 2017

[4] S. Destercke. "Imprecise probabilities to propagate uncertainties: a tour". In: *ETICS 2017*. Porquerolles, France, 2017

1 – Introduction

Common shared ideas

- **Sources of uncertainties**
  - Variability / randomness
    (e.g., soil mechanical properties, part manufacturing process)
  - Modeling errors / model form inadequacy
    (e.g., numerical approximation, simplified equations, scenarios)
  - Input modeling uncertainties
    (e.g., statistical uncertainty, measurement uncertainty)
- **Etymology**
  - Aleatory $\leftarrow$ *alea* (Latin) $\equiv$ "rolling of a dice"
  - Epistemic $\leftarrow$ $\epsilon\pi\iota\sigma\tau\eta\mu\eta$ (Greek) $\equiv$ "knowledge"

1 – Introduction

Common shared ideas

A first definition[5]

Risk analysts recognize two fundamentally distinct forms of uncertainty. The first is variability that arises from environmental stochasticity, inhomogeneity of materials, fluctuations in time, variation in space, or heterogeneity or other differences among components or individuals. Variability is sometimes called Type I uncertainty, or less cryptically, aleatory uncertainty to emphasize its relation to the randomness in gambling and games of chance. It is also sometimes called irreducible uncertainty because, in principle, it cannot be reduced by further empirical study (although it may be better characterized). The second kind of uncertainty is the incertitude that comes from scientific ignorance, measurement uncertainty, inobservability, censoring, or other lack of knowledge. This is sometimes called Type II uncertainty, or simply epistemic uncertainty. In contrast with aleatory uncertainty, epistemic uncertainty is sometimes called reducible uncertainty because it can generally be reduced by additional empirical effort at least in principle.

Two remarks

- EU is linked to: subjectivity, (lack-of-)knowledge, expert elicitation
- EU is still difficult to be properly defined ⇒ thus, it is hard to say what is the BEST method for EUM!

[5] S. Ferson et al. *Constructing Probability Boxes and Dempster-Shafer Structures, Sandia Report, No. SAND2002-4015*. Tech. rep. Sandia National Laboratories, USA, 2002

1 – Introduction

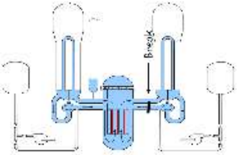
Common shared ideas

- **Aleatory vs. Epistemic: a pragmatic point of view?**[6]
  - ⟹ More a matter of "decision/choice/modeling" for the analyst than a matter of nature!
  - ⟹ Distinction useful for: "near-term" uncertainty reduction + transparency for decision-makers

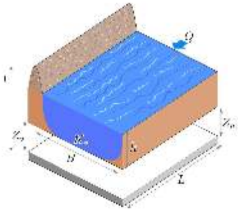
[6] A. Der Kiureghian and O. Ditlevsen. "Aleatory or epistemic? Does it matter?". In: *Structural Safety* 31.2 (2009), pp. 105–112

2 – Basics of rare event probability estimation

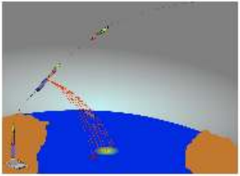
Industrial examples



(a) IBLOCA scenario (CEA - EDF)



(b) Flood protection dike (courtesy M. Keller, EDF)



(c) Launcher's fallout (ONERA)

Figure: Examples of critical systems.

Industrial contraints

- ▷ Critical systems: if failure ⇒ dramatic consequences
- ▷ Highly-safe systems: no/a few observed failure(s) ⇒ rare events

2 – Basics of rare event probability estimation

Problem specification

- Black-box model (i.e., non-intrusive UQ):

$$\mathcal{M} : \left| \begin{array}{ccc} \mathcal{D}_{\mathbf{X}} \subseteq \mathbb{R}^d & \longrightarrow & \mathcal{D}_Y \subseteq \mathbb{R} \\ \mathbf{X} & \longmapsto & Y = \mathcal{M}(\mathbf{X}) \end{array} \right. \quad (1)$$

  $\Longrightarrow$ Supposed to be "verified, validated, calibrated" (VV&UQ)[7]
- Probabilistic modeling of the d-dimensional input random vector $\mathbf{X} \sim f_{\mathbf{X}}$
  $\Longrightarrow$ Needs to identify the marginal distributions (type and parameters) and the copula[8,9]
- Definition of a **variable of interest** Y (scalar output)
- Focus on a specific **quantity of interest** (**QoI**)
  $\Longrightarrow$ central tendency (e.g., $\mathbb{E}[Y], \mathrm{Var}[Y]$) vs. tail (e.g., $\mathcal{R}(Y)$)

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[7] M. Couplet. "Validation of numerical models by statistics". In: *ETICS 2018*. Roscoff, France, 2018

[8] R. Lebrun. "Contributions à la modélisation de la dépendance stochastique". (in English). PhD thesis. Université Paris-Diderot – Paris VII, 2013

[9] N. Benoumechiara. "Traitement de la dépendance en analyse de sensibilité pour la fiabilité industrielle". (in English). PhD thesis. Sorbonne Université, 2019

2 – Basics of rare event probability estimation

Reliability assessment

- Limit-state function (LSF) ⇔ failure / safety of the system:

$$g : \mathbb{R}^d \to \mathbb{R}, \ \mathbf{X} \mapsto g(\mathbf{X}) = y_{th} - \mathcal{M}(\mathbf{X}) = y_{th} - Y, \quad y_{th} \in \mathbb{R} \tag{2}$$

  which allows to define the *failure domain*: $\mathcal{F}_{\mathbf{x}} = \{\mathbf{x} \in \mathcal{D}_{\mathbf{X}} \mid g(\mathbf{x}) \leq 0\}$

- Failure probability:

$$p_f = \mathbb{P}(Y \geq y_{th}) = \mathbb{P}\left(g(\mathbf{X}) \leq 0\right) \tag{3}$$

$$= \int_{\mathcal{F}_{\mathbf{x}}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = \int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = \mathbb{E}_{f_{\mathbf{X}}}\left[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X})\right] \tag{4}$$

- Rare event context ⇒ typically, $p_f \propto 10^{-2}$ to 10^{-6} (even less than that for some specific applications)

2 – Basics of rare event probability estimation

- **Mapping to the standard normal space**
  - **X**-space (physical space) vs. **U**-space (standard space):

$$\mathbf{U} = T(\mathbf{X}) \Leftrightarrow \mathbf{X} = T^{-1}(\mathbf{U}) \tag{5}$$

  with $\mathbf{U} = (U_1, \dots, U_d)^\top \sim \varphi_d$ such that $\varphi_d(\mathbf{u}) = \frac{1}{(2\pi)^{d/2}} \exp\left[-\frac{1}{2}||\mathbf{u}||^2\right]$

  - In practice: use Nataf or Rosenblatt transformations, but one needs to be careful to the underlying copula![10]
  - Transformation of the LSF:

$$\mathbf{u} \mapsto \mathring{g}(\mathbf{u}) = \left(g \circ T^{-1}\right)(\mathbf{u}) \tag{6}$$

  - Failure probability:

$$p_f = \mathbb{P}\left(\mathring{g}(\mathbf{U}) \leq 0\right) = \int_{\mathcal{F}_\mathbf{u}} \varphi_d(\mathbf{u}) d\mathbf{u} = \int_{\mathbb{R}^d} \mathbb{1}_{\mathcal{F}_\mathbf{u}}(\mathbf{u}) \varphi_d(\mathbf{u}) d\mathbf{u} = \mathbb{E}_{\varphi_d}\left[\mathbb{1}_{\mathcal{F}_\mathbf{u}}(\mathbf{U})\right] \tag{7}$$

[10] R. Lebrun and A. Dutfoy. "Do Rosenblatt and Nataf isoprobabilistic transformations really differ?". In: *Probabilistic Engineering Mechanics* 24 (2009), pp. 577–584

2 – Basics of rare event probability estimation

- **Failure probability estimation algorithms**
  - **Crude Monte Carlo** (CMC) sampling: the "reference", but often intractable!
  - **First-/Second-Order Reliability Methods** (FORM/SORM): geometric approximations of the LSF in the **U**-space!
  - **Importance Sampling** (IS): shifting the sampling PDF to reduce the variance of the estimator!
  - **Subset Sampling** (SS): splitting the rare event into less rare events!
  - . . . and so many others (variants or methods)[11,12,13]

[11] J. Morio and M. Balesdent. *Estimation of Rare Event Probabilities in Complex Aerospace and Other Systems: A Practical Approach*. Woodhead Publishing, Elsevier, 2015

[12] V. Moutoussamy. "Contributions to structural reliability analysis: accounting for monotonicity constraints in numerical models". PhD thesis. Université Toulouse III – Paul Sabatier, 2015

[13] C. Walter. "Using Poisson processes for rare event simulation". PhD thesis. Université Paris Diderot – Paris VII, 2016

2 – Basics of rare event probability estimation

To keep in mind!

Rare event probability estimation is a difficult task:

- Various methods with both advantages/drawbacks [14]
- The "Art of Tuning"[15]
- Switching between **U**-space (user-friendly) vs. **X**-space (physics-friendly)
- Rareness and complexity of the failure domain
- Constrained simulation budget & control of the estimation error (guarantees)
- High-dimensional input space / strong dependence in **X**
- ...
- **Robustness of the failure probability estimate regarding EU!**

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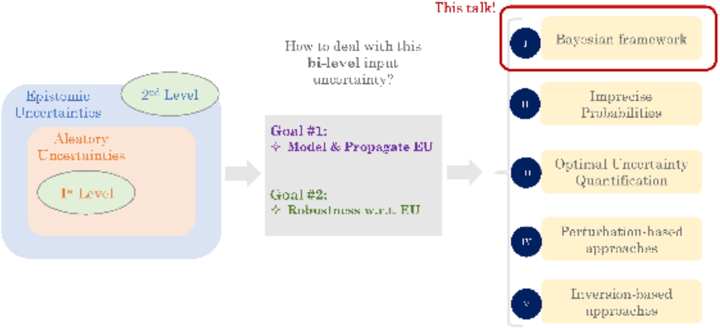
[14] R. Y. Rubinstein and D. P. Kroese. *Simulation and the Monte Carlo Method*. Second ed. Wiley, 2008

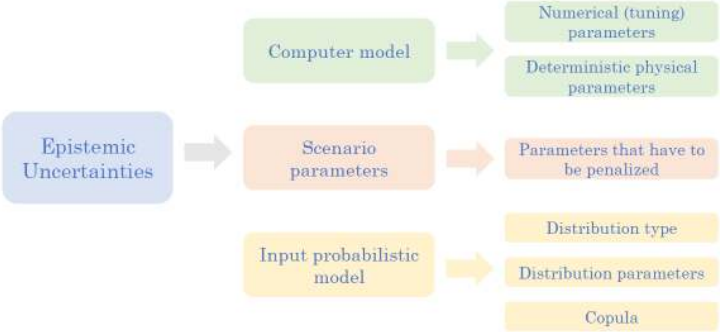
[15] M. Balesdent, J. Morio, and J. Marzat. "Recommendations for the tuning of rare event probability estimators". In: *Reliability Engineering and System Safety* 133 (2015), pp. 68–78

3 – EUM and rare event estimation: several tracks



- **Other topics at EDF**:
  Ongoing PhD Thesis of **Antoine AJENJO (II)**, recent PhD Thesis of **Jérôme STENGER (III)**, Work about **PLI indices** (Iooss, Verges, Stenger, Gauchy, Sueur, Delage) **(IV)**, **ICSCREAM** methodology (Marrel, Iooss, Chabridon) **(V)**

3 – EUM and rare event estimation: several tracks



4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- Bayesian framework $\Rightarrow$ hierarchical model in input
  (basic variables $\mathbf{X}$, distribution parameters $\boldsymbol{\Theta}$, hyperparameters $\boldsymbol{\xi}$)

$$\mathbf{X} \sim f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta}) : \mathcal{D}_{\mathbf{X}} \subseteq \mathbb{R}^d \to \mathbb{R}_+ \quad \text{(uncertainty level \#1)} \quad (8a)$$

$$\boldsymbol{\Theta} \sim f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi}) : \mathcal{D}_{\boldsymbol{\Theta}} \subseteq \mathbb{R}^{n_\theta} \to \mathbb{R}_+ \quad \text{(uncertainty level \#2)} \quad (8b)$$

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_{n_{\boldsymbol{\xi}}})^\top \in \mathbb{R}^{n_{\boldsymbol{\xi}}} \quad \text{(deterministic level)} \quad (8c)$$

- *Prior predictive* PDF $\widetilde{f}_{\mathbf{X}}$

$$\widetilde{f}_{\mathbf{X}}(\mathbf{x}) = \int_{\mathcal{D}_{\boldsymbol{\Theta}}} f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta}) f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\theta} \quad (9)$$

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

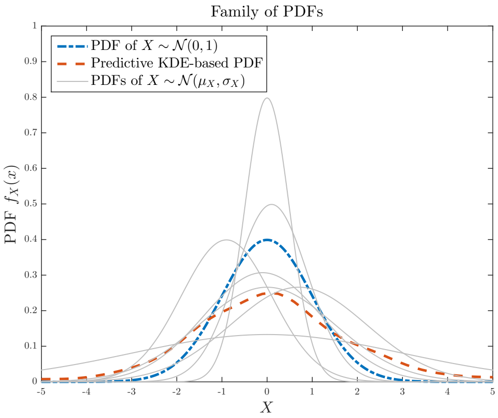


Figure: Example of a univariate predictive PDF.

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Conditional failure probability**

$$P_f(\boldsymbol{\theta}) = \mathbb{P}\left(g(\mathbf{X}) \leq 0 \mid \boldsymbol{\Theta} = \boldsymbol{\theta}\right) \tag{10a}$$

$$= \int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta}) \mathrm{d}\mathbf{x} = \mathbb{E}_{f_{\mathbf{X}|\boldsymbol{\Theta}}}\left[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X}) \mid \boldsymbol{\Theta} = \boldsymbol{\theta}\right] \tag{10b}$$

- **Predictive failure probability (PFP)**[16]

$$\widetilde{P_f}(\boldsymbol{\xi}) = \mathbb{E}_{f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}}\left[P_f(\boldsymbol{\Theta})\right] \tag{11a}$$

$$= \int_{\mathcal{D}_{\boldsymbol{\Theta}}} P_f(\boldsymbol{\theta}) f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\theta} \tag{11b}$$

$$= \int_{\mathcal{D}_{\boldsymbol{\Theta}}} \left(\int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta}) \mathrm{d}\mathbf{x}\right) f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\theta} \tag{11c}$$

[16] A. Der Kiureghian. "Analysis of structural reliability under parameter uncertainties". In: *Probabilistic Engineering Mechanics* 23.4 (2008), pp. 351–358

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Existing approaches to estimate the PFP

- **Nested Reliability Approach (NRA)**
  - "Double-loop" approach
  - Gives access to the conditional failure probabilities $P_f(\theta)$ (and thus to the distribution of $P_f(\boldsymbol{\Theta})$)
  - Investigated/used in several contexts[17,18,19]

[17] P. Limbourg, E. De Rocquigny, and G. Andrianov. "Accelerated uncertainty propagation in two-level probabilistic studies under monotony". In: *Reliability Engineering and System Safety* 95 (2010), pp. 998–1010

[18] N. Gayton et al. "APTA: advanced probability-based tolerance analysis of products". In: *Mechanics & Industry* 12 (2011), pp. 71–85

[19] M. Balesdent, J. Morio, and L. Brevault. "Rare Event Probability Estimation in the Presence of Epistemic Uncertainty on Input Probability Distribution Parameters". In: *Methodology and Computing in Applied Probability* 18.1 (2014), pp. 197–216

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Existing approaches to estimate the PFP

- **Augmented Reliability Approach (ARA)**
  - Consider an augmented input vector:

$$\mathbf{Z} := (\boldsymbol{\Theta}, \mathbf{X})^\top \sim f_{\mathbf{Z}}(\boldsymbol{\theta}, \mathbf{x}) = f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi}) f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta}) \tag{12}$$

  - Compute "simultaneously" the two integrals by sampling jointly over both domains (but respecting the conditioning $\mathbf{X}|\boldsymbol{\Theta}$)
  - ARA only investigated in the context of FORM analysis[20,21,22]

[20] A. Der Kiureghian. "Measures of Structural Safety Under Imperfect States of Knowledge". In: *Journal of Structural Engineering ASCE* 115.5 (1989), pp. 1119–1140

[21] H. P. Hong. "Evaluation of the Probability of Failure with Uncertain Distribution Parameters". In: *Civil Engineering Systems* 13 (1996), pp. 157–168

[22] M. Pendola. "Fiabilité des structures en contexte d'incertitudes statistiques et d'écarts de modélisation". (in French). PhD thesis. Université Blaise Pascal – Clermont II, 2000

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Proposed methodology

- **Basic idea of the proposed methodology**[23]
  - **Step #1**: use the ARA framework (Bayesian predictive modeling) + efficient rare event probability estimation algorithms
    $\Rightarrow$ **Modeling and propagation of bi-level input uncertainty**
  - **Step #2**: evaluate the (local) sensitivity of the PFP w.r.t. the hyperparameters of the prior
    $\Rightarrow$ **Local robustness of the PFP regarding the choice of the prior**
  - **Step #3**: perform a global sensitivity analysis using "disaggregated reliability-oriented Sobol indices"
    $\Rightarrow$ **Global study of the impact of both aleatory and epistemic uncertainties on the estimated reliability**

[23] V. Chabridon. "Reliability-oriented sensitivity analysis under probabilistic model uncertainty – Application to aerospace systems". PhD thesis. Université Clermont Auvergne, 2018

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #1: Rare event probability estimation using the ARA framework**[24]
  - One can describe NRA and ARA approaches in a unified framework (generic algorithms)
    $\Rightarrow$ **The key is in the transformation!**
  - Numerical comparison "NRA vs. ARA" and coupling with advanced rare event estimation methods (e.g., using SS or adaptive IS strategies)
    $\Rightarrow$ **NRA shows to have a smaller variance, but ARA offers a single loop with a single budget!**
  - Interesting industrial problems involving a "real" bi-level uncertainty[25]
  - {ARA framework + advanced algorithms} $\Rightarrow$ Efficient estimation of $\widetilde{P_f}$

[24] V. Chabridon et al. "Evaluation of failure probability under parameter epistemic uncertainty: application to aerospace system reliability assessment". In: *Aerospace Science and Technology* 69 (2017), pp. 526–537

[25] V. Chabridon et al. "Some Bayesian insights for statistical tolerance analysis". In: *Actes du 23ème Congrès Français de Mécanique (CFM 2017)*. Lille, France, 2017

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Application: a two d.o.f. primary/secondary damped oscillator**[26]

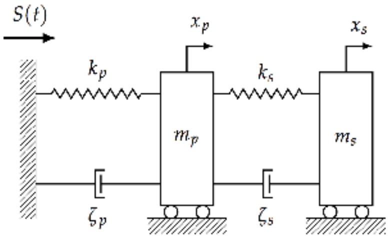


Figure: Two-degree-of-freedom damped oscillator with primary and secondary systems.

[26] A. Der Kiureghian and M. De Stefano. "Efficient Algorithm for Second-Order Reliability Analysis". In: *Journal of Engineering Mechanics ASCE* 117.12 (1991), pp. 2904–2923

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Application: a two d.o.f. primary/secondary damped oscillator**
  - LSF:

$$g(\mathbf{X}) = F_s - 3k_s\sqrt{\frac{\pi S_0}{4\zeta_s\omega_s^3}\left[\frac{\zeta_a\zeta_s}{\zeta_p\zeta_s\left(4\zeta_a^2+r^2\right)+\gamma\zeta_a^2}\frac{\left(\zeta_p\omega_p^3+\zeta_s\omega_s^3\right)\omega_p}{4\zeta\omega_a^4}\right]} = F_s - F_{\mathrm{acc}}$$

  - Input probabilistic model:

Variable	Distribution	Mean	C.v.
$X_1 = m_p$ (kg)	Lognormal	1.5	10%
$X_2 = m_s$ (kg)	Lognormal	0.01	10%
$X_3 = k_p$ $(\mathrm{N.m^{-1}})$	Lognormal	1	20%
$X_4 = k_s$ $(\mathrm{N.m^{-1}})$	Lognormal	0.01	20%
$X_5 = \zeta_p$ (1)	Lognormal	0.05	40%
$X_6 = \zeta_s$ (1)	Lognormal	0.02	50%
$X_7 = F_s$ (N)	Lognormal	μ_{X_7} uncertain	10%
$X_8 = S_0$ $(\mathrm{m.s^{-2}})$	Lognormal	100	10%
$\Theta = \mu_{X_7}$ (N)	Normal	21.5	10%

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Application: a two d.o.f. primary/secondary damped oscillator**
  - Numerical results:
    (RE ≡ relative error / RB ≡ relative bias / ν ≡ efficiency)

Approach	CMC [a]			SS [b]				
	$m_{\widetilde{P}_f}$	$s^2_{\widetilde{P}_f}$	RE	$m_{\widetilde{P}_f}$	$s^2_{\widetilde{P}_f}$	RE	RB	ν
NRA [c]	1.41×10^{-4}	1.41×10^{-10}	8.39×10^{-2}	1.59×10^{-4}	2.18×10^{-10}	9.27×10^{-2}	2.77×10^{-2}	0.18
ARA	1.53×10^{-4}	1.51×10^{-10}	6.00×10^{-2}	1.52×10^{-4}	3.01×10^{-9}	0.38	-1.06×10^{-2}	11.5

  - Reference value for the PFP: $\widetilde{P}_{f,ref} = 1.55 \times 10^{-4} > p_{f,ref} = 4.78 \times 10^{-5}$
  - ARA estimates ≈ NRA estimates
  - ARA/SS → efficiency $\nu = 11.5 \Rightarrow N_{\mathrm{CMC}}$ can be divided by 11.5

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #2**: **Evaluate the (local) sensitivity of the PFP w.r.t. the hyperparameters of the prior**
  - **Goal**: get an estimator of

$$\nabla \widetilde{P_f}(\boldsymbol{\xi}) = \left(\frac{\partial \widetilde{P_f}(\boldsymbol{\xi})}{\partial \xi_j}, j = 1, \dots, n_{\boldsymbol{\xi}} \right)^\top \tag{13}$$

  - **Constraint**: minimize the extra $g(\cdot)$-calls (or $\mathcal{M}(\cdot)$-evaluations)
  - **Idea**: use the nice properties of *score functions*[27,28,29]
  - **Remark**: two cases ⇒ **unbounded** vs. **bounded** support of the prior!

[27] R. Y. Rubinstein. "The score function approach for sensitivity analysis of computer simulation models". In: *Mathematics and Computers in Simulation* 28 (1986), pp. 351–379

[28] S. Rahman. "Stochastic sensitivity analysis by dimensional decomposition and score functions". In: *Probabilistic Engineering Mechanics* 24 (2009), pp. 278–287

[29] H. R. Millwater. "Universal properties of kernel functions for probabilistic sensitivity analysis". In: *Probabilistic Engineering Mechanics* 24 (2009), pp. 89–99

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Step #2 – Case 1: unbounded prior

- One can show that[30]:

$$S_j^{\text{loc}} := \frac{\partial \widetilde{P_f}(\boldsymbol{\xi})}{\partial \xi_j} = \int_{\mathcal{D}_{\boldsymbol{\Theta}}} P_f(\boldsymbol{\theta}) \frac{\partial f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi})}{\partial \xi_j} \mathrm{d}\boldsymbol{\theta} \tag{14a}$$

$$= \int_{\mathcal{D}_{\boldsymbol{\Theta}}} P_f(\boldsymbol{\theta}) \frac{\partial f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi}) / \partial \xi_j}{f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi})} f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\theta} \tag{14b}$$

$$= \int_{\mathcal{D}_{\boldsymbol{\Theta}}} \int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) \frac{\partial \ln f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi})}{\partial \xi_j} f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta}) f_{\boldsymbol{\Theta}|\boldsymbol{\xi}}(\boldsymbol{\theta}|\boldsymbol{\xi}) \mathrm{d}\mathbf{x} \mathrm{d}\boldsymbol{\theta} \tag{14c}$$

$$= \mathbb{E}_{f_{\mathbf{Z}}} \left[\mathbb{1}_{\mathcal{F}_{\mathbf{z}}}(\mathbf{Z}) \kappa_j(\boldsymbol{\Theta}, \boldsymbol{\xi}) \right] \tag{14d}$$

with $\kappa_j(\boldsymbol{\theta}, \boldsymbol{\xi})$ the *score function* of the prior distribution

[30] V. Chabridon et al. "Reliability-based sensitivity analysis of aerospace systems under distribution parameter uncertainty using an augmented approach". In: *Proc. of the 12th International Conference on Structural Safety and Reliability (ICOSSAR'17)*. Vienna, Austria, 2017

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #2 – Case 1: unbounded prior**
  - Monte Carlo estimators for an i.i.d. sample $\{\mathbf{Z}^{(i)}\}_{i=1}^{N} \sim f_{\mathbf{Z}}$:

$$\widetilde{P_f} \approx \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\mathcal{F}_{\mathbf{z}}}(\mathbf{Z}^{(i)}) \tag{15a}$$

$$S_j^{\mathrm{loc}} \approx \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\mathcal{F}_{\mathbf{z}}}(\mathbf{Z}^{(i)}) \kappa_j(\boldsymbol{\Theta}^{(i)}, \boldsymbol{\xi}) \tag{15b}$$

  - Simple post-treatment of the samples already used to estimate the PFP
- **Step #2 – Case 2: bounded prior**
  - Suppose $\Theta_j \sim \mathcal{U}([a, b])$, and $\# := a$ or b

$$S_{j,\#}^{\mathrm{loc}} \propto f_{X_j|\Theta_j}(x_j|\theta_j = \#) \left(\widetilde{P_f} \; - \; P_{f,aux}^{\theta_j = \#} \right) \tag{16}$$

  - Requires to estimate an "auxiliary" probability (flux integral)

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

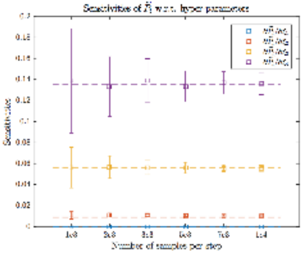
- **Step #2: In a nutshell**[31]
  - Derivation of local sensitivity estimators (unbounded and bounded cases)
    $\Rightarrow$ **If unbounded prior, just a simple post-processing of the samples!**
  - ARA framework + Advanced sampling methods (Adaptive IS strategies, using Cross-Entropy or KDE-based techniques)[32]
    $\Rightarrow$ **High sampling efficiency compared to CMC!**
  - {ARA framework + advanced algorithms + score functions}
    $\Rightarrow$ Efficient estimation of $\widetilde{P_f}$ + post-treatment to get S_j^{loc} for (almost) free!

[31] V. Chabridon et al. "Reliability-based sensitivity estimators of rare event probability in the presence of distribution parameter uncertainty". In: *Reliability Engineering and System Safety* 178 (2018), pp. 164–178

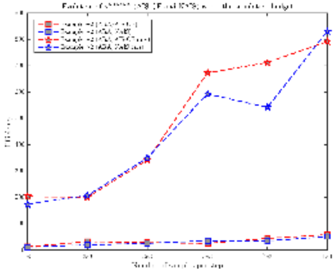
[32] P. Derennes et al. "Modeling and Optimization in Space Engineering. State of the Art and New Challenges". In: ed. by G. Fasano and J. D. (Eds.) Pintér. Switzerland: Springer International Publishing, 2019. Chap. 'Nonparametric Importance Sampling Techniques for Sensitivity Analysis and Reliability Assessment of a Launcher Stage Fallout', pp. 59–86

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Application: a two d.o.f. primary/secondary damped oscillator**



(a) Sensitivity estimates



(b) Efficiency

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #3**: **Global reliability-oriented sensitivity analysis under bi-level input uncertainty**
  - **Goal**: propose global indices to perform **reliability-oriented sensitivity analysis** (**ROSA**)
  - **Constraint**: minimize the extra $g(\cdot)$-calls (or $\mathcal{M}(\cdot)$-evaluations)
  - **Idea**: adapt the existing *Sobol indices on the indicator function*[33,34,35]

$$S_i^{\mathbb{1}_{\mathcal{F}}} = \frac{\mathrm{Var}\left[\mathbb{E}[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X})|X_i]\right]}{\mathrm{Var}\left[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X})\right]} \tag{17}$$

$$S_{T_i}^{\mathbb{1}_{\mathcal{F}}} = 1 - \frac{\mathrm{Var}\left[\mathbb{E}[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X})|\mathbf{X}^{-i}]\right]}{\mathrm{Var}\left[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X})\right]} = \frac{\mathbb{E}\left[\mathrm{Var}\left[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X})|\mathbf{X}^{-i}\right]\right]}{\mathrm{Var}\left[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X})\right]} \tag{18}$$

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[33] L. Li et al. "Moment-independent importance measure of basic variable and its state dependent parameter solution". In: *Structural Safety* 38 (2012), pp. 40–47

[34] P. Wei et al. "Efficient sampling methods for global reliability sensitivity analysis". In: *Computer Physics Communications* 183 (2012), pp. 1728–1743

[35] P. Lemaître. "Analyse de sensibilité en fiabilité des structures". (in English). PhD thesis. Université de Bordeaux, 2014

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #3: Wishes and Ideas**
  - Analyze the **separate effects** of aleatory and epistemic uncertainties (irreducible vs. reducible)[36]
    ⇒ **Need to separate / disaggregate both types of uncertainties**
  - Perform a **ranking** regarding the relative influence of the inputs on the failure probability
  - Get the indices with no extra calls
    ⇒ **Need to reuse efficiently the samples from the rare event estimation procedure**

[36] S. Sankararaman and S. Mahadevan. "Separating the contributions of variability and parameter uncertainty in probability distributions". In: *Reliability Engineering and System Safety* 112 (2013), pp. 187–199

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #3 – Task 1: disaggregate both types of uncertainties**
  - Consider a "disaggregated augmented vector", e.g., in the Gaussian case:
    - single-level uncertainty: $X = \mu_X + \sigma_X \; U_X, \quad \text{with } U_X \sim \mathcal{N}(0,1)$
    - bi-level uncertainty: $X = M_X + S_X \; U_X$
  - Augmented vector: $\mathbf{Z} = (\mathbf{V}_{\text{dis}}, \mathbf{X}_{\text{single}})^\top$
    - $\mathbf{V}_{\text{dis}} = (M_X, S_X, U_X)^\top$ the vector of disaggregated variables
    - $\mathbf{X}_{\text{mono}}$ the vector of mono-level variables
    - $f_{\mathbf{Z}} = f_{\mathbf{\Theta}} \; f_{\mathbf{U}} \; f_{\mathbf{X}_{\text{mono}}}$
  - Other transformations are available for various probability distributions[37]

[37] R. Schöbi and B. Sudret. "Global sensitivity analysis in the context of imprecise probabilities (p-boxes) using sparse polynomial chaos expansions". In: *Reliability Engineering and System Safety* 148 (2017), pp. 129–141

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #3 – Task 2: rank the inputs w.r.t. their influence on the failure probability**
  - Extend the $S_i^{\mathbb{1}_\mathcal{F}}$ and $S_{T_i}^{\mathbb{1}_\mathcal{F}}$ indices to the disaggregated augmented vector
  - Proposition of "Pick-Freeze" estimators for these indices[38]
- **Step #3 – Task 3: propose efficient estimators for the rare event context**
  - Smart rewriting of the indices using Bayes' theorem[39]

$$\widehat{S}_i^{\mathbb{1}_\mathcal{F}} = \frac{\widehat{\widetilde{P_f}}}{1-\widehat{\widetilde{P_f}}} \mathrm{Var}\left[\frac{\widehat{f}_{Z_i|\mathcal{F}}(Z_i)}{\widehat{f}_{Z_i}(Z_i)}\right] \quad \widehat{S}_{T_i}^{\mathbb{1}_\mathcal{F}} = 1 - \frac{\widehat{\widetilde{P_f}}}{1-\widehat{\widetilde{P_f}}} \mathrm{Var}\left[\frac{\widehat{f}_{\mathbf{Z}^{-i}|\mathcal{F}}(\mathbf{Z}^{-i})}{\widehat{f}_{\mathbf{Z}^{-i}}(\mathbf{Z}^{-i})}\right] \tag{19}$$

[38] V. Chabridon. "Reliability-oriented sensitivity analysis under probabilistic model uncertainty – Application to aerospace systems". PhD thesis. Université Clermont Auvergne, 2018

[39] G. Perrin and G. Defaux. "Efficient Evaluation of Reliability-Oriented Sensitivity Indices". In: *Journal of Scientific Computing* (2019), pp. 1–23

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #3 – Task 3: propose efficient estimators for the rare event context**
  - Smart rewriting of the indices using Bayes' theorem[40]

$$\widehat{S}_i^{\mathbb{1}_\mathcal{F}} = \frac{\widehat{\widehat{P_f}}}{1 - \widehat{\widehat{P_f}}} \mathrm{Var}\left[\frac{\widehat{f}_{Z_i|\mathcal{F}}(Z_i)}{\widehat{f}_{Z_i}(Z_i)}\right] \quad \widehat{S}_{T_i}^{\mathbb{1}_\mathcal{F}} = 1 - \frac{\widehat{\widehat{P_f}}}{1 - \widehat{\widehat{P_f}}} \mathrm{Var}\left[\frac{\widehat{f}_{\mathbf{Z}^{-i}|\mathcal{F}}(\mathbf{Z}^{-i})}{\widehat{f}_{\mathbf{Z}^{-i}}(\mathbf{Z}^{-i})}\right] \tag{20}$$

  - +++ Can be estimated with a post-processing phase of the failure samples obtained from, e.g., using Subset Sampling or Moving Particles
  - — Requires to estimate a 1D PDF (easy) $\widehat{f}_{Z_i|\mathcal{F}}$ and a high-dimensional one $\widehat{f}_{\mathbf{Z}^{-i}|\mathcal{F}}$ (not easy)

[40] G. Perrin and G. Defaux. "Efficient Evaluation of Reliability-Oriented Sensitivity Indices". In: *Journal of Scientific Computing* (2019), pp. 1–23

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Basic principles of Gaussian kernel density estimation (G-KDE)

- Let $\mathcal{Z} = \{\mathbf{z}^{(1)}, \dots, \mathbf{z}^{(N)}\}$ be a N-sample drawn according to an unknown parent density $f_{\mathbf{Z}}$
- Thus, its kernel density estimator $\widehat{f}_{\mathbf{Z}}$ is given by:

$$\widehat{f}_{\mathbf{Z}}(\mathbf{z}) = \frac{\det(\mathbf{H})^{-1/2}}{N} \sum_{j=1}^{N} K_d\left(\mathbf{H}^{-1/2}(\mathbf{z} - \mathbf{z}^{(j)})\right) \quad (21)$$

with $K_d(\mathbf{z}) = \frac{1}{(2\pi)^{d/2}} \exp\left[-\frac{1}{2}\mathbf{z}^\top \mathbf{z}\right]$ (Gaussian kernel)

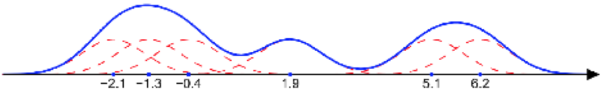


Figure: Example of a 1D KDE.

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Basic principles of Gaussian kernel density estimation (G-KDE)

- $\mathbf{H}$ is the *bandwidth matrix* → can be optimized using a MISE criterion:

$$\mathrm{MISE}(\mathbf{H}) = \mathbb{E}\left[\int_{\mathbb{R}^d} \left(\widehat{f}_{\mathbf{Z}}(\mathbf{z}) - f_{\mathbf{Z}}(\mathbf{z})\right)^2 \mathrm{d}\mathbf{z}\right] \tag{22}$$

- $\mathbf{H}$ → use the "*Silverman's rule of thumb*" (approximation):

$$\mathbf{H} = \eta_{\mathrm{Silv}}^2 \begin{bmatrix} \widehat{\sigma}_1^2 & 0 & \dots & 0 \\ 0 & \widehat{\sigma}_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \widehat{\sigma}_d^2 \end{bmatrix} \quad \text{with} \quad \eta_{\mathrm{Silv}} = \left(\frac{1}{N}\frac{4}{(d+2)}\right)^{\frac{1}{d+4}} \tag{23}$$

⇒ Known to be inefficient regarding high-dimensional ($d > 2$) or complex multivariate densities

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #3 – Task 3: Use of a Data-driven Tensorized G-KDE**[41]
  - **New Feature #1** $\rightarrow$ "*data-driven*": estimate the empirical moments $\widehat{\mathbf{m}}_{\mathcal{Z}}$ and $\widehat{\mathbf{\Sigma}}_{\mathcal{Z}}$ + take them into account in the computation of **H**
    $\Rightarrow$ **replace η_{Silv} by η_{ML} (maximum likelihood)**
  - **New Feature #2** $\rightarrow$ "*tensorized*": account for the possible dependence structure of the data points at failure
    $\Rightarrow$ ***block-by-block decomposition* for H**

[41] G. Perrin, C. Soize, and N. Ouhbi. "Data-driven kernel representations for sampling with an unknown block dependence structure under correlation constraints". In: *Journal of Computational Statistics and Data Analysis* 119 (2018), pp. 139–154

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

- **Step #3**: **Proposed methodology (in a nutshell)**[42]
  1. Construct the augmented vector $\mathbf{Z} = (\mathbf{V}_{\mathrm{dis}}, \mathbf{X}_{\mathrm{single}})^\top$
  2. Use ARA/SS $\rightarrow$ get $\widetilde{P_f}$ + failure points at last iteration
  3. Fit $\widehat{f}_{Z_i|\mathcal{F}}$ and $\widehat{f}_{\mathbf{Z}_{-i}|\mathcal{F}}$ using the Data-driven Tensorized G-KDE
  4. Estimate the $S^{\mathbb{1}_\mathcal{F}}$-indices adapted to the bi-level uncertainty

  ▷ {ARA framework + disaggregated vector + advanced algorithms + DT-G-KDE}
  $\Rightarrow$ Efficient estimation of $\widetilde{P_f}$ + post-treatment to get $S^{\mathbb{1}_\mathcal{F}}$ for free!

[42] V. Chabridon et al. "Mechanical Engineering Under Uncertainties". In: ed. by C. (Editor) Gogu. Wiley - ISTE Ltd, 2020. Chap. 'Global reliability-oriented sensitivity analysis under distribution parameter uncertainty', pp. 1–43

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Description

- **Application: fallout phase of an expendable space launcher**[43]

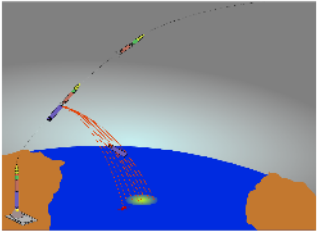


Figure: Fallout phase of the first stage.

[43] V. Chabridon et al. "Mechanical Engineering Under Uncertainties". In: ed. by C. (Editor) Gogu. Wiley - ISTE Ltd, 2020. Chap. 'Global reliability-oriented sensitivity analysis under distribution parameter uncertainty', pp. 1–43

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Description

- **Application: problem specification**
  - ▷ Black-box computer code $\mathcal{M} : \mathbb{R}^{d=6} \to \mathbb{R}$
  - ▷ Independent inputs:
    - X_1: stage altitude perturbation at separation Δa (m)
    - X_2: velocity perturbation at separation Δv $(\mathrm{m.s}^{-1})$
    - X_3: flight path angle perturbation at separation $\Delta\gamma$ (rad)
    - X_4: azimuth angle perturbation at separation $\Delta\psi$ (rad)
    - X_5: propellant mass residual perturbation at separation Δm (kg)
    - X_6: drag force error perturbation ΔC_d (dimensionless)
  - ▷ $g(\mathbf{X}) = \mathrm{d_{safe}} - \mathcal{M}(\mathbf{X}) = 15,000(\mathrm{m}) - D_{\mathrm{code}}(\mathrm{m})$
  - ▷ **Single-level problem**: $p_{f,ref} = 1.36 \times 10^{-4}$
  - ▷ **Augmented problem**: 6 independent basic variables in $\mathbf{X}$, 2 independent parameters in $\boldsymbol{\Theta}$, 4 hyper-parameters in $\boldsymbol{\xi}$

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Description

- **Application: problem specification**
  - Input probabilistic model:

Variable	Distribution	Mean	Std
$X_1 = \Delta h$ (m)	Normal	0	$1,650$
$X_2 = \Delta v$ ($\mathrm{m.s}^{-1}$)	Normal	μ_{X_2} uncertain	3.7
$X_3 = \Delta\gamma$ (rad)	Normal	μ_{X_3} uncertain	0.001
$X_4 = \Delta\psi$ (rad)	Normal	0	0.0018
$X_5 = \Delta m$ (kg)	Normal	0	70
$X_6 = \Delta C_d$ (1)	Normal	0	0.1
$\Theta_2 = \mu_{X_2}$ ($\mathrm{m.s}^{-1}$)	Normal	$\xi_1 = 0$	$\xi_2 = 3.7$
$\Theta_3 = \mu_{X_3}$ (rad)	Normal	$\xi_3 = 0$	$\xi_4 = 0.001$

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Step #1 and Step #2

- **Predictive failure probability and local sensitivities**
  - Numerical results:

	ARA/CMC ($N_{x,\theta} = 10^6$ samples)		ARA/AIS-CE ($N_{x,\theta} = 10^4$ samples/threshold)		ARA/NAIS ($N_{x,\theta} = 10^4$ samples/threshold)	
	Estimate	cv	Estimate	cv	Estimate	cv
$\widehat{\widetilde{P}}_f$	4.40×10^{-3}	(1.38 %)	4.41×10^{-3}	(10.3 %)	4.40×10^{-3}	(2.08 %)
$\widehat{\partial \widetilde{P}_f / \partial \xi_1}$	-9.13×10^{-4}	(3.44 %)	-8.68×10^{-4}	(27.7 %)	-9.12×10^{-4}	(5.90 %)
$\widehat{\partial \widetilde{P}_f / \partial \xi_2}$	2.95×10^{-3}	(2.32 %)	3.02×10^{-3}	(14.8 %)	2.95×10^{-3}	(3.22 %)
$\widehat{\partial \widetilde{P}_f / \partial \xi_3}$	-2.31	(3.88 %)	-2.29	(25.7 %)	-2.30	(5.82 %)
$\widehat{\partial \widetilde{P}_f / \partial \xi_4}$	6.43	(2.18 %)	6.26	(14.6 %)	6.41	(3.77 %)
$\nu^{ARA/AIS}$	–	–	0.5	–	13	–

  - Reference value for the PFP: $\widetilde{P}_{f,ref} = 4.4 \times 10^{-3} > p_{f,ref} = 1.36 \times 10^{-4}$
  - ξ_4 and ξ_3 are the most influent hyper-parameters
  - Decision: investigate the probabilistic model of the flight path angle (EU reduction)

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Step #1 and Step #2

- **Influence of the rareness of the failure event**
  - $g(\mathbf{X}) = \mathrm{d}_{\mathrm{safe}} - \mathcal{M}(\mathbf{X}) = 20,000(\mathrm{m}) - D_{\mathrm{code}}(\mathrm{m})$
  - Numerical results:

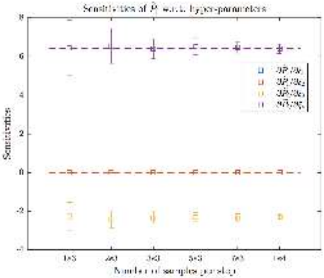
	ARA/AIS-CE (★) ($N_{x,\theta} = 10^4$ samples/step)		ARA/NAIS (★) ($N_{x,\theta} = 10^4$ samples/step)	
	Estimate	cv	Estimate	cv
$\widehat{\widetilde{P}}_f$	1.00×10^{-4}	(29.7 %)	1.19×10^{-4}	(2.85 %)
$\widehat{\partial \widetilde{P}_f / \partial \xi_1}$	-4.65×10^{-5}	(46.9 %)	-3.66×10^{-5}	(7.00 %)
$\widehat{\partial \widetilde{P}_f / \partial \xi_2}$	1.21×10^{-4}	(40.3 %)	1.41×10^{-4}	(3.62 %)
$\widehat{\partial \widetilde{P}_f / \partial \xi_3}$	-1.19×10^{-1}	(34.4 %)	-9.18×10^{-2}	(7.67 %)
$\widehat{\partial \widetilde{P}_f / \partial \xi_4}$	2.51×10^{-1}	(25.2 %)	3.10×10^{-1}	(4.24 %)
$\nu^{\mathrm{ARA/AIS}}$	2	–	207	–

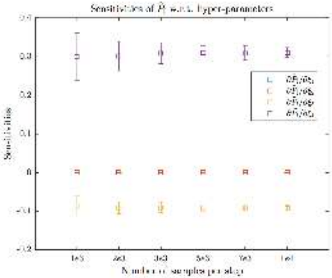
  - Reference value for the PFP: $\widetilde{P}_{f,ref} = 1.20 \times 10^{-4} > p_{f,ref} = 2.31 \times 10^{-7}$
  - ARA/AIS-CE gives poor results in terms of efficiency
  - ARA/NAIS is very efficient compared to ARA/CMC

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Step #1 and Step #2

- **Convergence of sensitivities estimation results**





4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Step #2

- **Global reliability-oriented sensitivity analysis**
  - Disaggregated random variables:

$$X_2 = M_{X_2} + \sigma_{X_2} \; U_{X_2} \tag{24}$$
$$X_3 = M_{X_3} + \sigma_{X_3} \; U_{X_3} \tag{25}$$

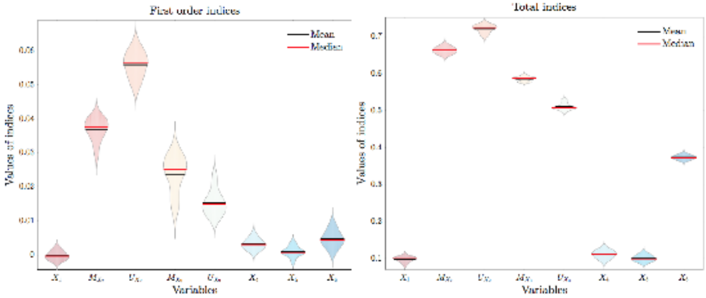
  - Augmented random vector:

$$\mathbf{Z} = (\mathbf{V}_{\mathrm{dis}}, \mathbf{X}_{\mathrm{mono}})^\top = (X_1, M_{X_2}, U_{X_2}, M_{X_3}, U_{X_3}, X_4, X_5, X_6)^\top \tag{26}$$

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Step #2

- **Reference results for the $S^{\Vert\mathcal{F}}$-indices estimated using ARA/CMC**

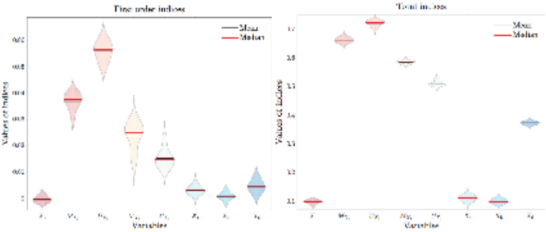


(a) (b)

4 – A Bayesian framework for reliability assessment under distribution parameter uncertainty

Step #1 and Step #2

- **Results for the $S^{\mathbb{1}_{\mathcal{F}}}$-indices obtained by ARA/SS + G-KDE**
  - $g(\mathbf{X}) = \mathrm{d}_{\mathrm{safe}} - \mathcal{M}(\mathbf{X}) = 20,000(\mathrm{m}) - D_{\mathrm{code}}(\mathrm{m})$
  - Numerical results:



(a) (b)

  - Reference value for the PFP: $\widetilde{P}_{f,ref} = 3.98 \times 10^{-2} > p_{f,ref} = 6.10 \times 10^{-3}$
  - First-order $S^{\mathbb{1}_{\mathcal{F}}}$-indices: ✓✓✓ / Sucess rates: ✓✓✓
  - Total-order $S^{\mathbb{1}_{\mathcal{F}}}$-indices: ✓ / Sucess rates: ~

5 – Conclusion

- **About this talk**
  - ✓ A full methodology has been proposed to handle EUM in a standard UQ (and reliability analysis) framework
  - ✓ {ARA framework + advanced rare event estimation algorithms + score functions + DT-G-KDE}
    $\Rightarrow$ Efficient estimation of $\widetilde{P_f}$ + S_j^{loc} + $S^{\mathbb{1}_{\mathcal{F}}}$ for free!
  - ✗ There are still difficulties to handle complex industrial use-cases! (dimensionality, complexity of the LSF)

5 – Conclusion

- **Some "Take-Home Messages"**
  - ▷ Before doing EUM for rare events, you need to ensure the rare event estimation!
  - ▷ EU can be properly handled using several paradigms
  - ▷ Rare event estimation should always be conducted along with dedicated sensitivity analysis
- **Research perspectives**
  - ▷ Rationale for a "pure" probabilistic EUM[44]
  - ▷ Comparison between the several paradigms (I, II, III, IV, V)
  - ▷ Links with Decision Theory[45]
  - ▷ Investigate the use of advanced 2^{nd}-level indices[46]

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