Model Order Reduction and Bayesian Optimization for MDO problems

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Static aeroelastic optimization of an aircraft wing

- Initial guess ($x_0$)
- Update design space point ($x$)
- Aerodynamics model: $f_a = M(x, u_s)$
- Structural model: $u_s = M(x, f_a)$
- Stopping criteria met?
- Multidisciplinary Analysis (MDA)
- Optimum point ($x^*$)
Static aeroelastic optimization of an aircraft wing

The solution of the MDA is the **displacement field** \( u_s \) and the **vector of aerodynamic forces** \( f_a \).

Both disciplinary solvers depend on the **design variables** \( x \), handled by an **optimization algorithm**.
Static aeroelastic optimization of an aircraft wing

Introduction
Application Example
Reference Framework
DPOD+I & SLSQP
DPOD+I & EGMD
Conclusion

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Assumptions:

- Partitioned approach: non-intrusive coupling between the disciplinary solvers.
- Multidisciplinary Feasible (MDF) approach: optimization problem and non-linear coupled problem solved independently.
- The coupling variables are high dimensional vectors.
Problem statement

**Problem:**
- When using high fidelity solvers (e.g. FEM or CFD solvers), the computational cost may become intractable.

**Goal:**
- Reduce the computational cost, by reducing the number of disciplinary solver calls made.
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![Diagram showing the optimization process](Image)
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V. Conclusion
Aerodynamics and Structural solvers

- Common Research Model (configuration uCRM-9) [1]:

Aerodynamics mesh (VLM solver – 2100 degrees of freedom)  
Structural mesh (FEM solver – 43416 degrees of freedom)

Aerodynamics and Structural solvers

- Common Research Model (configuration uCRM-9) [1]:

![Aerodynamics mesh](image1) ![Structural mesh](image2)

Aerodynamics mesh (VLM solver – 2100 degrees of freedom)  Structural mesh (FEM solver – 43416 degrees of freedom)

- Four considered **design variables** \( \{\alpha, V_\infty, t_{sk}, t_{sp}\} \):

<table>
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<tr>
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<th>( t_{sk}[m] )</th>
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<tr>
<td><strong>Designation</strong></td>
<td>Angle of attack</td>
<td>Air freestream velocity</td>
<td>Skin thickness</td>
<td>Spar thickness</td>
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<td><strong>Range of variation</strong></td>
<td>[1, 9]</td>
<td>[220, 250]</td>
<td>[0.003, 0.01]</td>
<td>[0.01, 0.1]</td>
</tr>
</tbody>
</table>

The design variables were scaled to take values in the range \([0,1]\).

Objective function

- Objective function chosen as an inverse problem:

\[ x^* = \arg \min_{x \in \mathcal{X}} f_{\text{obj}}(x) \quad \text{with} \quad f_{\text{obj}}(x) = \frac{\| f_a(x_{\text{ref}}) - f_a(x) \|_2}{\| f_a(x_{\text{ref}}) \|_2} + \frac{\| u_s(x_{\text{ref}}) - u_s(x) \|_2}{\| u_s(x_{\text{ref}}) \|_2} \]

with \( x_{\text{ref}} \) the design space point that results in the maximum wing tip displacement \( x_{\text{ref}} = \{1,1,0,0\} \).
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The MDA is solved via non-linear block Gauss-Seidel using Aitken acceleration.

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Starting point chosen at the center of the design space \( x_0 = \{0.5,0.5,0.5,0.5\} \).

\( \Rightarrow \) The algorithm needed 17 iterations and 286 calls to each disciplinary solver in order to find the reference point.
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**DPOD+I & SLSQP framework**

- Replacement of the disciplinary solvers by Disciplinary Proper Orthogonal Decomposition + Interpolation (DPOD+I) surrogate models [2].

DPOD+I & SLSQP framework

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  Disciplinary Proper Orthogonal Decomposition + Interpolation (DPOD+I) surrogate models [2].

⇒ Model order reduction by **Disciplinary Proper Orthogonal Decomposition (DPOD):**

\[
\hat{f}_a \approx \phi_0^a + \sum_{i=1}^{n_a} \alpha_i^a(x, u_s) \phi_i^a
\]

\[
\hat{u}_s \approx \phi_0^s + \sum_{i=1}^{n_s} \alpha_i^s(x, f_o) \phi_i^s
\]

where \( \phi_0 \) is a constant vector, \( \phi_i \) are the POD basis vectors, \( \alpha_i \) are the POD coefficients, and \( n_a \) and \( n_s \) give the number of terms retained in the POD approximations.

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• Replacement of the disciplinary solvers by Disciplinary Proper Orthogonal Decomposition + Interpolation (DPOD+I) surrogate models [2].

⇒ Model order reduction by Disciplinary Proper Orthogonal Decomposition (DPOD) followed by the interpolation of each coefficient by Gaussian Processes (GP):

\[ \hat{\alpha}_i^q \sim \text{GP}_{\text{DoE}_a} (\mu_{\text{DoE}_a}, k_{\text{DoE}_a}) \]
\[ \hat{\alpha}_i^s \sim \text{GP}_{\text{DoE}_s} (\mu_{\text{DoE}_s}, k_{\text{DoE}_s}) \]

where the GP approximation of the coefficients is denoted by \( \hat{\alpha}_i \) and is characterized by a mean value \( \mu \) and a covariance kernel \( k \).

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- The MDO is solved by a gradient-based optimizer (SLSQP). Gradient computation uses the Gaussian Process derivatives.

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DPOD+I & SLSQP results

- The GP surrogates are built upon random initial disciplinary DoEs ⇒ 10 runs are performed.
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- Structural POD basis composed of an average of 6 coefficients and aerodynamics POD basis composed of an average of 5 coefficients.

⇒ Initial DoE: average of 42 points for the structural discipline and 52 for the aerodynamics discipline.
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Comparison between DPOD+I & SLSQP framework and reference framework

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- The GP surrogates are built upon random initial disciplinary DoEs ⇒ **10 runs are performed.**

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where $n^a$ and $n^s$ are, respectively, the number of aerodynamics solver calls and the number of structural solver calls.

- **Reduction by a factor of 5** in the number of necessary disciplinary solver calls.

- An average of **only 10 calls** was made to each disciplinary solver during the optimization process.
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The MDA remains the same as in the previous framework.

Replacement of the optimizer by the **EGMDO (Efficient Global Multidisciplinary Optimization)** algorithm [3].

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The point must have some likelihood of being the minimum to be enriched.

Possible to add new points to the DoE.

A modified Expected Improvement (EI) criterion is used.

Due to the non-Gaussian nature of $f_{obj}$ the EI is estimated via Monte Carlo Simulation.

DPOD+I & EGMDO results

- Random initial disciplinary DoEs and random initial optimizer DoE ⇒ **10 runs are performed.**

- **Initial DoE:** average of 49 points for the structural discipline and 59 for the aerodynamics discipline.
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The optimization of a two-disciplinary problem requires many disciplinary solver calls if the real solvers are used for the MDA.
Conclusion and perspectives

• The optimization of a two-disciplinary problem requires many disciplinary solver calls if the real solvers are used for the MDA.

• DPOD+I surrogates allow us to **perform multi-disciplinary optimization using high-fidelity solvers**, at a reduced computational cost.
Conclusion and perspectives

- The optimization of a two-disciplinary problem requires many disciplinary solver calls if the real solvers are used for the MDA.

- DPOD+I surrogates allow us to perform multi-disciplinary optimization using high-fidelity solvers, at a reduced computational cost.

- The EGMDO algorithm allows to perform global optimization when the disciplinary solvers are replaced by disciplinary Gaussian Processes, reducing the disciplinary solver calls during the optimization process.
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- The EGMDO algorithm allows to **perform global optimization when the disciplinary solvers are replaced by disciplinary Gaussian Processes**, reducing the disciplinary solver calls during the optimization process.

Some perspectives to the proposed framework include the implementation of **other dimension reduction techniques**, for instance via local POD basis or non-linear model order reduction, to account for more complex disciplinary models. **Other approximation models**, such as the Kriging with Partial Least Squares model [4] could allow the construction of GPs for a greater number of design variables.

**Bibliography:**


