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Estimation of seismic fragility curves by sequential design of experiments

Clément GAUCHY (CEA, CMAP), Cyril FEAU (CEA), Josselin GARNIER (CMAP)

Seismic safety studies

Probabilistic safety studies in earthquake engineering

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- **1)** The estimation of an annual occurrence probability of a seismic excitation of specific intensity
- **2)** The estimation of the probability of failure of a structure conditional to the seismic intensity (seismic fragility curve)
- **3)** The evaluation of the annual failure probability of the structure, evaluated thanks to the 2 steps above

↪ This presentation will focus on the step **2)**.

Schematic representation

Mechanical parameters

$\mathbf{x} \in \mathcal{D}_{\mathbf{x}} \subset \mathbb{R}^d$

$\hookrightarrow$ Young modulus, boundary conditions

Computer code of the structure

$\hookrightarrow$ Piping system of a nuclear power plant.

Numerical simulation output

$z(a, \mathbf{x}) \in \mathbb{R}$

$\hookrightarrow$ Out of plane rotation of a pipe elbow

Seismic ground motion

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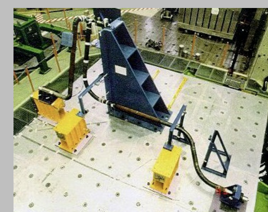

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The output of the simulation $z(a, \mathrm{x})$ is stochastic (i.e. for a same value of (a, x) the value of the output can change)

Seismic intensity measurement

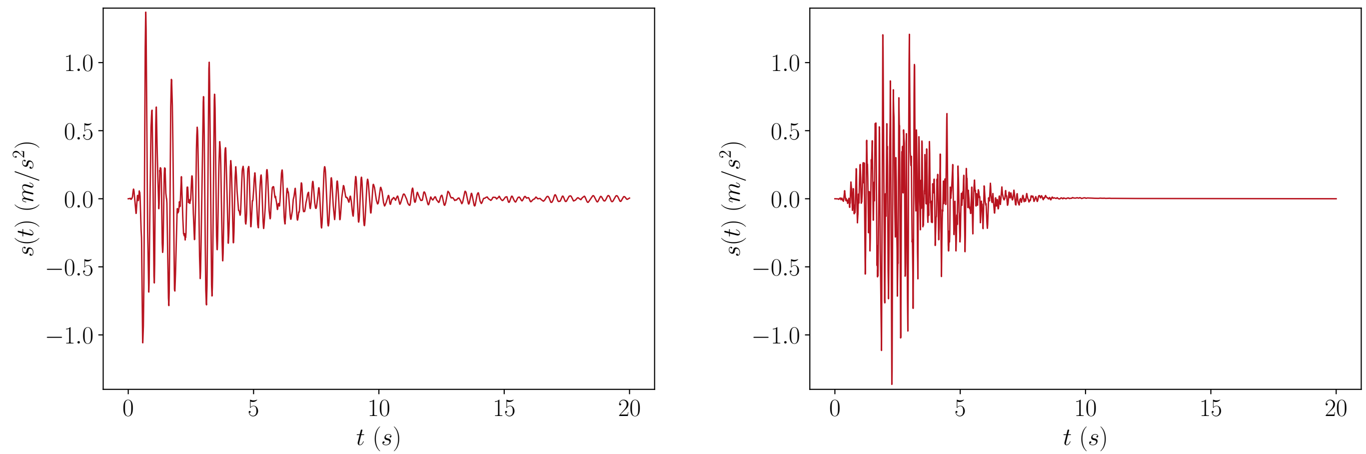

Figure: Two earthquake accelerograms with the same peak ground acceleration

Generally a measure of seismic intensity is a scalar quantity derived from the time signal. Example: the peak ground acceleration of a seismic ground motion with accelerogram $t \to s(t)$.

$$a = \max_{t \in [0,T]} |s(t)| .$$

$\hookrightarrow$ There is no uniqueness between a seismic measurement intensity value and a seismic signal.

Seismic fragility curve

The fragility curve is the conditional probability of failure of the structure:

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$\mathbf{X}$: Mechanical parameters of the structure.

$\boldsymbol{C}$: Critical level above which the structure is considered in failure state.

$\boldsymbol{z}$: Computer model output

Scientific goals

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With a Monte Carlo sample $(\mathbf{X}_i)_{1\leq i\leq M}$ such that $\mathbf{X}_i \sim \mathbb{P}_{\mathrm{X}}$ we can propagate the uncertainty on the fragility curve by studying the distribution of $(\boldsymbol{a} \to \boldsymbol{\Psi}(\boldsymbol{a}, \mathbf{X}_i))_{1\leq i\leq M}$.

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Problem: About $\mathbf{10^4}$ simulations to estimate a curve $\boldsymbol{a} \to \boldsymbol{\Psi}(\boldsymbol{a}, \mathbf{x})$ for a $\mathbf{x}$ fixed in the classical way. For $\boldsymbol{M} = \mathbf{1000}$ it would be necessary to do 10^7 simulations.

10^7 CAST3M simulations $\approx$ 100 days of computation time.

Goal: Provide a best estimate of $\Psi(a, \mathrm{x})$ with a fixed budget of CAST3M simulations.

Sequential design of experiments

Statistical model

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$$y(a, \mathrm{x}) = g(a, \mathrm{x}) + \varepsilon ,$$

where $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ and $y(a, \mathrm{x}) = \log(z(a, \mathrm{x}))$.

The fragility curve with this model writes:

$$\Psi(a, x; g) = \Phi\left(\frac{g(a, x) - \log(C)}{\sigma_\varepsilon}\right) ,$$

where Φ is the cdf of the standard Gaussian distribution

Gaussian process

We define a Gaussian process $\boldsymbol{G} : \mathrm{x} \to \boldsymbol{G}(\mathrm{x})$ thanks to the Gaussian vectors. Let $(\mathrm{x}^{(1)}, \ldots, \mathrm{x}^{(p)})$, the process $\boldsymbol{G}$ is said to be Gaussian if the vector $(\boldsymbol{G}(\mathrm{x}^{(1)}), \ldots, \boldsymbol{G}(\mathrm{x}^{(p)}))$ is Gaussian.

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A Gaussian process is entirely defined by its mean function:

$$\boldsymbol{m}(\mathrm{x}) = \mathbb{E}[\boldsymbol{G}(\mathrm{x})] \ ,$$

and its covariance function:

$$\boldsymbol{\Sigma}(\mathrm{x}, \tilde{\mathrm{x}}) = \mathbb{E}[(\boldsymbol{G}(\mathrm{x}) - \boldsymbol{\mu}(\mathrm{x}))(\boldsymbol{G}(\tilde{\mathrm{x}}) - \boldsymbol{\mu}(\tilde{\mathrm{x}}))].$$

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$$\boldsymbol{Y} \sim \mathcal{GP}(\boldsymbol{m}, \Sigma) \, .$$

The Gaussian vector $(\boldsymbol{G}(\mathrm{x}^{(1)}), \ldots, \boldsymbol{G}(\mathrm{x}^{(p)}))$ has mean vector $\boldsymbol{\mu} = (\boldsymbol{m}(\mathrm{x}^{(i)}))_{1 \leq i \leq p}$ and covariance matrix $\boldsymbol{K} = (\Sigma(\mathrm{x}^{(i)}, \mathrm{x}^{(j)}))_{1 \leq i,j \leq p}$.

A Bayesian model

We will model an uncertainty on $\boldsymbol{g}$ by a Gaussian process prior $\boldsymbol{G}$ defined on some probabilistic space $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{B}}, \mathbb{P}_0)$. The statistical model then becomes:

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We can propose a Bayesian estimator of the fragility curve. Define $\boldsymbol{\mathcal{F}}_n$ the $\boldsymbol{\sigma}$-algebra defined by $(\boldsymbol{A}_i, \mathrm{X}_i, \boldsymbol{y}(\boldsymbol{A}_i, \mathrm{X}_i))_{1\leq i\leq n}$. The *posterior mean* writes:

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We have $(\boldsymbol{G}(\boldsymbol{a}, \mathrm{x}) | \boldsymbol{\mathcal{F}}_n) \sim \boldsymbol{\mathcal{N}}(\widehat{\boldsymbol{G}}_n(\boldsymbol{a}, \mathrm{x}), \widehat{\boldsymbol{\sigma}}_n(\boldsymbol{a}, \mathrm{x})^2)$. Thus:

$$\widehat{\boldsymbol{\Psi}}_n(\boldsymbol{a}, \mathrm{x}) = \boldsymbol{\Phi}\left(\frac{\widehat{\boldsymbol{G}}_n(\boldsymbol{a}, \mathrm{x}) - \log(\boldsymbol{C})}{\boldsymbol{\sigma}_n(\boldsymbol{a}, \mathrm{x})}\right) \ ,$$

where $\boldsymbol{\sigma}_n(\boldsymbol{a}, \mathrm{x})^2 = \widehat{\boldsymbol{\sigma}}_n(\boldsymbol{a}, \mathrm{x})^2 + \boldsymbol{\sigma}_{\varepsilon}^2$

A decision problem

For a given $\boldsymbol{n} \geq \mathbf{1}$, we have to choose a design $\boldsymbol{\mathcal{D}}_n = (\boldsymbol{A}_i, \mathrm{X}_i)_{1 \leq i \leq n}$ where we compute the mechanical response $\boldsymbol{y}(\boldsymbol{A}_i, \mathrm{X}_i)$. **How to measure the quality of the design $\mathcal{D}_n$?**

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The *Bayes risk* r_B of a Bayesian estimator of the fragility curve $\widehat{\Psi}_n$ based on a design $\mathcal{D}_n = (A_i, \mathrm{X}_i)_{1\leq i\leq n}$ and computer experiments $y(A_i, \mathrm{X}_i)$ is defined by:

$$r_B(\mathcal{D}_n, G) = \mathbb{E}_{\mathbb{P}_0}\left[\int_{\mathcal{A}\times\mathcal{X}} (\Psi(\alpha, u; G) - \widehat{\Psi}_n(\alpha, u))^2 dh(\alpha) d\mathbb{P}_{\mathrm{X}}(u)\right] .$$

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The optimal design $\mathcal{D}_n^* = (A_i^*, \mathrm{X}_i^*)_{1 \leq i \leq n}$ is obtained by minimizing the Bayes risk:

$$\mathcal{D}_n^* = \underset{\mathcal{D}_n \in \mathcal{S}_n}{\operatorname{argmin}}\, r_B(\mathcal{D}_n, G) ,$$

where $\mathcal{S}_n$ is the set of all admissible sequential strategies of size n.

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- A loss function (here $\int_{\mathcal{A}\times\mathcal{X}}(\boldsymbol{\Psi}(\boldsymbol{\alpha}, \boldsymbol{u}; \boldsymbol{G}) - \widehat{\boldsymbol{\Psi}}_n(\boldsymbol{\alpha}, \boldsymbol{u}))^2 \boldsymbol{dh}(\boldsymbol{\alpha})\boldsymbol{d}\mathbb{P}_{\mathrm{X}}(\boldsymbol{u})$)

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These **3** parts are necessary to define the Bayes risk and hence the optimal design $\boldsymbol{\mathcal{D}}_{\boldsymbol{n}}^*$

Optimal design of experiments

The optimal design $\mathcal{D}_n^*$ can be obtained formally using dynamic programming. Define

$$R_n = \mathbb{E}_{\mathbb{P}_0}\left[\int_{\mathcal{A}\times\mathcal{X}} (\Psi(\alpha, u; G) - \widehat{\Psi}_n(\alpha, u))^2 dh(\alpha) d\mathbb{P}_{\mathrm{X}}(u) \middle| \mathcal{F}_n\right] ,$$

and by reverse induction

$$R_k = \min_{a,\mathrm{x}\in\mathcal{A}\times\mathcal{X}} \mathbb{E}_{\mathbb{P}_0}\left[R_{k+1} \middle| A_{k+1} = a, \mathrm{X}_{k+1} = \mathrm{x}, \mathcal{F}_k\right] ,$$

$$A_{k+1}^*, \mathrm{X}_{k+1}^* = \operatorname*{argmin}_{a,\mathrm{x}\in\mathcal{A}\times\mathcal{X}} \mathbb{E}_{\mathbb{P}_0}\left[R_{k+1} \middle| A_{k+1} = a, \mathrm{X}_{k+1} = \mathrm{x}, \mathcal{F}_k\right] .$$

for $0 \leq k \leq n-1$.

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The exact optimal design is intractable to obtain in practice.

Stepwise uncertainty reduction (SUR)

The optimal design is intractable to obtain due to nested nonconvex optimization problem.

[1] J. Bect, D. Ginsbourger, L. Li, V. Picheny, and E. Vazquez. Sequential design of computer experiments for the estimation of a probability of failure.
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$$\boldsymbol{A}_{n+1}^{\mathrm{SUR}}, \mathrm{X}_{n+1}^{\mathrm{SUR}} = \underset{a,\mathrm{x} \in \mathcal{A} \times \mathcal{X}}{\operatorname{argmin}} J_n(a, \mathrm{x})$$

$$J_n(a, \mathrm{x}) = \mathbb{E}_{\mathbb{P}_0}\left[\int_{\mathcal{A}\times\mathcal{X}} (\Psi(\alpha, u; G) - \widehat{\Psi}_{n+1}(\alpha, u))^2 dh(\alpha) d\mathbb{P}_{\mathrm{X}}(u) \middle| A_{n+1} = a, \mathrm{X}_{n+1} = \mathrm{x}, \mathcal{F}_n\right]$$

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$$\boldsymbol{J}_n(\boldsymbol{a}, \mathbf{x}) = \mathbb{E}_{\mathbb{P}_0}\left[\int_{\mathcal{A} \times \mathcal{X}} (\boldsymbol{\Psi}(\boldsymbol{\alpha}, \boldsymbol{u}; \boldsymbol{G}) - \widehat{\boldsymbol{\Psi}}_{n+1}(\boldsymbol{\alpha}, \boldsymbol{u}))^2 d\boldsymbol{h}(\boldsymbol{\alpha}) d\mathbb{P}_{\mathbf{X}}(\boldsymbol{u}) \middle| \boldsymbol{A}_{n+1} = \boldsymbol{a}, \mathbf{X}_{n+1} = \mathbf{x}, \mathcal{F}_n\right]$$

Remark that $\boldsymbol{J}_n(\boldsymbol{a}, \boldsymbol{x})$ is an expectation w.r.t. $(\boldsymbol{Y}(\boldsymbol{A}_{n+1}, \mathbf{X}_{n+1}) | \mathcal{F}_n)$.

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Practical computation of J_n

It is possible to rewrite $\boldsymbol{J_n(a,x)}$ to perform a simple Monte-Carlo loop[2]:

$$\boldsymbol{J_n(a,x) = \int_{\mathcal{A}\times\mathcal{X}} \mathbb{E}_{\mathbb{P}_0}\left[\Psi(\alpha,u;G)^2\middle|\mathcal{F}_n\right] - \mathbb{E}_{\mathbb{P}_0}\left[\widehat{\Psi}_{n+1}(\alpha,u)^2\middle|A_{n+1}=a, \mathrm{X}_{n+1}=\mathrm{x}, \mathcal{F}_n\right] dh(\alpha)d\mathbb{P}_{\mathrm{X}}(u)}$$

[2] Clement Gauchy, Cyril Feau, and Josselin Garnier. Estimation of seismic fragility curves by sequential design of experiments.
February 2022.
hal-03588974

Practical computation of J_n

First, remark that $\boldsymbol{J_n(a,x)}$ is an expectation w.r.t. $(\boldsymbol{G}(\boldsymbol{A}_{n+1}, \mathbf{X}_{n+1})|\boldsymbol{\mathcal{F}_n})$.

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where $\widehat{G}_{n+1}(\alpha,u;Z)$ is the conditional mean of the GP with "virtual" output Z at design point $(A_{n+1}, \mathrm{X}_{n+1})$.

Practical computation of J_n

The first integral is approximated using Monte-Carlo simulation with a sample $(\boldsymbol{\alpha}_i, \boldsymbol{U}_i)_{1 \leq i \leq N}$ drawn from the product measure $\boldsymbol{h} \otimes \mathbb{P}_{\mathbf{X}}$.

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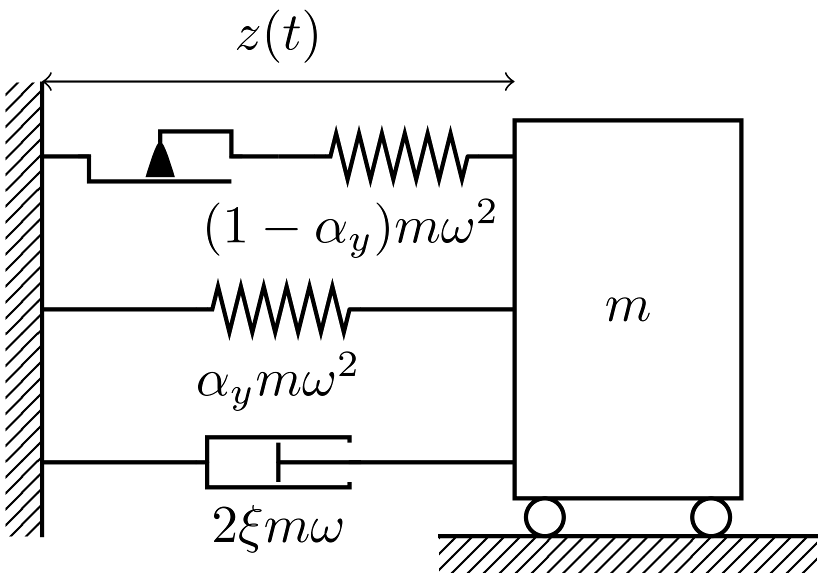
The two expectations are approximated using Gauss-Hermite quadrature:

$$\mathbb{E}_{Z \sim \mathcal{N}(\mu, \sigma^2)}[f(Z)] \approx \frac{1}{\sqrt{\pi}} \sum_{q=1}^{Q} \omega_q f(z_q) \,,$$

$$z_q = \mu + \sqrt{2}\sigma u_q \,,$$

Numerical application



Nonlinear oscillator with kinematic hardening

$$\ddot{\boldsymbol{z}}(\boldsymbol{t}) + 2\boldsymbol{\xi}\boldsymbol{\omega}\dot{\boldsymbol{z}}(\boldsymbol{t}) + \boldsymbol{f}^{\mathrm{NL}}(\boldsymbol{z}(\boldsymbol{t})) = -\boldsymbol{s}(\boldsymbol{t})\ , \tag{1}$$

where $\boldsymbol{f}^{\mathrm{NL}}$ is a nonlinear restoring force.

Uncertain mechanical parameters

Table: Probabilistic model of $\mathbf{X}$ for the nonlinear oscillator.

Variable	Name	Mean
$\boldsymbol{m}$ (kg)	Mass of the system	**300**
$\boldsymbol{k}$ (N/m)	Stiffness	$\mathbf{2.7\ 10^5}$
$\boldsymbol{\xi}$ (1)	Damping ratio	**0.015**
$\boldsymbol{z_d}$ (m)	Yield displacement	$\mathbf{5\ 10^{-3}}$
$\boldsymbol{\alpha_y}$ (1)	Post-yield stiffness	$\mathbf{2\ 10^{-4}}$

The marginal distributions are uniforms with 15% coefficient of variation. The parameters are considered independent.

Benchmark settings

The input variables of the GP is composed of the subset $(\boldsymbol{PGA}, \boldsymbol{k}, \boldsymbol{m})$ (these are the most influential variables). $\mathbf{10^5}$ artificial ground motions and random draws of $\boldsymbol{k}$ and $\boldsymbol{m}$ are generated and the nonlinear oscillator is evaluated for each realization.

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10 randomly chosen realizations are used for initialization.

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At step $\boldsymbol{n}$, $\boldsymbol{m} = \mathbf{1000}$ candidate points $(\boldsymbol{A}_i, \mathbf{X}_i)_{1\leq i\leq m}$ are subsampled in the dataset of $\mathbf{10^5}$ computations. We define:

$$(\boldsymbol{A}_{n+1}^{\mathrm{SUR}}, \mathbf{X}_{n+1}^{\mathrm{SUR}}) = \underset{1\leq i\leq m}{\operatorname{argmin}} \boldsymbol{J}_n(\boldsymbol{A}_i, \mathbf{X}_i)\ .$$

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The Gaussian process hyperparameters are updated every $\mathbf{10}$ iterations using a MAP estimator with a jointly robust prior [3].

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Performance metrics

A numerical benchmark is carried out to compare the performance of SUR strategy and Monte-Carlo designs in terms of posterior variance:

$$\boldsymbol{v_n} = \mathbb{E}_{\mathbb{P}_0}\left[\int_{\mathcal{A}\times\mathcal{X}} (\boldsymbol{\Psi}(\boldsymbol{\alpha}, \boldsymbol{u}; \boldsymbol{G}) - \widehat{\boldsymbol{\Psi}}_n(\boldsymbol{\alpha}, \boldsymbol{u}))^2 \boldsymbol{dh}(\boldsymbol{\alpha})\boldsymbol{d}\mathbb{P}_{\mathrm{X}}(\boldsymbol{u}) \middle| \mathcal{F}_n\right] ,$$

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and in terms of bias using a reference fragility curve $\boldsymbol{\Psi}_{\mathrm{ref}}$:

$$\boldsymbol{b_n} = \int_{\boldsymbol{\mathcal{A}}\times\boldsymbol{\mathcal{X}}} (\boldsymbol{\Psi}_{\mathrm{ref}}(\boldsymbol{\alpha}, \boldsymbol{u}) - \widehat{\boldsymbol{\Psi}}_{\boldsymbol{n}}(\boldsymbol{\alpha}, \boldsymbol{u}))^2 \boldsymbol{dh}(\boldsymbol{\alpha}) \boldsymbol{d}\mathbb{P}_{\mathrm{X}}(\boldsymbol{u}) ,$$

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The integral is evaluated with a Monte-Carlo sample of size 5000 and the expectation on $\mathbb{P}_0$ using 4000 realizations of the GP surrogate.

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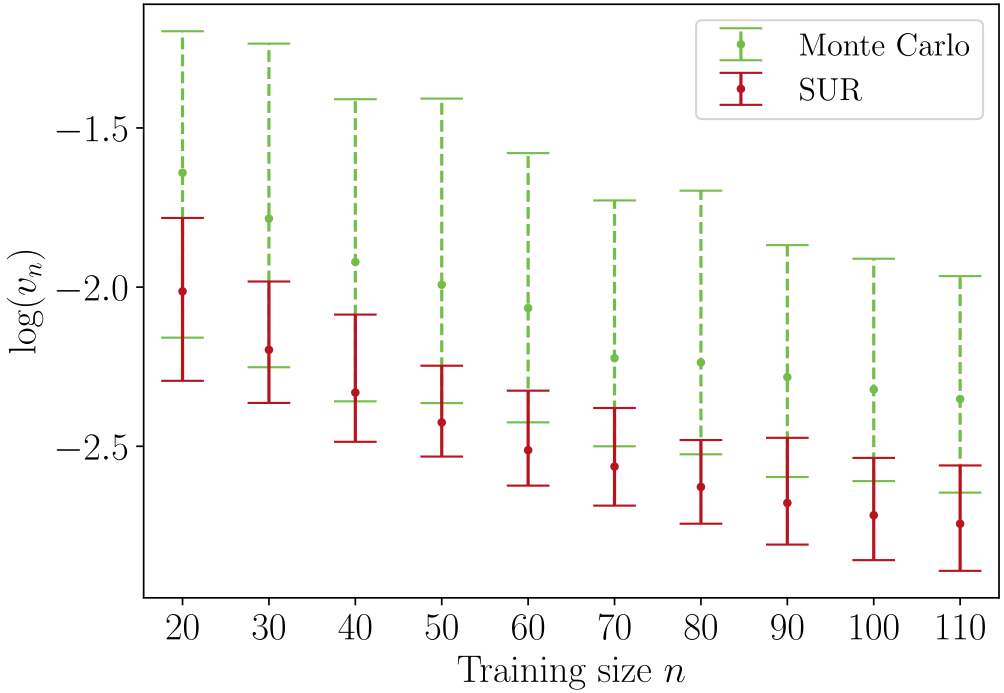
Due to the randomness induced in the SUR algorithm by choosing the candidate points at each step, **100** runs of the SUR strategy are carried out using HPC.

Performance assessment



Comparison of the posterior variance v_n between 100 Monte-Carlo designs and 100 runs of the SUR strategy for a failure threshold $C = 2.1 m_{z_d}$.

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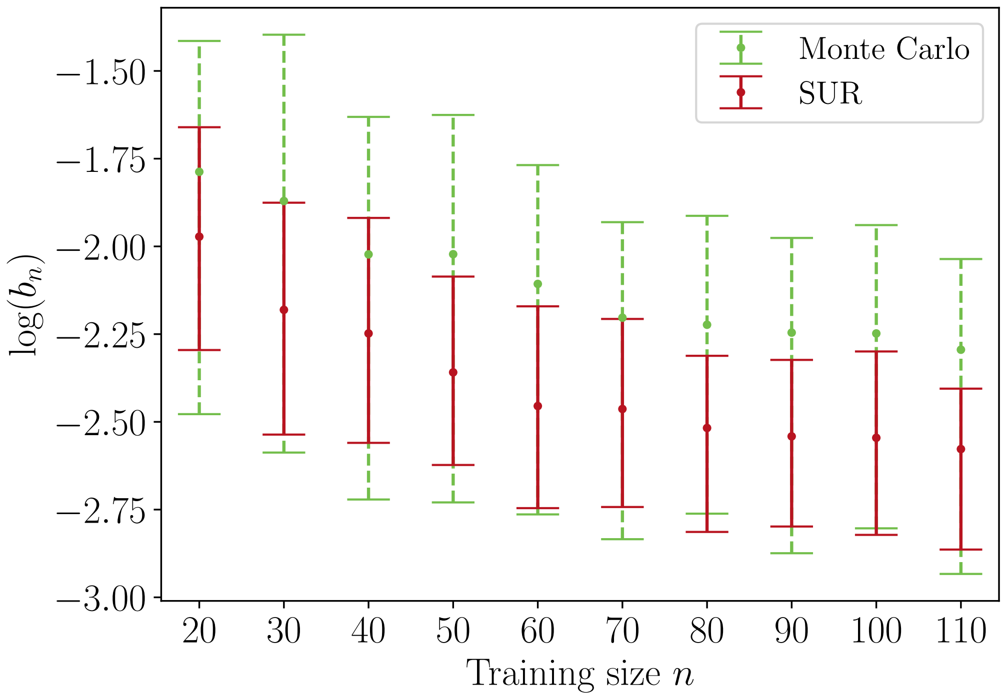

Comparison of the posterior bias $\boldsymbol{b_n}$ between 100 Monte-Carlo designs and 100 runs of the SUR strategy for a failure threshold $\boldsymbol{C = 2.1 m_{z_d}}$.

Conclusion

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Drawback: Very sensitive to the dimension of the input parameter, difficult optimization problem.

Merci pour votre attention !

`clgch.github.io`

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