Efficient Estimation of A-basis and B-basis Values under Epistemic Uncertainty Using Importance Sampling and Control Variates

Elton DONFACK-SIEWE, doctorant 2A

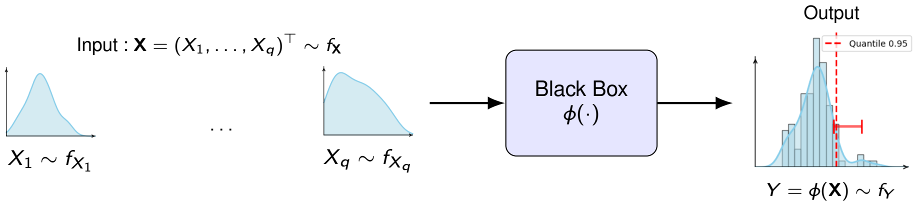
Directeur(s) : Jérôme Morio[1], Sylvain Dubreuil[1]
Encadrant(s) : Christian Fagiano[2], Jean-Philippe Navarro[3]

Financement(s) : AIRBUS

[1] ONERA/DTIS, [2] ONERA/DMAS, [3] AIRBUS OPERATIONS SAS

Introduction

Goals



- We consider the continuous random vector input $\mathbf{X} \sim f_{\mathbf{X}}$ of dimension q, defined by :

$$\mathbf{X} : \Omega \to \mathcal{X} \subseteq \mathbb{R}^q, \quad \omega \mapsto \mathbf{x} = \mathbf{X}(\omega)$$

- The problem of interest consists in the estimation of the q_α associated to a level of probability $\alpha \in]0, 1[$ defined by :

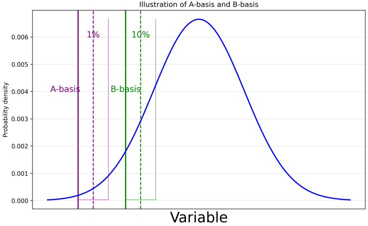
$$q_\alpha = \inf \{ y \in \mathbb{R} \mid F_Y(y) \geq \alpha \}$$

- F_Y is CDF of Y and the quantile is estimated directly from this CDF.

Introduction

Goals : A-Basis and B-basis

- **A-basis** is the 95% lower confidence bound on the 1st percentile, and **B-basis** is the 95% lower confidence bound on the 10th percentile.



- They are introduced in aerospace and defense standards (e.g., MIL-HDBK-17 / CMH-17) and required by certification rules for composite allowables [1, 2, 3].

Problem : How to estimate A-basis and B-basis in the presence of epistemic uncertainties, within a small-data context and under limited budget constraints ?

[1] CMH-17, Composite Materials Handbook, Volume 17 : Polymer Matrix Composites, ASTM International (2012).
[2] F. A. Administration, Advisory Circular AC 20-107B : Composite Aircraft Structure, Rapport Technique FAA (2003).
[3] E. U. A. S. Agency, Certification Specifications for Large Aeroplanes : CS-25, Rapport Technique EASA (2019).

Table of contents

1. State of the art
2. Uncertainties in statistical estimation
3. Propagation under three epistemic uncertainties
4. Application
5. Conclusion

State of the art

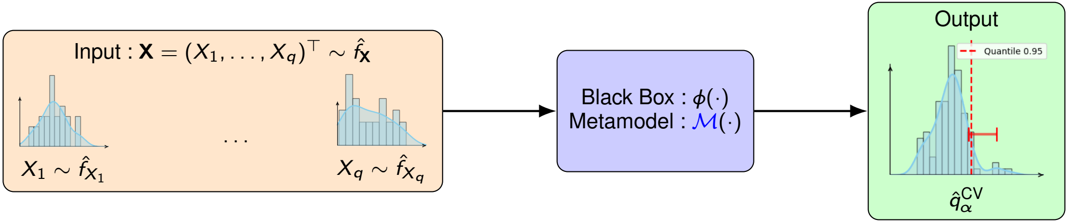
- Traditionally estimated using the Wilks method (one-sided tolerance bounds), ensuring coverage without strict distributional assumptions [4].
- Monte Carlo method [5].
- Methods also rely on safety margins to compensate for limited data and variability, providing conservative design allowables [6].

Limitations of classical approaches :

- Assume data are representative.
- Do not fully account for epistemic uncertainties : probabilistic model identification, statistical estimation uncertainty, surrogate modeling error (if used).

[4] S. S. Wilks, Determination of Sample Sizes for Setting Tolerance Limits, Annals of Mathematical Statistics, 12 (1), pp. 91–96 (1941).
[5] R. A. S. Cardoso et al., A-Basis and B-Basis Buckling Allowables for Composite Semi-Wing –Application of MonteCarlo Simulations and Sensitivity Analysis, Proceedings of the 34th Congress of the International Council of the Aeronautical Sciences (ICAS) (2024).
[6] O. L. D. Weck, C. Eckert et P. J. Clarkson, Uncertainty and Design Allowables for Aerospace Composites, Encyclopedia of Aerospace Engineering. Wiley (2019).

Uncertainties in statistical estimation



Three sources of epistemic uncertainty :

- Probabilistic model identification
- Surrogate modeling error
- Statistical estimation uncertainty

Uncertainties in statistical estimation

Identification of input density functions from experimental test data

- The true input distribution $f_{\mathbf{X}}$ is unknown
- Only a limited dataset $\tilde{\mathbf{X}}_{\text{obs}} = \{\mathbf{X}_{\text{obs}(i)}, i = 1, \cdots, N_{\text{obs}}\}$ is available

Physical test → $\tilde{\mathbf{X}}_{\text{obs}} = \{\mathbf{X}_{\text{obs}(i)},\ i = 1, \ldots, N_{\text{obs}}\}$, $\mathbf{X}_{\text{obs}(i)} \overset{\text{i.i.d.}}{\sim} f_{\mathbf{X}}$ → $\hat{f}_{\mathbf{X}|\tilde{\mathbf{X}}_{\text{obs}}}$

- Limited N_{obs} induces **a first source of epistemic uncertainty** in model identification

How do we take it into account ? by bootstrapping [7, 8] the test samples that we have

$\tilde{\mathbf{X}}_{\text{obs}}$ —Bootstrap (BS)→ $\tilde{\mathbf{X}}^*_{\text{obs}} = \{\mathbf{X}^*_{\text{obs}(i)},\ i = 1, \ldots, N_{\text{obs}}\}$

[7] C. H. Yu, Resampling methods : concepts, applications, and justification, Practical Assessment, Research, and Evaluation, 8 (1), p. 19 (2002).
[8] B. Efron, The Jackknife, the Bootstrap and Other Resampling Plans, SIAM (1982).

Uncertainties in statistical estimation

Estimation of a quantity of interest : Monte Carlo

- We generate a Monte Carlo simulation sample : $\tilde{\mathbf{X}} = \{\mathbf{X}_j,\ i = 1, \ldots, N_{\text{mc}}\} \quad , \mathbf{X}_j \sim \hat{f}_{\mathbf{X}|\tilde{\mathbf{X}}_{\text{obs}}}$
- The variance of the quantile estimator depends on N_{mc} as

$$\mathbb{V}_g\left[\hat{q}_\alpha\right] = \frac{\alpha(1-\alpha)}{N_{\text{mc}}\, f_Y^2(q_\alpha)},$$

- Limited N_{mc} induces **a second source of epistemic uncertainty** in estimation method

How do we take it into account ? by bootstrapping [7, 8] the initial simulations samples

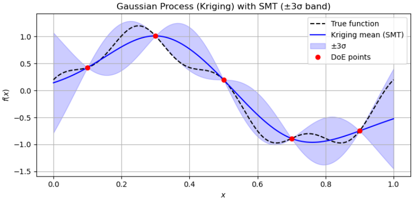
$\tilde{\mathbf{X}}$ → Bootstrap (BS) → $\tilde{\mathbf{X}}^* = \left\{\mathbf{X}_j^*,\ i = 1, \ldots, N_{\text{mc}}\right\}$

[7] C. H. Yu, Resampling methods : concepts, applications, and justification, Practical Assessment, Research, and Evaluation, 8 (1), p. 19 (2002).
[8] B. Efron, The Jackknife, the Bootstrap and Other Resampling Plans, SIAM (1982).

Uncertainties in statistical estimation

The surrogate model

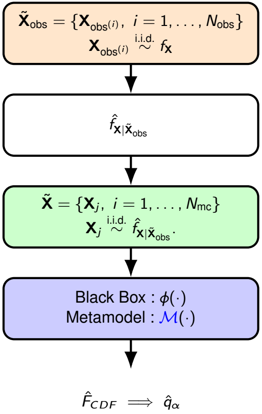
- They are built from a Design of Experiments (DoE), or they may be reduced models of the original one (e.g., analytical approximations)



- Common metamodels : Gaussian Process (Kriging), Polynomial Chaos Expansions (PCE), Machine Learning / Deep Neural Networks.
- Uncertainty sources : finite sample size, generalization errors (overfitting, truncation), model class limitations
- Modelisation : $(\omega, \mathbf{X}) \mapsto \hat{Y} = \mathcal{M}(\mathbf{X}, \omega)$

Uncertainties in statistical estimation

Summary



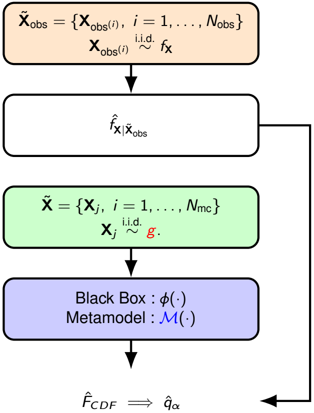
Presence of 3 sources of epistemic uncertainties (identification of laws, estimation of the QoI, and metamodel) but :

The identification uncertainty is linked to the Monte Carlo sampling uncertainty. How can we treat them independently ?

The metamodel can often poorly approximate the true black box. How can we use the metamodel, including its uncertainty, without biasing our estimator, regardless of the metamodel's quality ?

Uncertainties in statistical estimation

Summary



We use Importance Sampling (IS)[9] : It allows independent treatment of two sources of epistemic uncertainty probabilistic model identification and Monte Carlo sampling.

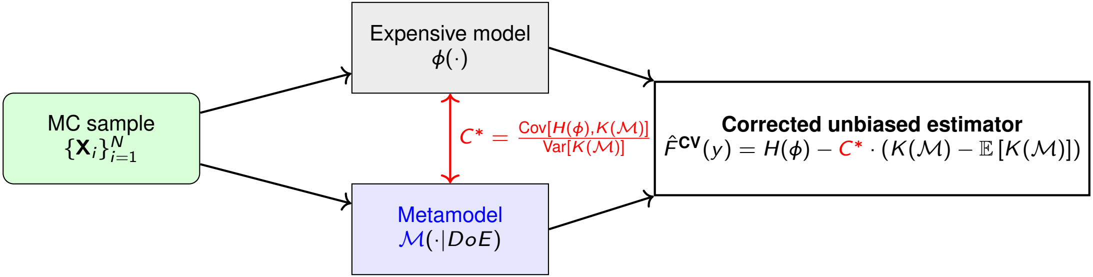
We use Control Variate (CV)[10] : CV leverages the metamodel to reduce variance without introducing bias or compromising estimator accuracy.

We need and estimator with CV and IS.

[9] S. Tokdar et R. Kass, Importance Sampling : A Review, Technical Report Department of Statistical Science, Duke University no TR-09-01 (2009).
[10] C. Cannamela, J. Garnier et B. Iooss, Controlled Stratification for Quantile Estimation, Annals of Applied Statistics, 2 (4), pp. 1554–1580 (2008).

Uncertainties in statistical estimation

Control Variate principle



$$\mathbb{V}[\hat{F}^{\text{CV}}(y)] = \mathbb{V}[H(\phi)][1 - \textbf{Corr}(H(\phi), K(\mathcal{M}))^2]$$

- **Good correlation :** If **Corr**$(H(\phi), K(\mathcal{M})) \approx \pm 1$, the variance is greatly reduced.
- **Poor correlation :** If **Corr**$(H(\phi), K(\mathcal{M})) \approx 0$, there is little to no variance reduction.

Control variates allow us to **weight the use of the metamodel** according to its quality **without risk**.
Now weed need to combined it with IS

Proposed quantile estimation using CV and IS

Assumption

We suppose :

- A metamodel $\mathcal{M}$ (Gaussian Process models, Polynomial Chaos Expansions, a Neural Network), for which the quantile z_α is assumed to be known and easy to estimate.
- The identified input distribution is denoted by $\hat{f}_{\mathbf{X}}$.
- We introduce a new auxiliary density g from which the Monte Carlo samples will be drawn. Its construction will be detailed after.

Proposed quantile estimation using CV and IS

CDF estimator using CV and IS

- We propose the following unbiased CDF estimator, coupling CV and IS (CV-IS) :

$$
\begin{aligned}
\hat{F}^{\text{CV-IS}}(y) &= \frac{1}{N_{\text{mc}}} \sum_{i=1}^{N_{\text{mc}}} \mathbf{1}_{\phi(\mathbf{X}_i)\leq y} \frac{\hat{f}_{\mathbf{X}}(\mathbf{X}_i)}{g(\mathbf{X}_i)} - C(y) \left(\frac{1}{N_{\text{mc}}} \sum_{i=1}^{N_{\text{mc}}} \mathbf{1}_{\mathcal{M}(\mathbf{X}_i)\leq z_\alpha} \frac{\hat{f}_{\mathbf{X}}(\mathbf{X}_i)}{g(\mathbf{X}_i)} - \mathbb{E}_{\hat{f}_{\mathbf{X}}}\left[\mathbf{1}_{\mathcal{M}(\mathbf{X})\leq z_\alpha}\right] \right) \\
&= \hat{F}^{\text{IS}}_{\text{EE}}(y) - C(y)(\hat{h}^{\text{IS}}_n - \mathbb{E}_{\hat{f}_{\mathbf{X}}}\left[\mathbf{1}_{\mathcal{M}(\mathbf{X})\leq z_\alpha}\right])
\end{aligned}
$$

With

$$
\mathbf{X}_i \sim g, \quad \mathbb{E}_{\hat{f}_{\mathbf{X}}}\left[\mathbf{1}_{\mathcal{M}(\mathbf{X})\leq z_\alpha}\right] = \mathbb{E}_g\left[\mathbf{1}_{\mathcal{M}(\mathbf{X})\leq z_\alpha} \lambda^{\text{IS}}(\mathbf{X})\right] = \alpha \quad \text{and} \quad \lambda^{\text{IS}}(\mathbf{X}) = \frac{\hat{f}_{\mathbf{X}}(\mathbf{X})}{g(\mathbf{X})}
$$

- The optimal parameter $C(y)$ (in terms of variance reduction) is defined by :

$$
C(y) = \frac{\text{Cov}_g[\hat{F}^{\text{IS}}_{\text{EE}}(y), \hat{h}^{\text{IS}}_n]}{\mathbb{V}_g[\hat{h}^{\text{IS}}_n]}
$$

Proposed quantile estimation using CV and IS

Quantile estimator using CV and IS

- Using the CV-IS estimator Eq. (9) of the CDF of Y, the α-quantile estimator is :

$$\hat{q}_\alpha^{\text{CV-IS}} = \inf \left\{ q \in \mathbb{R} \;\middle|\; \hat{F}^{\text{CV-IS}}(q) \geq \alpha \right\} \tag{1}$$

- This estimator is asymptotically normal with the reduced variance :

$$\mathbb{V}_g[\hat{q}_\alpha^{\text{CV-IS}}] = \frac{\mathbb{E}_g[\mathbf{1}_{\phi(\mathbf{x}) \leq q_\alpha} \lambda^{\text{IS}}(\mathbf{X})^2] - \alpha^2}{N_{\text{mc}} f_Y^2(q_\alpha)} [1 - \rho_I^2(q_\alpha)] \tag{2}$$

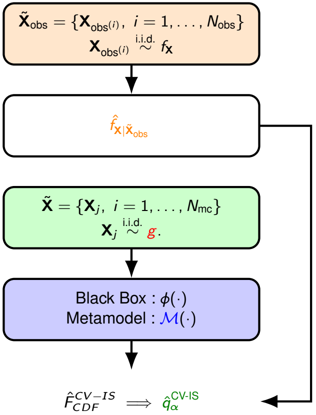
With

$$\rho_I(q_\alpha) = \text{Cor}_g[\mathbf{1}_{\phi \leq q_\alpha} \lambda^{\text{IS}}, \mathbf{1}_{\mathcal{M} \leq z_\alpha} \lambda^{\text{IS}}]$$

- **Good correlation :** $\rho_I(q_\alpha) \approx \pm 1$ and for **Poor correlation :** $\rho_I(q_\alpha) \approx 0$

Proposed quantile estimation using CV and IS

Summary



We now have an estimator combining two ingredients, CV and IS, which allows us to :

- Use IS to generate Monte Carlo samples from an auxiliary density, different from the identified one, thus disentangling the sources of uncertainty associated with each.
- Use CV to exploit the metamodel safely, regardless of its accuracy.

The next step is to propagate all sources of uncertainty.

Propagation under three epistemic uncertainties

Estimation of a quantity of interest

Given the available data and model, we aim to estimate the **quantile** of the quantity of interest using the **Importance Sampling with Control Variates (CV-IS)** method introduced in slide 14.

- The nominal input distribution $f_{\mathbf{X}}$ is replaced by its conditional estimate $\hat{f}_{\mathbf{X}|\tilde{\mathbf{x}}_{\text{obs}}=\tilde{\mathbf{x}}_{\text{obs}}}$
- The N_{mc}-simulation samples are generated from an auxiliary density function g
- The deterministic model output $\mathcal{M}$ is replaced by the metamodel realization m_k

Under these assumptions, the CV-IS estimator can be written as :

$$\hat{q}_{\alpha}^{\text{CV-IS}}(m_k, \tilde{\mathbf{x}}_{obs}, \tilde{\mathbf{X}}) = \inf\left\{q \in \mathbb{R} \;\middle|\; \hat{F}^{\text{CV-IS}}(q \mid m_k, \tilde{\mathbf{x}}_{obs}, \tilde{\mathbf{X}}) \geq \alpha\right\} \tag{3}$$

Propagation under three epistemic uncertainties

Propagation

We estimate the quantile by accounting for **three sources of epistemic uncertainty** within a augmented space : $(\mathcal{M}_n, \tilde{\mathbf{X}}_{\text{obs}}, \tilde{\mathbf{X}}) \in \Omega_q \times \mathcal{X}^{\otimes}_{\text{obs}} \times \mathcal{X}^{\otimes}$

- Ω_q : set of realizations of the metamodel $\mathcal{M}_n$
- $\mathcal{X}^{\otimes}_{\text{obs}}$: set of observed inputs (N_{obs} samples)
- $\mathcal{X}^{\otimes}$: set of simulated inputs for MC estimation (N_{mc} samples)

In the augmented space :

$$\mathbb{E}_{M_n,\tilde{X}_{obs},\tilde{X}}[q_\alpha] = \int_{\Omega_q \times \mathcal{X}^{\otimes}_{obs} \times \mathcal{X}^{\otimes}} \hat{q}_\alpha^{\text{CV-IS}}(m_n, \tilde{\mathbf{x}}_{obs}, \tilde{\mathbf{x}}) f_{(M_n,\tilde{X}_{obs},\tilde{X})}(m_n, \tilde{\mathbf{x}}_{obs}, \tilde{\mathbf{x}})\, d(m_n, \tilde{\mathbf{x}}_{obs}, \tilde{\mathbf{x}}) \qquad (4)$$

An estimator : $\hat{q}_\alpha^{\text{A-CV-IS}} = \frac{1}{N}\sum_{i=1}^{N} \hat{q}_\alpha^{\text{CV-IS}}(M_i, \tilde{X}_{obs^i}, \tilde{X}_i)$

$$\mathbb{V}_{(\mathcal{M}_n,\tilde{X}_{\text{obs}},\tilde{X})}\left[\hat{q}_\alpha^{\text{CV-IS}}\right] \approx \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{q}_\alpha^{\text{CV-IS}}(M_i, \tilde{X}_{obs^i}, \tilde{X}_i) - \hat{q}_\alpha^{\text{A-CV-IS}}\right)^2$$

Propagation under three epistemic uncertainties

Confidence interval

The estimated asymptotic confidence interval for the CV-IS quantile estimator $\hat{q}_\alpha^{\text{CV-IS}}$, at confidence level β is directly computed from the empirical quantiles of the sample $(\hat{q}_\alpha^{\text{CV-IS}}(M_i, \tilde{X}_{\text{obs}^i}, \tilde{X}_i))_{1\leq i\leq N}$:

$$\left[\hat{q}_\alpha^{\text{CV-IS}}{}_{\text{inf}}(\beta),\ \hat{q}_\alpha^{\text{CV-IS}}{}_{\text{sup}}(\beta)\right] = \left[\hat{\theta}_{1-\beta/2},\ \hat{\theta}_{\beta/2}\right], \tag{5}$$

where $\hat{\theta}_{1-\beta/2}$, and $\hat{\theta}_{\beta/2}$ are the empirical quantiles at levels $1-\beta/2$ and $\beta/2$ of the sample $(\hat{q}_\alpha^{\text{CV-IS}}(M_i, \tilde{X}_{\text{obs}^i}, \tilde{X}_i))_{1\leq i\leq N}$.

Propagation under three epistemic uncertainties

Variance decomposition and sensitivity indice (Sobol)

The following sensitivity indices can be estimated at no additional cost :

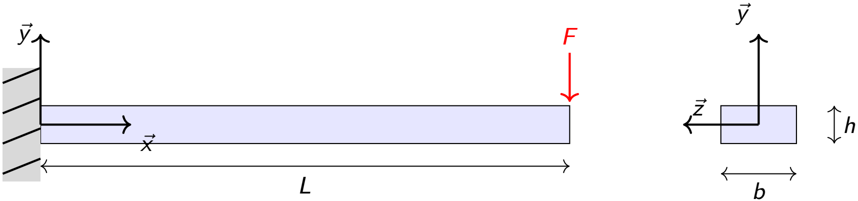
$$S_{\tilde{X}} = \frac{\mathbb{V}\left[\mathbb{E}[\hat{q}_\alpha^{\text{CV-IS}} \mid \tilde{X}]\right]}{\mathbb{V}[\hat{q}_\alpha^{\text{CV-IS}}]}, \quad S_{\tilde{X}}^T = 1 - \frac{\mathbb{V}\left[\mathbb{E}[\hat{q}_\alpha^{\text{CV-IS}} \mid \mathcal{M}_n, \tilde{X}_{obs}]\right]}{\mathbb{V}[\hat{q}_\alpha^{\text{CV-IS}}]}$$

$$S_{\tilde{X}_{obs}} = \frac{\mathbb{V}\left[\mathbb{E}[\hat{q}_\alpha^{\text{CV-IS}} \mid \tilde{X}_{obs}]\right]}{\mathbb{V}[\hat{q}_\alpha^{\text{CV-IS}}]}, \quad S_{\tilde{X}_{obs}}^T = 1 - \frac{\mathbb{V}\left[\mathbb{E}[\hat{q}_\alpha^{\text{CV-IS}} \mid \mathcal{M}_n, \tilde{X}]\right]}{\mathbb{V}[\hat{q}_\alpha^{\text{CV-IS}}]}$$

$$S_{\mathcal{M}_n} = \frac{\mathbb{V}\left[\mathbb{E}[\hat{q}_\alpha^{\text{CV-IS}} \mid \mathcal{M}_n]\right]}{\mathbb{V}[\hat{q}_\alpha^{\text{CV-IS}}]}, \quad S_{\mathcal{M}_n}^T = 1 - \frac{\mathbb{V}\left[\mathbb{E}[\hat{q}_\alpha^{\text{CV-IS}} \mid \tilde{X}_{obs}, \tilde{X}]\right]}{\mathbb{V}[\hat{q}_\alpha^{\text{CV-IS}}]}$$

Application : deflection of a cantilever beam

Description of the model



Cantilever beam (2D view) **Cross-section**

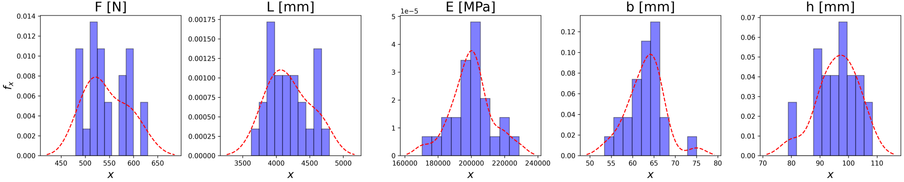
The function ϕ writes in the following analytical form with q = 5 uncertain inputs :

$$\phi(F, L, E_{\text{YM}}, b, h) = \frac{4FL^3}{E_{\text{YM}} b h^3} \quad (6)$$

Application : deflection of a cantilever beam

Assumptions

- We assume that the input distribution is known, but we have drawn N_{test} samples to perform the analysis.



$$\forall \mathbf{x} \in \mathbb{R}^d, \quad g(\mathbf{x}) = \frac{1}{N} \sum_{\ell=1}^{N} \hat{f}_{X|\tilde{X}_{obs^\ell}}(\mathbf{x} \mid \tilde{X}_{obs^\ell})$$

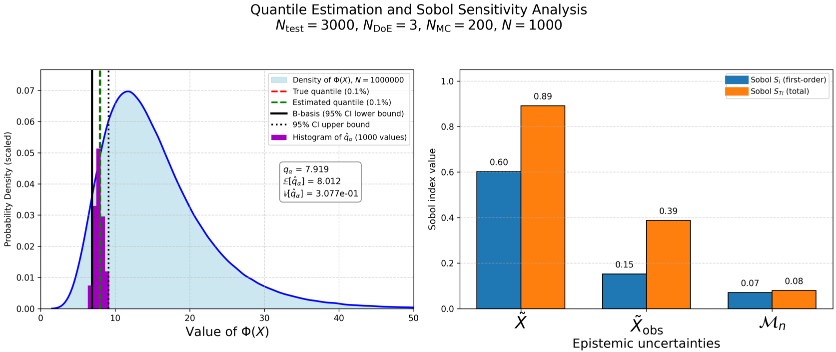
- The metamodel used is a Gaussian process trained on N_{DoE} samples.
- We perform uncertainty propagation using $N_{IS} = N_{mc}$ samples drawn from an auxiliary density.

We will study the influence of these three parameters on the estimators.

Application : deflection of a cantilever beam

Results

Simple Monte Carlo sampling uncertainties only, good input identification and poor metamodel.

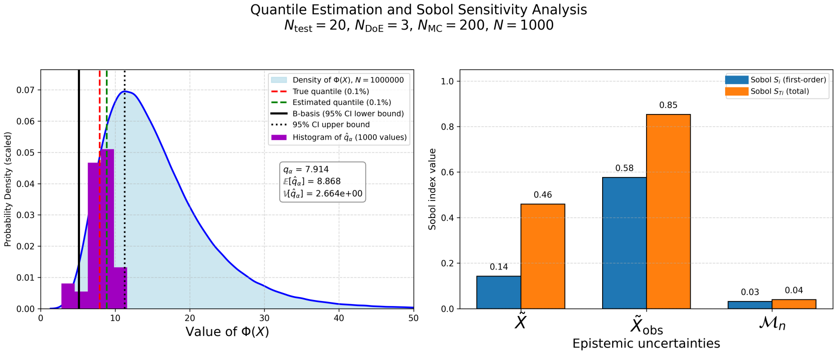


$\mathbb{V}_{(\mathcal{M}_n, \tilde{X}_{\text{obs}}, \tilde{X})}\left[\hat{q}_\alpha^{\text{CV-IS}}\right]$ not affected by the quality of the metamodel.

Application : deflection of a cantilever beam

Results

Simple Monte Carlo sampling uncertainties, bad input identification and poor metamodel.

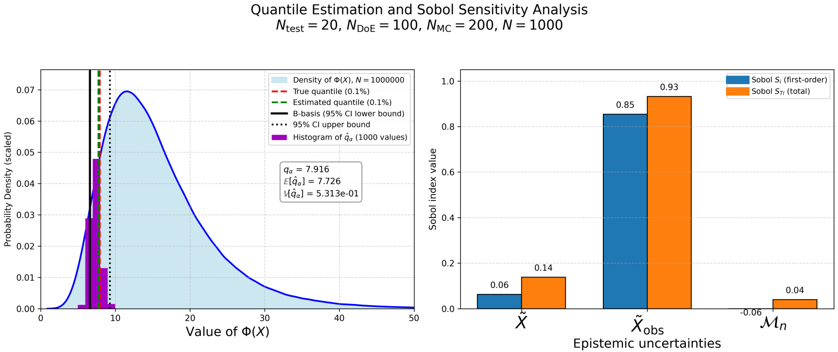


$\mathbb{V}_{(\mathcal{M}_n, \tilde{X}_{\text{obs}}, \tilde{X})}\left[\hat{q}_\alpha^{\text{CV-IS}}\right] \nearrow$ and not affected by the quality of the metamodel.

Application : deflection of a cantilever beam

Results

Simple Monte Carlo sampling uncertainties, bad input identification and good metamodel.



$\mathbb{V}_{(\mathcal{M}_n, \tilde{X}_{\text{obs}}, \tilde{X})}\left[\hat{q}_\alpha^{\text{CV-IS}}\right] \searrow$ due to the good quality of the metamodel.

Conclusion

Unified Quantile Estimation under Uncertainty

- Rigorous framework handling both *aleatory* and *epistemic* uncertainties : probabilistic model identification, statistical estimation uncertainty, metamodel error.

For a fixed budget and limited data context

- Robust alternative to A/B-basis methods, integrating uncertainties directly into quantile estimation which enables more efficient and justified designs (reduced safety margin).

Variance Reduction via Control Variates

- Metamodel used as a control variate not a replacement, ensures unbiased and consistent estimators.
- Free global sensitivity analysis (Sobol indices) on epistemic uncertainties, enabling potential enrichment strategies.

The application of this methodology is currently underway in a multiscale context of stiffened composite panels at Airbus.

Merci de votre attention !

Des questions ?

www.onera.fr

Bibliographie

[1] CMH-17,
Composite Materials Handbook, Volume 17 : Polymer Matrix Composites,
ASTM International (2012).

[2] FAA,
Advisory Circular AC 20-107B : Composite Aircraft Structure,
Rapport Technique Federal Aviation Administration (2003).

[3] EASA,
Certification Specifications for Large Aeroplanes : CS-25,
Rapport Technique EASA (2019).

[4] S. S. Wilks,
Determination of Sample Sizes for Setting Tolerance Limits,
Annals of Mathematical Statistics, 12 (1), pp. 91–96 (1941).

[5] R. A. S. Cardoso et al.,
A-Basis and B-Basis Buckling Allowables for Composite Semi-Wing – Application of Monte Carlo Simulations and Sensitivity Analysis,
Dans *Proceedings of the 34th Congress of the International Council of the Aeronautical Sciences (ICAS)* (2024).

[6] O. L. D. Weck, C. Eckert et P. J. Clarkson,
Uncertainty and Design Allowables for Aerospace Composites,
Dans *Encyclopedia of Aerospace Engineering*. Wiley (2019).

[7] C. H. Yu,
Resampling methods : concepts, applications, and justification,
Practical Assessment, Research, and Evaluation, 8 (1), p. 19 (2002).

[8] B. Efron,
The Jackknife, the Bootstrap and Other Resampling Plans,
SIAM (1982).

[9] S. Tokdar et R. Kass,
Importance Sampling : A Review,
Technical Report Department of Statistical Science, Duke University n° TR-09-01 (2009).

[10] C. Cannamela, J. Garnier et B. Iooss,
Controlled Stratification for Quantile Estimation,
Annals of Applied Statistics, 2 (4), pp. 1554–1580 (2008).

Appendix 1 -

$$\hat{F}^{\text{CV-IS}}(y) = \hat{F}^{\text{IS}}_{\text{EE}}(y) - \hat{C}(y)(\hat{h}_n - \alpha) = \sum_{j=1}^{N_{\text{mc}}} W_j \mathbf{1}_{\phi(\mathbf{x}_j)\leq y} \tag{7}$$

$$W_j = \left[\frac{1}{N_{\text{mc}}} + \frac{(\hat{h}_n - \alpha)(\hat{h}_n - \mathbf{1}_{\mathcal{M}(\mathbf{x}_j)\leq z_\alpha} \lambda^{\text{IS}}(\mathbf{X}_j))}{\sum_{i=1}^{N_{\text{mc}}} (\mathbf{1}_{\mathcal{M}(\mathbf{x}_i)\leq z_\alpha} \lambda^{\text{IS}}(\mathbf{X}_i) - \hat{h}_n)^2} \right] \lambda^{\text{IS}}(\mathbf{X}_j) \tag{8}$$

$$\text{if } W_j < 0, \text{ then } W_j := \frac{1}{N_{\text{mc}}} \lambda^{\text{IS}}(X_j),$$

Appendix 2 -

- With the optimal value of $C(y)$, we have :

$$\hat{F}^{\text{CV-IS}}(y) = \hat{F}^{\text{IS}}_{\text{EE}}(y) - \hat{C}(y)(\hat{h}^{\text{IS}}_n - \alpha) = \sum_{j=1}^{N_{\text{mc}}} W_j \mathbf{1}_{\phi(\mathbf{x}_j) \leq y} \tag{9}$$

With

$$W_j = \left[\frac{1}{N_{\text{mc}}} + \frac{(\hat{h}^{\text{IS}}_n - \alpha)(\hat{h}^{\text{IS}}_n - \mathbf{1}_{\mathcal{M}(\mathbf{x}_j) \leq z_\alpha} \lambda^{\text{IS}}(\mathbf{X}_j))}{\sum_{i=1}^{N_{\text{mc}}} (\mathbf{1}_{\mathcal{M}(\mathbf{x}_i) \leq z_\alpha} \lambda^{\text{IS}}(\mathbf{X}_i) - \hat{h}^{\text{IS}}_n)^2} \right] \lambda^{\text{IS}}(\mathbf{X}_j) \tag{10}$$

- We can show that :

$$\sum_{j=1}^{N_{\text{mc}}} W_j \xrightarrow[N_{\text{mc}} \to \infty]{a.s.} 1$$

Appendix 3 -

$$\rho_I(q_\alpha) = \mathrm{Cor}_g[\mathbf{1}_{\phi \le q_\alpha} \lambda^{\mathrm{IS}}, \mathbf{1}_{\mathcal{M} \le z_\alpha} \lambda^{\mathrm{IS}}] = \frac{\mathbb{E}_g[\mathbf{1}_{\phi(\mathbf{X}) \le q_\alpha} \mathbf{1}_{\mathcal{M}(\mathbf{X}) \le z_\alpha} \lambda^{\mathrm{IS}}(\mathbf{X})^2] - \alpha^2}{\sqrt{\mathbb{E}_g[\mathbf{1}_{\phi(\mathbf{X}) \le q_\alpha} \lambda^{\mathrm{IS}}(\mathbf{X})^2] - \alpha^2} \sqrt{\mathbb{E}_g[\mathbf{1}_{\mathcal{M}(\mathbf{X}) \le z_\alpha} \lambda^{\mathrm{IS}}(\mathbf{X})^2] - \alpha^2}}$$

$$C(y) = \frac{\mathrm{Cov}[\hat{F}^{\mathrm{IS}}_{\mathrm{EE}}(y), \hat{h}^{\mathrm{IS}}_n]}{\mathbb{V}[\hat{h}^{\mathrm{IS}}_n]} \approx \frac{\sum_{j=1}^{N_{\mathrm{mc}}} (\mathbf{1}_{\phi(\mathbf{x}_j) \le y} \lambda^{\mathrm{IS}}(\mathbf{X}_j) - \hat{F}^{\mathrm{IS}}_{\mathrm{EE}}(y))(\mathbf{1}_{\mathcal{M}(\mathbf{x}_j) \le z_\alpha} \lambda^{\mathrm{IS}}(\mathbf{X}_j) - \hat{h}^{\mathrm{IS}}_n)}{\sum_{i=1}^{N_{\mathrm{mc}}} (\mathbf{1}_{\mathcal{M}(\mathbf{x}_i) \le z_\alpha} \lambda^{\mathrm{IS}}(\mathbf{X}_i) - \hat{h}^{\mathrm{IS}}_n)^2}$$

Appendix 4 -

$(N_{\text{MC}}, N_{\text{test}}, N_{\text{DoE}})$	$\hat{\mathbb{E}}[q_\alpha^{\text{CV-IS}}]$	$\hat{\mathbb{V}}[q_\alpha^{\text{CV-IS}}]$	$\text{CI}_{0.95}$	$[\hat{S}_{\tilde{X}}, \hat{S}^T_{\tilde{X}}, \hat{S}_{\tilde{X}_{\text{obs}}}, \hat{S}^T_{\tilde{X}_{\text{obs}}}, \hat{S}_{\mathcal{M}_n}, \hat{S}^T_{\mathcal{M}_n}]$
(100, 700, 3)	5.05	1.47	[3.95, 7.16]	(0.50, 0.81, 0.21, 0.49, 0.01, 0.01)
(100, 700, 70)	4.87	0.89	[3.04, 5.92]	(0.24, 0.89, 0.09, 0.78, 0.04, 0.20)
(300, 30, 3)	4.82	0.53	[3.60, 5.80]	(0.21, 0.56, 0.38, 0.82, 0.03, 0.15)
(300, 30, 100)	4.55	0.45	[3.59, 5.60]	(0.23, 0.75, 0.28, 0.82, 0.02, 0.26)
(300, 300, 3)	3.92	0.60	[2.73, 5.29]	(0.20, 0.64, 0.33, 0.77, -0.01, 0.15)
(300, 300, 100)	4.22	0.37	[3.20, 5.23]	(0.04, 0.84, 0.18, 0.98, 0.01, 0.21)

True quantile : $q_\alpha = 4.89$ with $\alpha = 0.01$, $\beta = 0.95$

CI lower bound = A-basis value

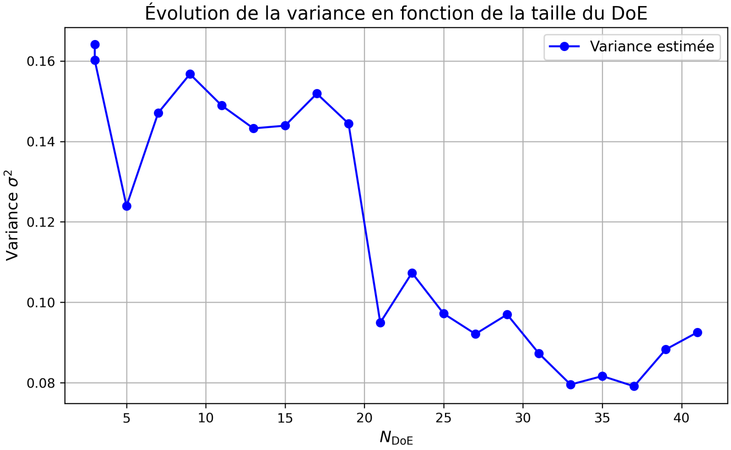
Appendix 5 -

$(N_{\text{MC}}, N_{\text{test}}, N_{\text{DoE}})$	$\hat{\mathbb{E}}[q_\alpha^{\text{CV-IS}}]$	$\hat{\mathbb{V}}[q_\alpha^{\text{CV-IS}}]$	$\text{CI}_{0.95}$	$[\hat{S}_{\tilde{X}}, \hat{S}^T_{\tilde{X}}, \hat{S}_{\tilde{X}_{\text{obs}}}, \hat{S}^T_{\tilde{X}_{\text{obs}}}, \hat{S}_{\mathcal{M}_n}, \hat{S}^T_{\mathcal{M}_n}]$
(100, 700, 3)	7.32	0.35	[6.17, 8.50]	(0.50, 0.85, 0.16, 0.41, 0.02, 0.14)
(100, 700, 70)	7.43	0.16	[6.80, 8.27]	(0.02, 0.75, 0.24, 0.89, 0.03, 0.53)
(300, 30, 3)	7.58	1.70	[4.78, 9.97]	(0.09, 0.25, 0.72, 0.93, 0.01, 0.06)
(300, 30, 100)	7.45	0.95	[5.47, 9.42]	(0.01, 0.10, 0.90, 0.99, 0.03, 0.09)
(300, 300, 3)	7.70	0.40	[6.60, 9.06]	(0.35, 0.67, 0.28, 0.63, -0.01, 0.07)
(300, 300, 100)	7.44	0.17	[6.61, 8.15]	(0.01, 0.49, 0.55, 0.93, 0.00, 0.47)

True quantile : $q_\alpha = 7.89$ with $\alpha = 0.10$, $\beta = 0.95$

CI lower bound = B-basis value

Appendix 6 -



$\alpha = 0.10$