Decision-making on critical physical systems using risk measures

ETICS 2025 (Évian-les-Bains)

Marie Temple-Boyer[1,2], Guillaume Perrin[1], Vincent Chabridon[2], Julien Pelamatti[2] and Emmanuel Remy[2].

[1] Université Gustave Eiffel, COSYS, 14-20 boulevard Newton, 77447 Champs-sur-Marne, France.

[2] EDF R&D, 6 quai Watier, 78401 Chatou, France.

Contact: marie.temple-boyer@edf.fr

October 10th, 2025

Introduction and context

Main concern: To guarantee the reliability of critical physical systems.

Introduction and context

Main concern: To guarantee the reliability of critical physical systems.

Example: Flood model.
There is a flood if:

- the *altitude of the surface* (Z_c) is higher than the *altitude of the top of the dyke* (Z_d).

▶ Physical variable of interest $Y := Z_c - Z_d$.

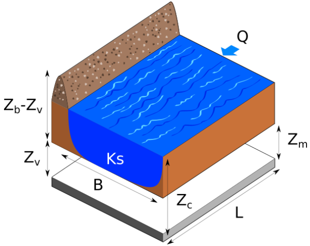


Figure 1: Flood model from [OpenTURNS, 2024]

Introduction and context

Main concern: To guarantee the reliability of critical physical systems.

Example: Flood model.
There is a flood if:

- the *altitude of the surface* (Z_c) is higher than the *altitude of the top of the dyke* (Z_d).

▶ Physical variable of interest $Y := Z_c - Z_d$.

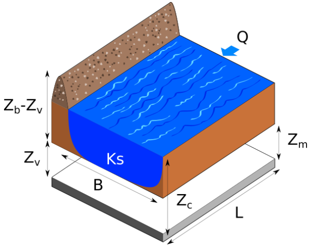


Figure 1: Flood model from [OpenTURNS, 2024]

Mathematical setting: Variable of interest $Y \in \mathbb{R}$.

- The greater Y, the less reliable the system.
- $Y = g(X)$, where input X is random to account for uncertainties.

Risk measures for engineering reliability

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space.
- V a subspace of the real valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1 (Risk measure)

A *risk measure* on V is an application

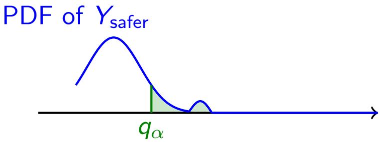
$$\mathcal{R} : \begin{cases} V & \rightarrow & \mathbb{R} \\ Y & \mapsto & \mathcal{R}(Y) \end{cases}.$$

Goals of the presentation

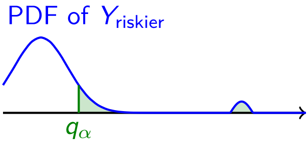
1. Categorization of risk measures:
   - Y-homogeneous risk measures,
   - $[0, 1]$-homogeneous risk measures.
2. Decision-making framework using risk measures.

Y-homogeneous risk measures

- **Quantile** $\boxed{q_\alpha(Y) := \inf\{y \in \mathbb{R} \mid F_Y(y) \geq \alpha\}}$ (F_Y CDF of Y).



(a) Safer random variable.



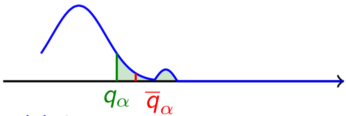
(b) Riskier random variable.

$$q_\alpha(Y_{\text{safer}}) = q_\alpha(Y_{\text{riskier}})$$

Y-homogeneous risk measures

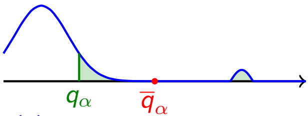
- **Quantile** $\boxed{q_\alpha(Y) := \inf\{y \in \mathbb{R} \mid F_Y(y) \geq \alpha\}}$ (F_Y CDF of Y).
- **Superquantile** $\boxed{\overline{q}_\alpha := \mathbb{E}(Y|Y \geq q_\alpha(Y))}$ [Rockafellar and Uryasev, 2002].

PDF of Y_{safer}



(a) Safer random variable.

PDF of Y_{riskier}

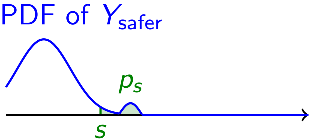


(b) Riskier random variable.

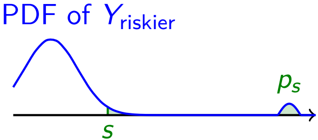
$$q_\alpha(Y_{\text{safer}}) = q_\alpha(Y_{\text{riskier}})$$
$$\overline{q}_\alpha(Y_{\text{safer}}) < \overline{q}_\alpha(Y_{\text{riskier}})$$

$[0,1]$-homogeneous risk measures

- **Failure probability** $\boxed{p_s(Y) := \mathbb{P}(Y > s)}$.



(a) Safer random variable.



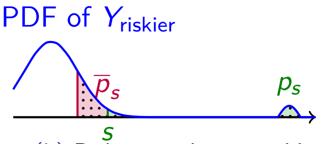
(b) Riskier random variable.

$$p_s(Y_{\text{safer}}) = p_s(Y_{\text{riskier}})$$

[0, 1]-homogeneous risk measures

- **Failure probability** $\boxed{p_s(Y) := \mathbb{P}(Y > s)}$.
- **Buffered failure probability** $\boxed{\overline{p}_s(Y) := 1 - \alpha}$ where $\alpha \in [0,1)$ is such that $\boxed{\overline{q}_\alpha(Y) = s}$ [Mafusalov and Uryasev, 2018].

(a) Safer random variable.



(b) Riskier random variable.

$$p_s(Y_{\text{safer}}) = p_s(Y_{\text{riskier}})$$
$$\overline{p}_s(Y_{\text{safer}}) < \overline{p}_s(Y_{\text{riskier}})$$

Duality

- $(\mathcal{R}_\alpha)_{0<\alpha<1}$ family of Y-homogeneous risk measures. $(\mathcal{R}_\alpha = q_\alpha)$
- $(\mathcal{P}_s)_{s\in\mathbb{R}}$ family of $[0,1]$-homogeneous risk measures. $(\mathcal{P}_s = p_s)$

Definition 2 ([Temple-Boyer et al., 2025])

The family of risk measures $(\mathcal{R}_\alpha)_{0<\alpha<1}$ and $(\mathcal{P}_s)_{s\in\mathbb{R}}$ are *dual* if, for any $Y \in V$, we have:

$$\forall(\alpha, s) \in (0,1) \times \mathbb{R},\ \mathcal{R}_\alpha(Y) \leq s \iff \mathcal{P}_s(Y) \leq 1-\alpha.$$

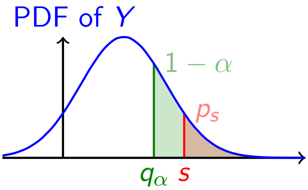
Duality

- $(\mathcal{R}_\alpha)_{0<\alpha<1}$ family of Y-homogeneous risk measures. $(\mathcal{R}_\alpha = q_\alpha)$
- $(\mathcal{P}_s)_{s\in\mathbb{R}}$ family of $[0,1]$-homogeneous risk measures. $(\mathcal{P}_s = p_s)$

Definition 2 ([Temple-Boyer et al., 2025])

The family of risk measures $(\mathcal{R}_\alpha)_{0<\alpha<1}$ and $(\mathcal{P}_s)_{s\in\mathbb{R}}$ are *dual* if, for any $Y \in V$, we have:

$$\forall(\alpha, s) \in (0,1) \times \mathbb{R},\ \mathcal{R}_\alpha(Y) \leq s \iff \mathcal{P}_s(Y) \leq 1 - \alpha.$$



Quantile and failure probability:

$$\boxed{q_\alpha(Y) \leq s \iff p_s(Y) \leq 1 - \alpha}.$$

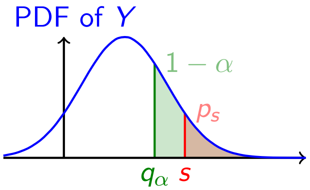
Duality

- $(\mathcal{R}_\alpha)_{0<\alpha<1}$ family of Y-homogeneous risk measures. $(\mathcal{R}_\alpha = q_\alpha)$
- $(\mathcal{P}_s)_{s\in\mathbb{R}}$ family of $[0,1]$-homogeneous risk measures. $(\mathcal{P}_s = p_s)$

Definition 2 ([Temple-Boyer et al., 2025])

The family of risk measures $(\mathcal{R}_\alpha)_{0<\alpha<1}$ and $(\mathcal{P}_s)_{s\in\mathbb{R}}$ are *dual* if, for any $Y \in V$, we have:

$$\forall(\alpha, s) \in (0,1) \times \mathbb{R},\ \mathcal{R}_\alpha(Y) \leq s \iff \mathcal{P}_s(Y) \leq 1-\alpha.$$



Quantile and failure probability:

$$\boxed{q_\alpha(Y) \leq s \iff p_s(Y) \leq 1-\alpha}.$$

Superquantile and buffered failure probability:

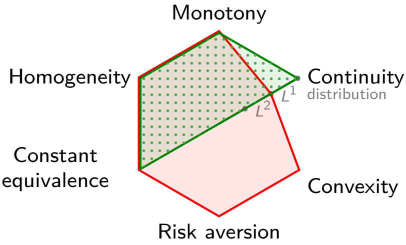
$$\boxed{\overline{q}_\alpha(Y) \leq s \iff \overline{p}_s(Y) \leq 1-\alpha}.$$

Desired properties for risk measures

Y-homogeneous risk measures	[0,1]-homogeneous risk measures
Quantile	Failure probability
Superquantile	Buffered failure probability

Desired properties for risk measures

Y-homogeneous risk measures	[0,1]-homogeneous risk measures
Quantile	Failure probability
Superquantile	Buffered failure probability

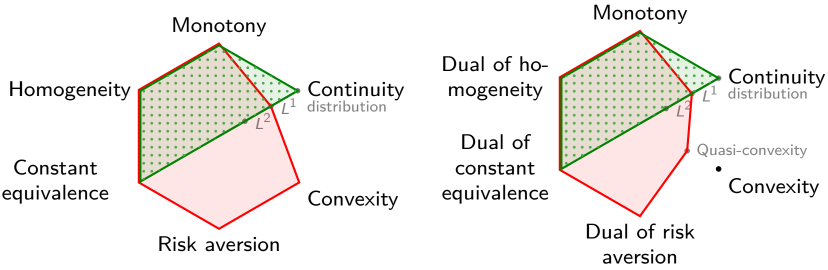


Properties for risk measures:

- [Artzner et al., 1999, Rockafellar and Royset, 2015].

Desired properties for risk measures

Y-homogeneous risk measures	[0,1]-homogeneous risk measures
Quantile	Failure probability
Superquantile	Buffered failure probability



Properties for risk measures:

- [Artzner et al., 1999, Rockafellar and Royset, 2015].
- Dual properties for $[0, 1]$-homogeneous case: [Temple-Boyer et al., 2025].

Decision-making using risk measures

- $(\mathcal{R}_\alpha)_{0<\alpha<1}$ and $(\mathcal{P}_s)_{s\in\mathbb{R}}$ dual families of risk measures.
- $1-\alpha \in (0,1)$ an acceptable risk and $s \in \mathbb{R}$ a security threshold.

Decision-making using risk measures

- $(\mathcal{R}_\alpha)_{0<\alpha<1}$ and $(\mathcal{P}_s)_{s\in\mathbb{R}}$ dual families of risk measures.
- $1-\alpha \in (0,1)$ an acceptable risk and $s \in \mathbb{R}$ a security threshold.

Decision Criterion 1 (Theoretical)

The system is safe if $\mathcal{R}_\alpha(Y) \leq s$ (or equivalently if $\mathcal{P}_s(Y) \leq 1-\alpha$).

Decision-making using risk measures

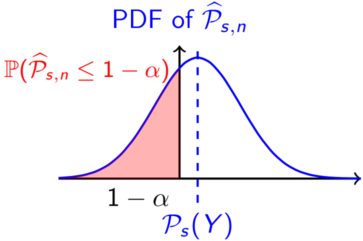
- $(\mathcal{R}_\alpha)_{0<\alpha<1}$ and $(\mathcal{P}_s)_{s\in\mathbb{R}}$ dual families of risk measures.
- $1-\alpha \in (0,1)$ an acceptable risk and $s \in \mathbb{R}$ a security threshold.

Decision Criterion 1 (Theoretical)

The system is safe if $\mathcal{R}_\alpha(Y) \leq s$ *(or equivalently if* $\mathcal{P}_s(Y) \leq 1-\alpha$*).*

- $\widehat{\mathcal{P}}_{s,n}$ numerical estimator of $\mathcal{P}_s(Y)$ from an i.i.d. sample $(Y_1, ..., Y_n)$.

Practical decision criterion: $\widehat{\mathcal{P}}_{s,n} \leq 1-\alpha$,



Decision-making using risk measures

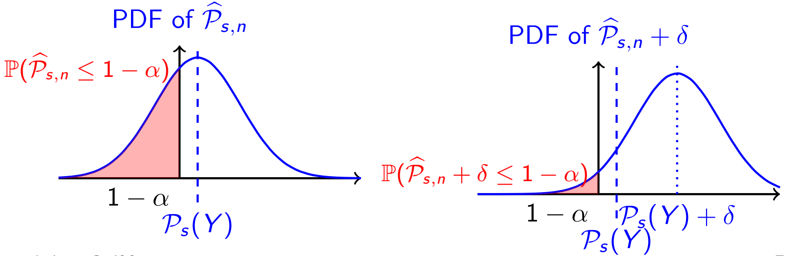
- $(\mathcal{R}_\alpha)_{0<\alpha<1}$ and $(\mathcal{P}_s)_{s\in\mathbb{R}}$ dual families of risk measures.
- $1-\alpha \in (0,1)$ an acceptable risk and $s \in \mathbb{R}$ a security threshold.

Decision Criterion 1 (Theoretical)

The system is safe if $\mathcal{R}_\alpha(Y) \leq s$ (or equivalently if $\mathcal{P}_s(Y) \leq 1-\alpha$).

- $\widehat{\mathcal{P}}_{s,n}$ numerical estimator of $\mathcal{P}_s(Y)$ from an i.i.d. sample $(Y_1, ..., Y_n)$.

Practical decision criterion: $\widehat{\mathcal{P}}_{s,n} \leq 1-\alpha$, or rather $\boxed{\widehat{\mathcal{P}}_{s,n} + \delta \leq 1-\alpha}$.



Main reliability result

- $1 - \beta \in (0, 1)$ a confidence level.

Main reliability result

- $1 - \beta \in (0, 1)$ a confidence level.

Security factor $\delta_*(\alpha, s, \beta, n) \in [0, 1 - \alpha]$ defined by:

$$\boxed{\delta_*(\alpha, s, \beta, n) := \inf \left\{ \delta > 0 \mid \sup_{Z \in V_{\text{unsafe}}} \mathbb{P}(\widehat{\mathcal{P}}_{s,n}(Z) + \delta \leq 1 - \alpha) < \beta \right\}}.$$

Main reliability result

- $1-\beta \in (0,1)$ a confidence level.

Security factor $\delta_*(\alpha, s, \beta, n) \in [0, 1-\alpha]$ defined by:

$$\delta_*(\alpha, s, \beta, n) := \inf \left\{ \delta > 0 \mid \sup_{Z \in V_{\text{unsafe}}} \mathbb{P}(\widehat{\mathcal{P}}_{s,n}(Z) + \delta \leq 1-\alpha) < \beta \right\}.$$

Decision Criterion 2 (Practical)

The system is labeled as safe at confidence level $1-\beta$ *if*

$$\widehat{\mathcal{P}}_{s,n} + \delta_*(\alpha, s, \beta, n) < 1-\alpha.$$

Proposition 1 ([Temple-Boyer et al., 2025])

If Y *is unsafe (i.e.* $\mathcal{R}_\alpha(Y) > s$*) then the probability that Criterion 2 labels the system as safe is below* β*.*

Main reliability result

- $1-\beta \in (0,1)$ a confidence level.

Security factor $\delta_*(\alpha, s, \beta, n) \in [0, 1-\alpha]$ defined by:

$$\delta_*(\alpha, s, \beta, n) := \inf \left\{ \delta > 0 \mid \sup_{Z \in V_{\text{unsafe}}} \mathbb{P}(\widehat{\mathcal{P}}_{s,n}(Z) + \delta \leq 1-\alpha) < \beta \right\}.$$

Decision Criterion 2 (Practical)

The system is labeled as safe at confidence level $1-\beta$ if $\widehat{\mathcal{P}}_{s,n} + \delta_*(\alpha, s, \beta, n) < 1-\alpha$.

Proposition 1 ([Temple-Boyer et al., 2025])

If Y is unsafe (i.e. $\mathcal{R}_\alpha(Y) > s$) then the probability that Criterion 2 labels the system as safe is below β.

Limits:

- $\delta_*(\alpha, s, \beta, n)$ not easy to compute in general.
- When $\delta_*(\alpha, s, \beta, n) = 1-\alpha$, Criterion 2 is non informative.

Summary of the constants

Constants to be chosen:

- $s \in \mathbb{R}$ threshold: homogeneous to variable of interest Y.
- $\alpha \in (0,1)$: $1-\alpha$ is an acceptable risk. ($\mathcal{P}_s(Y) \leq 1-\alpha$ is an acceptable situation.)
- $\beta \in (0,1)$ maximum error for safe labeling: $1-\beta$ is a confidence level.
- $n \in \mathbb{N}$ sample size.

Summary of the constants

Constants to be chosen:

- $s \in \mathbb{R}$ threshold: homogeneous to variable of interest Y.
- $\alpha \in (0,1)$: $1-\alpha$ is an acceptable risk. ($\mathcal{P}_s(Y) \leq 1-\alpha$ is an acceptable situation.)
- $\beta \in (0,1)$ maximum error for safe labeling: $1-\beta$ is a confidence level.
- $n \in \mathbb{N}$ sample size.

Optimal security factor $\delta_*(\alpha, s, \beta, n) \in [0, 1-\alpha]$:
security factor for Criterion 2 ($\widehat{\mathcal{P}}_{s,n} + \delta_*(\alpha, s, \beta, n) < 1-\alpha$)

- ▶ Probability of wrongly labeling as safe is below β.

Application to the case quantile/failure probability (1/2)

- Crude Monte Carlo estimator: $\widehat{p}_{s,n} := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{Y_i > s\}}$.

  - $\delta_*(\alpha, s, \beta, n) = 1 - \alpha - \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))$ ($\mathcal{B}(\cdot,\cdot)$ binomial distribution).

Decision Criterion 3 (Practical)

The system is labeled as safe at confidence level $1-\beta$ if

$$\boxed{\widehat{p}_{s,n} < \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))}.$$

Application to the case quantile/failure probability (1/2)

- Crude Monte Carlo estimator: $\widehat{p}_{s,n} := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{Y_i > s\}}$.

▶ $\delta_*(\alpha, s, \beta, n) = 1 - \alpha - \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))$ ($\mathcal{B}(\cdot,\cdot)$ binomial distribution).

Decision Criterion 3 (Practical)

The system is labeled as safe at confidence level $1-\beta$ if

$$\boxed{\widehat{p}_{s,n} < \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))}.$$

1. Choose α, s, β, n.
2. Simulate a sample $(Y_1, ..., Y_n)$
3. Compute $\widehat{p}_{s,n} := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{Y_i > s\}}$.
4. Use Criterion 3.

Application to the case quantile/failure probability (2/2)

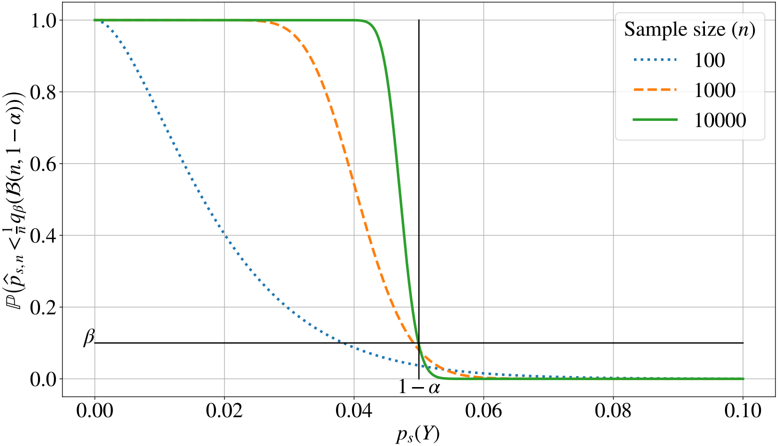


Figure 4: Probability that the system is labeled as safe using Criterion 3.

Link with Wilks' method

- $Y_{(1,n)} \leq ... \leq Y_{(n,n)}$ order statistics of the sample $(Y_1, ..., Y_n)$.
- $k_*(\alpha, \beta, n) := n - q_\beta(\mathcal{B}(n, 1-\alpha)) + 1$.

Proposition 2 ([Temple-Boyer et al., 2025])

Criterion 3 is equivalent to $Y_{(k_*(\alpha,\beta,n),n)} \leq s$.

Criterion 3: $\widehat{p}_{s,n} < \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))$.

Link with Wilks' method

- $Y_{(1,n)} \leq ... \leq Y_{(n,n)}$ order statistics of the sample $(Y_1, ..., Y_n)$.
- $k_*(\alpha, \beta, n) := n - q_\beta(\mathcal{B}(n, 1-\alpha)) + 1$.

Proposition 2 ([Temple-Boyer et al., 2025])

Criterion 3 is equivalent to $Y_{(k_*(\alpha,\beta,n),n)} \leq s$.

Criterion 3: $\widehat{p}_{s,n} < \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))$.

Wilks' method: quantile estimator $Y_{(k_W,n)}$ where

- $k_W := \min\left\{1 \leq k \leq n \mid \mathbb{P}(Y_{(k,n)} < q_\alpha(Y)) < \beta\right\}$.

If Y admits a PDF, then $k_W = k_*(\alpha, \beta, n)$.

Link with Wilks' method

- $Y_{(1,n)} \leq ... \leq Y_{(n,n)}$ order statistics of the sample $(Y_1, ..., Y_n)$.
- $k_*(\alpha, \beta, n) := n - q_\beta(\mathcal{B}(n, 1-\alpha)) + 1$.

Proposition 2 ([Temple-Boyer et al., 2025])

Criterion 3 is equivalent to $Y_{(k_*(\alpha,\beta,n),n)} \leq s$.

Criterion 3: $\widehat{p}_{s,n} < \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))$.

Wilks' method: quantile estimator $Y_{(k_W,n)}$ where

- $k_W := \min \left\{1 \leq k \leq n \mid \mathbb{P}(Y_{(k,n)} < q_\alpha(Y)) < \beta\right\}$.

If Y admits a PDF, then $k_W = k_*(\alpha, \beta, n)$.

- ▶ **For reliability-oriented decision making , it is equivalent to work with a quantile or a failure probability.**

Application to the flood model

Question: Is the flood probability below the acceptable risk $1 - \alpha$?

Threshold	Acceptable risk	Confidence level	Sample size
s	$1 - \alpha$	$1 - \beta$	n
0m	5.10^{-4}	$1 - 10^{-5}$	3.10^{4}

Table 1: Table of parameters used in the flood model test case.

Application to the flood model

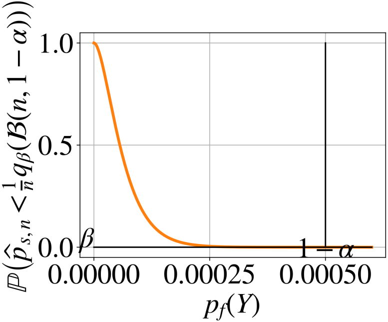
Question: Is the flood probability below the acceptable risk $1-\alpha$?

Threshold	Acceptable risk	Confidence level	Sample size
s	$1-\alpha$	$1-\beta$	n
0m	5.10^{-4}	$1-10^{-5}$	3.10^4

Table 1: Table of parameters used in the flood model test case.

Criterion 3:

- $\widehat{p}_n < 1.67 \times 10^{-4}$.
- $Y_{(29\ 999,30\ 000)} \leq 0$m.

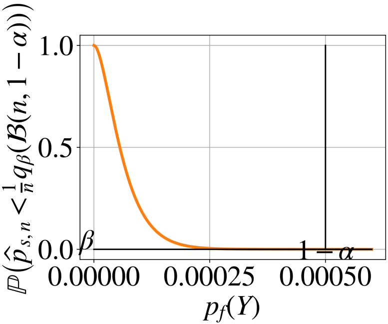


Application to the flood model

Question: Is the flood probability below the acceptable risk $1-\alpha$?

Threshold	Acceptable risk	Confidence level	Sample size
s	$1-\alpha$	$1-\beta$	n
0m	5.10^{-4}	$1-10^{-5}$	3.10^4

Table 1: Table of parameters used in the flood model test case.



Criterion 3:

- $\widehat{p}_n < 1.67 \times 10^{-4}$.
- $Y_{(29\,999,30\,000)} \leq 0\text{m}$.

Scenario	$\widehat{p}_n$	$Y_{(k_*(\alpha,\beta,n),n)}$	Criterion 3
Dyke is high	6.10^{-5}	-2.41m	yes
Dyke is low	5.67×10^{-4}	7.06m	no

Table 2: Numerical results on the flood case.

Conclusion and perspectives

On risk measures:

- Duality between *Y-homogeneous risk measures* and $[0,1]$-*homogeneous risk measures*.
- The couple *superquantile/buffered failure probability*: better theoretical properties than *quantile/failure probability*.

Conclusion and perspectives

On risk measures:

- Duality between Y-*homogeneous risk measures* and $[0,1]$-*homogeneous risk measures*.
- The couple *superquantile/buffered failure probability*: better theoretical properties than *quantile/failure probability*.

On the decision-making framework:

- General decision-making framework.
  - *Limit: Not straightforward to a generic risk measure.*
- Operational when the risk measures are quantile/failure probability.
  - *Limit: Crude MC estimator not applicable in the rare event context.*

Conclusion and perspectives

On risk measures:

- Duality between *Y-homogeneous risk measures* and $[0,1]$-*homogeneous risk measures*.
- The couple *superquantile/buffered failure probability*: better theoretical properties than *quantile/failure probability*.

On the decision-making framework:

- General decision-making framework.
  - *Limit: Not straightforward to a generic risk measure.*
- Operational when the risk measures are quantile/failure probability.
  - *Limit: Crude MC estimator not applicable in the rare event context.*

Perspectives:

- Working on the decision-making framework using the superquantile.
- Including more efficient failure probability estimators (e.g., importance splitting), to this decision-making framework.

Thank you for your listening.

Article submitted and available in HAL: [Temple-Boyer et al., 2025]

Contact: temple-boyer.marie@edf.fr

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999).
Coherent Measures of Risk.
Mathematical Finance, 9(3):203–228.

Mafusalov, A. and Uryasev, S. (2018).
Buffered Probability of Exceedance: Mathematical Properties and Optimization.
SIAM Journal on Optimization, 28(2):1077–1103.

OpenTURNS (2024).
OpenTURNS – A flood model.

Rockafellar, R. and Royset, J. (2015).
Engineering Decisions under Risk Averseness.
ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering, 1(2):04015003.

Rockafellar, R. T. and Uryasev, S. (2002).
Conditional value-at-risk for general loss distributions.
Journal of Banking & Finance, 26(7):1443–1471.

Temple-Boyer, M., Cussenot, E., Perrin, G., Chabridon, V., Pelamatti, J., Remy, E., and Iooss, B. (2025).
Risk measures in engineering reliability: comparison, duality and decision-making.

Superquantile

- $Y \in L^1(\Omega)$ a real-valued random variable.
- $0 < \alpha < 1$ a conservatism level.

Definition 3 (Superquantile)

$\overline{q}_\alpha(Y) := \min_{c \in \mathbb{R}} \left(c + \frac{1}{1-\alpha} \mathbb{E}([Y - c]_+) \right).$

Proposition 3

$q_\alpha(Y) = \frac{1}{1-\alpha} \int_\alpha^1 q_\beta(Y) d\beta.$

Proposition 4

Assume that Y admits a PDF. Then $\overline{q}_\alpha(Y) = \mathbb{E}(Y | Y > q_\alpha(Y)).$

Buffered failure probability (1/2)

- $Y \in L^1(\Omega)$ a real-valued random variable.
- $s \in \mathbb{R}$ a threshold.

Definition 4 (Buffered failure probability)

The buffered failure probability $\overline{p}_s(Y)$ is defined as follow.

- If $s < \mathbb{E}(Y)$ then $\overline{p}_s(Y) := 1$.
- If $s \geq \sup(Y)$ then $\overline{p}_s(Y) = 0$.
- Else, there exists a unique $\alpha \in [0, 1)$ such that $\overline{q}_\alpha(Y) = s$ and we define $\overline{p}_s(Y) := 1 - \alpha$.

Proposition 5

Assume that $s < \sup(Y)$, *then* $\overline{p}_s(Y) = \inf_{c \geq 0} \mathbb{E}([c(Y - s] + 1]_+)$.

Buffered failure probability (2/2)

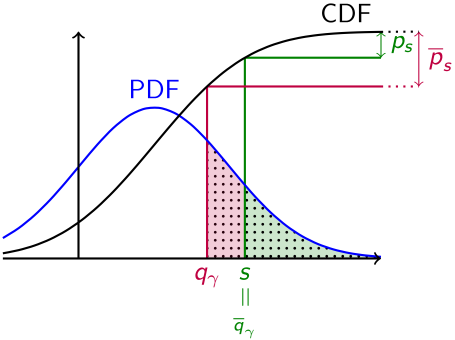


Figure 5: Illustration of the buffered failure probability definition.

Desired properties for Y-homogeneous risk measures

Definition 5 ([Artzner et al., 1999, Rockafellar and Royset, 2015])

Let $\mathcal{R}$ be a risk measure. Let define the following properties :

1. **(Monotony).** If $Y_1 \leq Y_2$ a.s. then $\mathcal{R}(Y_1) \leq \mathcal{R}(Y_2)$.
2. **(Continuity).** If $(Y_n)_{n\in\mathbb{N}}$ converges to Y then $(\mathcal{R}_{Y_n})_{n\in\mathbb{N}}$ converges to $\mathcal{R}(Y)$.
3. **(Homogeneity).** Pour $\lambda > 0$ et $c \in \mathbb{R}$, $\mathcal{R}(\lambda Y + c) = \lambda\mathcal{R}(Y) + c$.
4. **(Constant equivalence).** If $Y = c$ a.s. then $\mathcal{R}(Y) = c$.
5. **(Risk aversion).** If Y is non-constant then $\mathcal{R}(Y) > \mathbb{E}(Y)$.
6. **(Subadditivity).** $\mathcal{R}(Y_1 + Y_2) \leq \mathcal{R}(Y_1) + \mathcal{R}(Y_2)$.
7. **(Convexity).** For $0 \leq \lambda \leq 1$, $\mathcal{R}(\lambda Y_1 + (1-\lambda)Y_2) \leq \lambda\mathcal{R}(Y_1) + (1-\lambda)\mathcal{R}(Y_2)$

Dual properties for $[0,1]$-homogeneous risk measures

Definition 6 ([Temple-Boyer et al., 2025])

Let $(\mathcal{P}_s)_{s\in\mathbb{R}}$ be a family of $[0,1]$-homogeneous risk measures. We define the following dual properties:

1. **(Dual of monotony).** For all $s \in \mathbb{R}$ and $(Y_1, Y_2) \in V^2$ such that $Y_1 \leq Y_2$ a.s., then $\mathcal{P}_s(Y_1) \leq \mathcal{P}_s(Y_2)$.
3. **(Dual of homogeneity).** For all $Y \in V$, $s \in \mathbb{R}$, $\lambda > 0$ and $c \in \mathbb{R}$, we have $\mathcal{P}_{\lambda s+c}(\lambda Y + c) = \mathcal{P}_s(Y)$.
4. **(Dual of constant equivalence).** If $Y \in V$ and $c \in \mathbb{R}$ are such that $Y = c$ a.s., then for all $s \in \mathbb{R}$, we have $\mathcal{P}_s(Y) = \mathbb{1}_{\{s<c\}}$.
5. **(Dual of risk aversion).** For all Y that is not constant a.s., then for any $s \leq \mathbb{E}(Y)$, we have $\mathcal{P}_s(Y) = 1$.

Proposition 6

$(\mathcal{R}_\alpha)_{0<\alpha<1}$ *respects a properties if and only if its dual risk measure* $(\mathcal{P}_s)_{s\in\mathbb{R}}$ *respects the corresponding dual properties.*

Probability of labeling as safe

Criterion 3: $\widehat{p}_{s,n} < \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha)$

Since $n\widehat{p}_{s,n} = \sum_{i=1}^{n} \mathbb{1}_{\{Y_i > s\}} \sim \mathcal{B}(n, p_s(Y))$, we have

$$\boxed{\mathbb{P}\left(\widehat{p}_{s,n} < \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha)\right) = F_{\mathcal{B}(n, p_s(Y))}(q_\beta(\mathcal{B}(n, 1-\alpha)) - 1)}.$$

(F. CDF of the corresponding distribution)

Study of the security bound bound $\frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))$ (1/3)

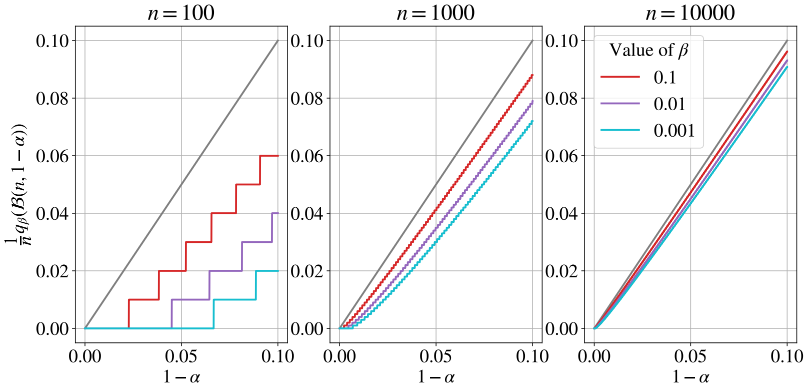


Figure 6: Security bound with respect to risk acceptable $1 - \alpha$.

Criterion 3: $\widehat{p}_{s,n} < \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))$

Study of the security bound bound $\frac{1}{n}q_\beta(\mathcal{B}(n, 1-\alpha))$ (2/3)

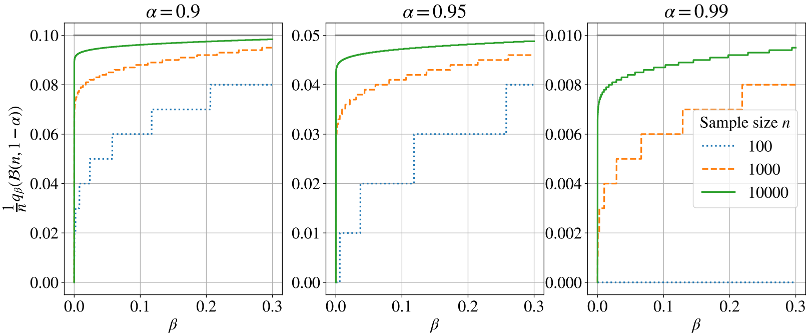


Figure 7: Security bound with respect to β.

Criterion 3: $\widehat{p}_{s,n} < \frac{1}{n}q_\beta(\mathcal{B}(n, 1-\alpha))$

Study of the security bound bound $\frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha))$ (3/3)

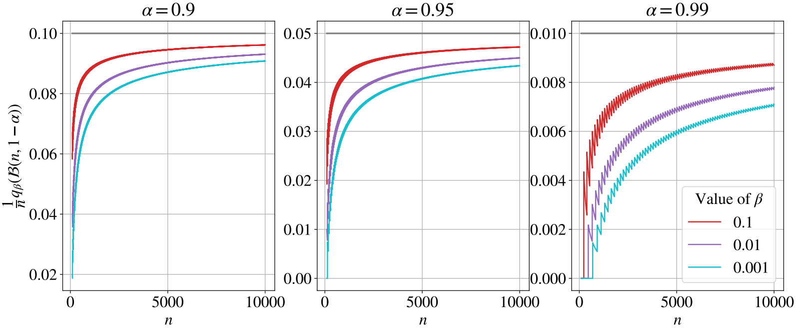


Figure 8: Security bound with respect to sample size n.

Criterion 3: $\widehat{p}_{s,n} < \frac{1}{n} q_\beta(\mathcal{B}(n, 1-\alpha)$