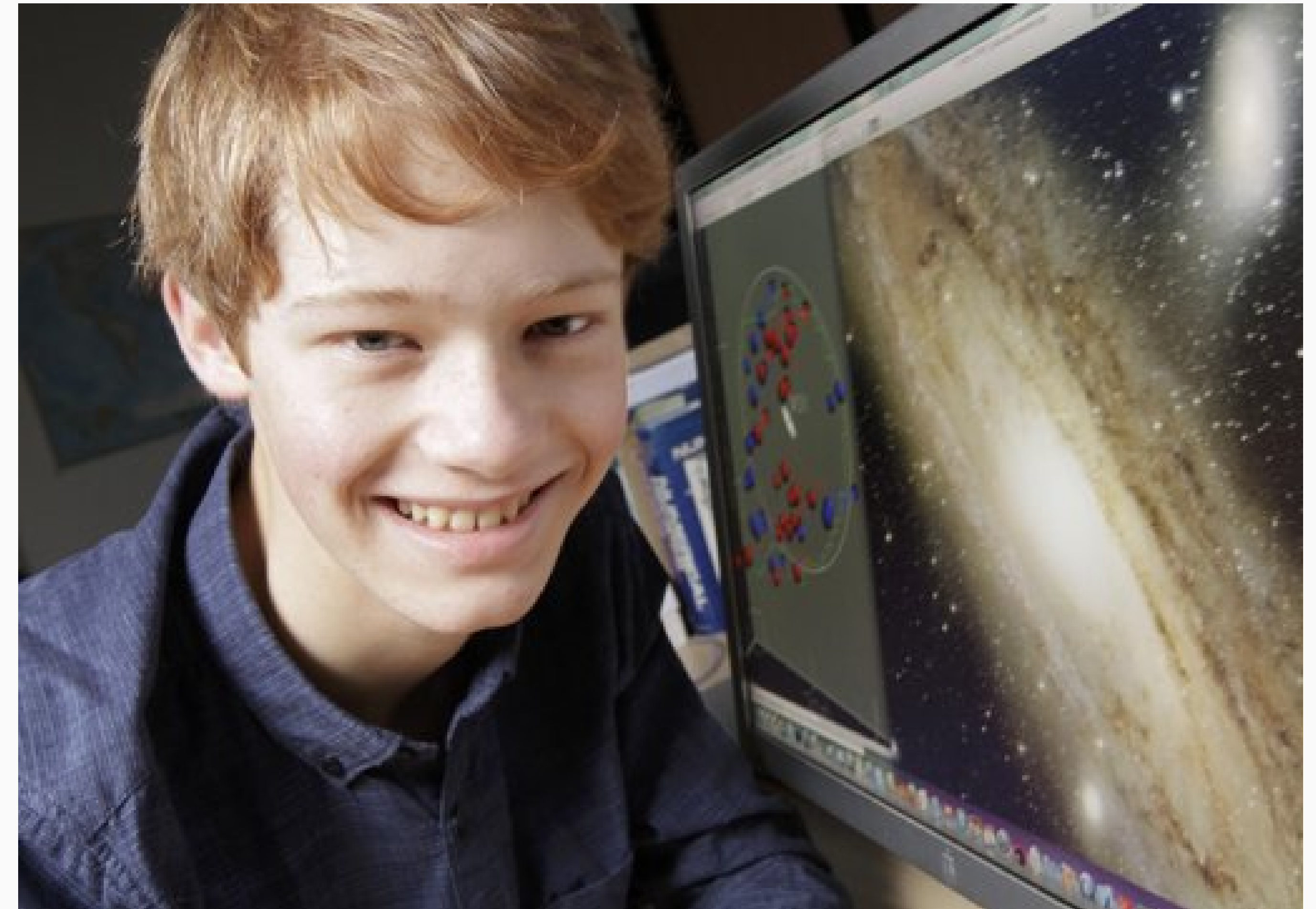
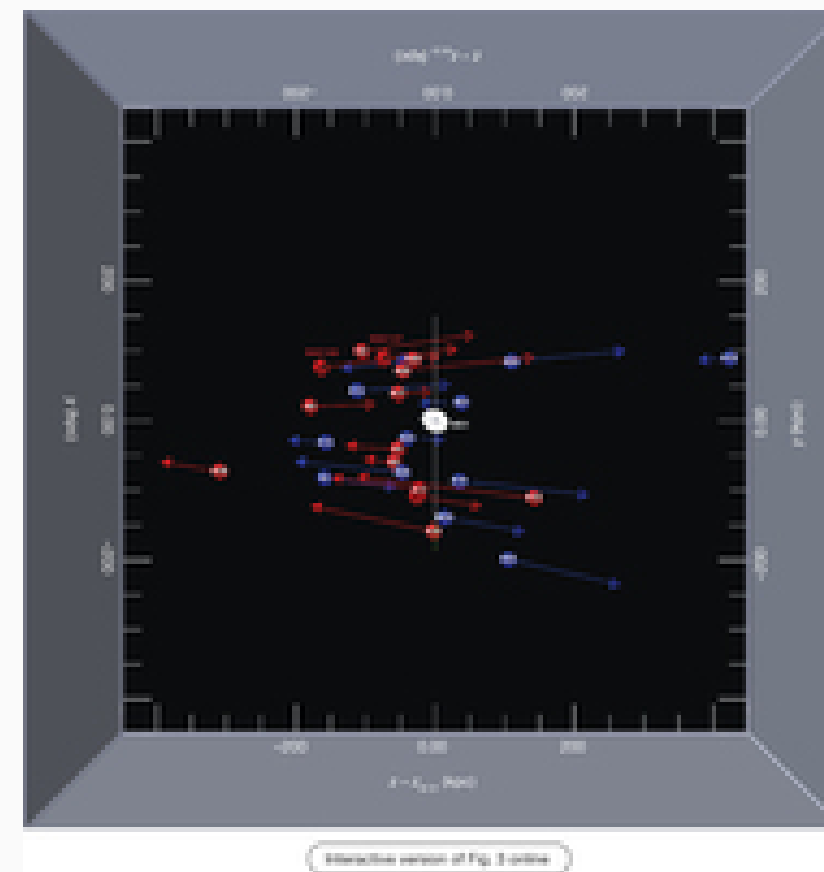
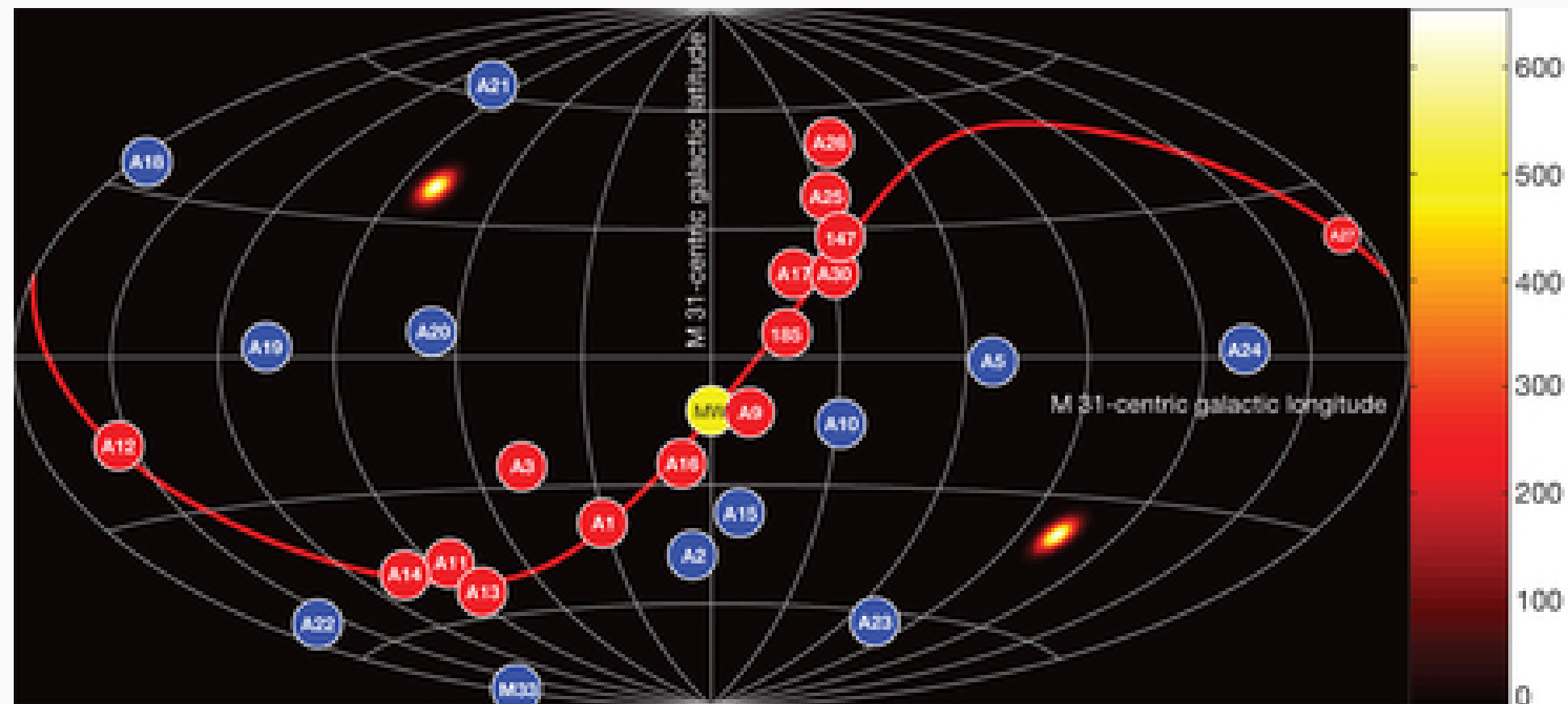


Visualization: what for?

- Dwarf galaxies orbiting the Andromeda Galaxy
 - *Nature*, January 3rd, 2013



Temp Volume Time: 0

- builtin:
 - disk_out_ref_ex2
 - StreamTracer1
 - Glyph2
 - GenerateTubes1
 - Glyph3
 - disk_out_ref_surface.vtp

Object Inspector

Properties Display Information

Root

- Element Blocks (1)
 - Unnamed block ID: 1 Type: HEX8 Size: 747...
- Face Blocks (3)
- Edge Blocks (4)

Interpolation: Gouraud

Material: None

Point size: 2.00

Line width: 1.00

Opacity: 1.00

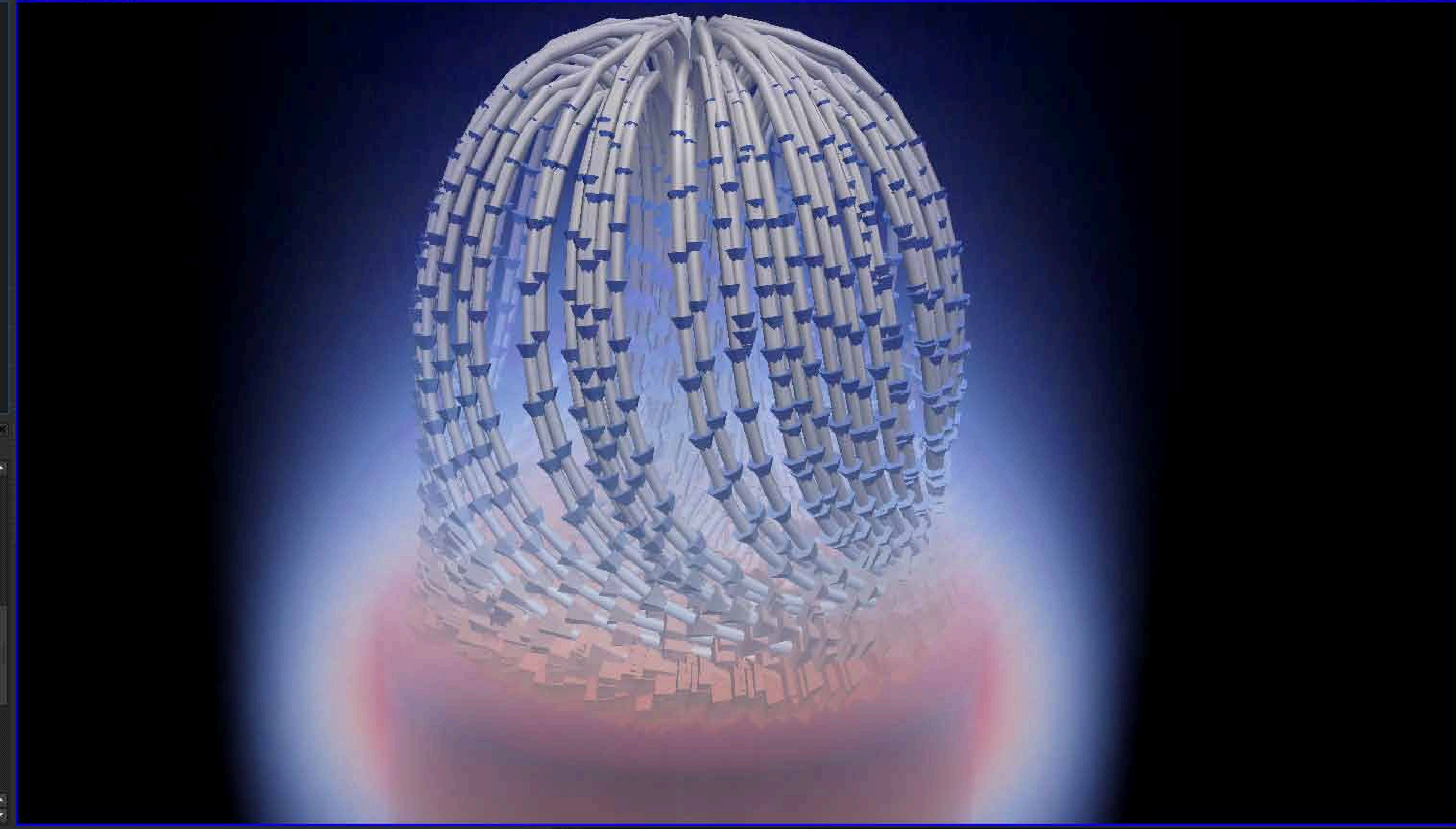
Volume mapper: Projected tetra

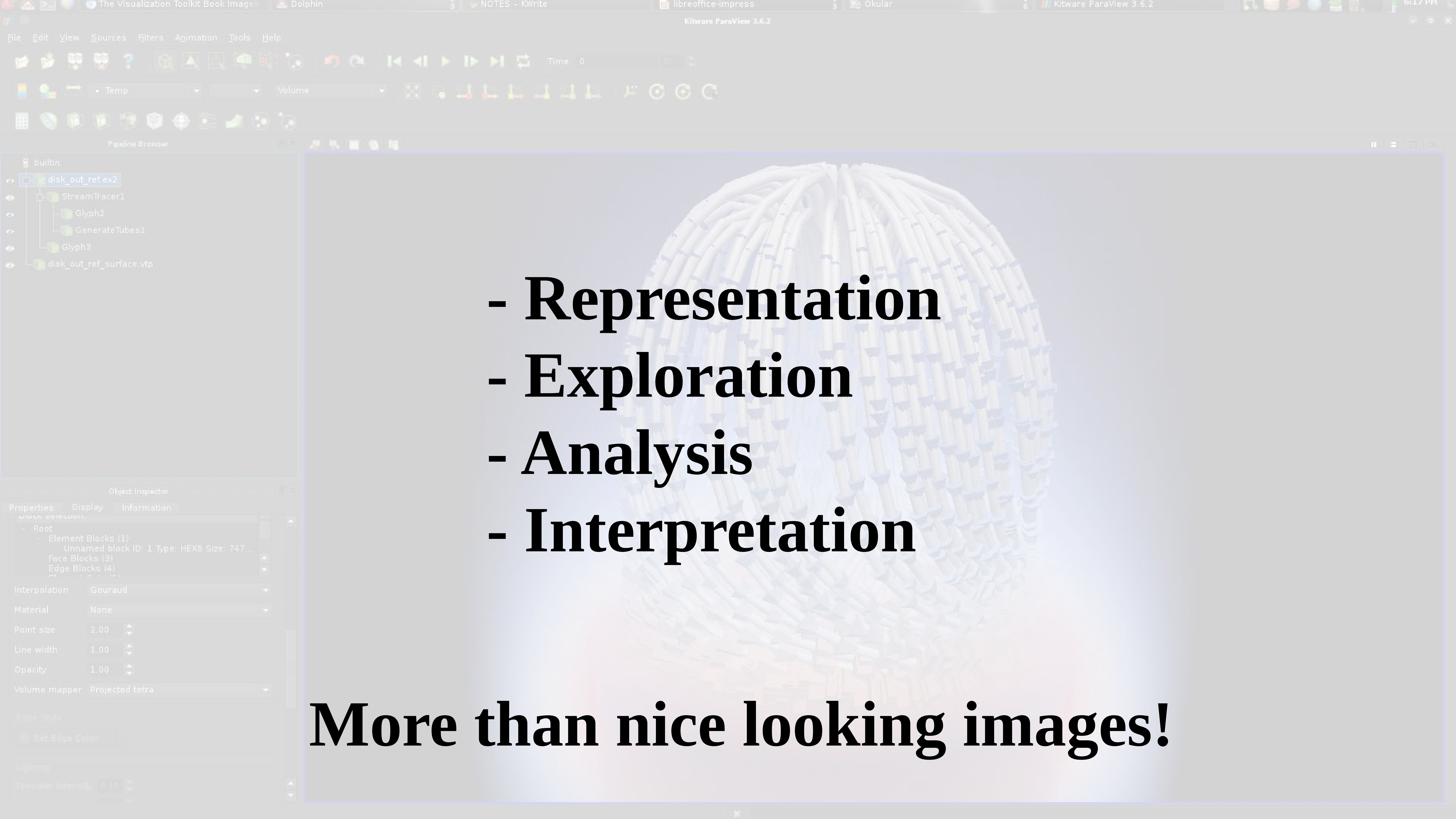
Edge Style

- Set Edge Color

Lighting

- Specular Intensity: 0.10

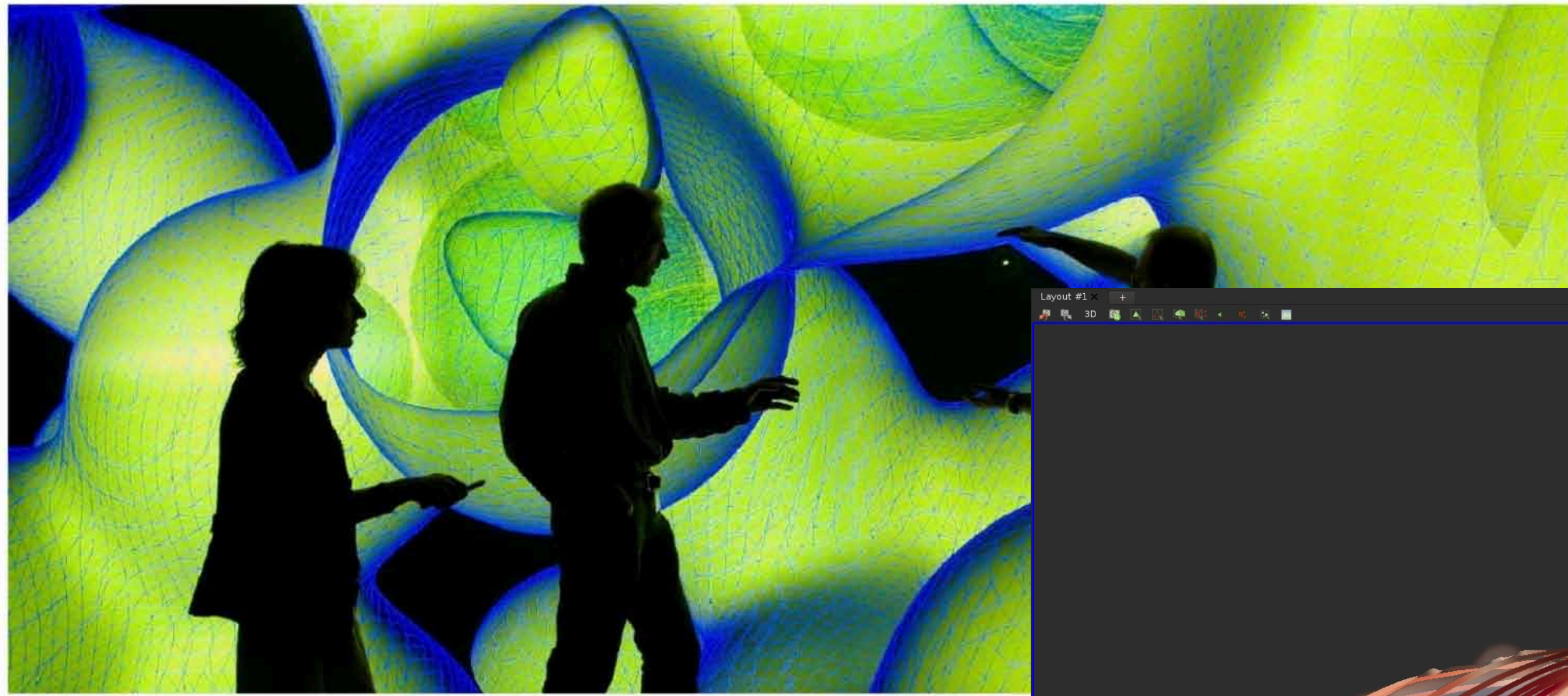




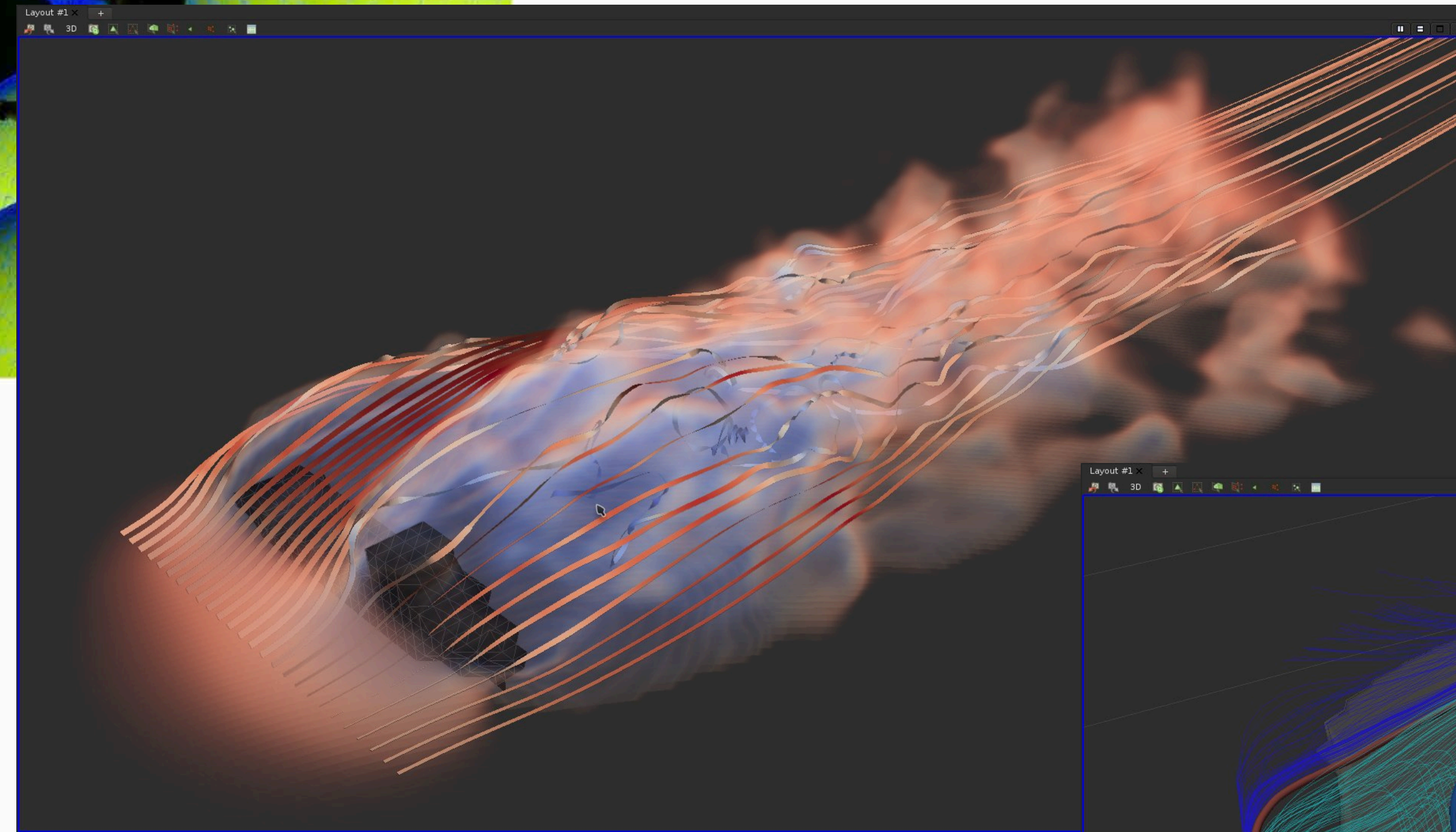
- **Representation**
- **Exploration**
- **Analysis**
- **Interpretation**

More than nice looking images!

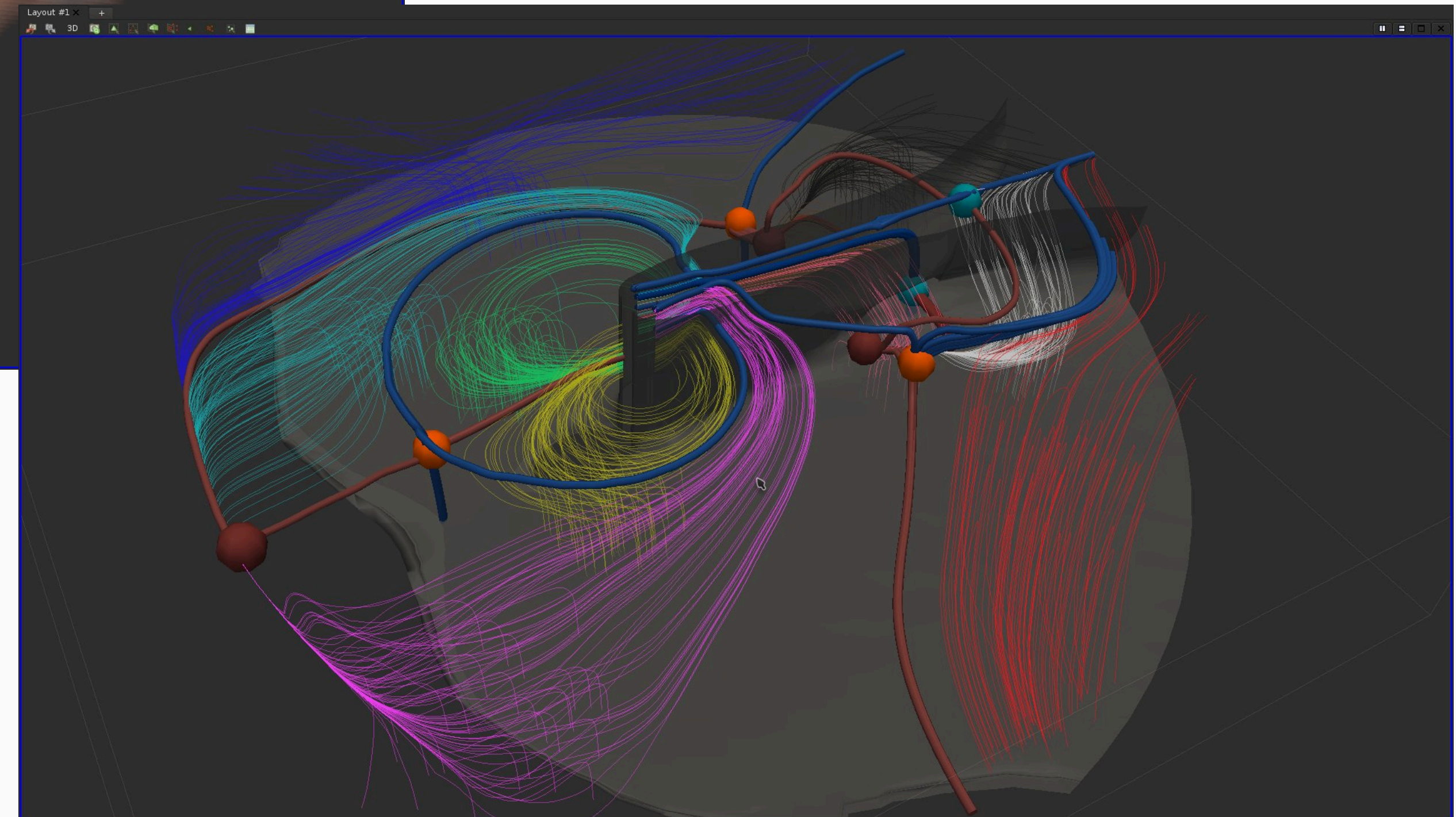
Main Research Themes



User Interactions



Interactive Rendering



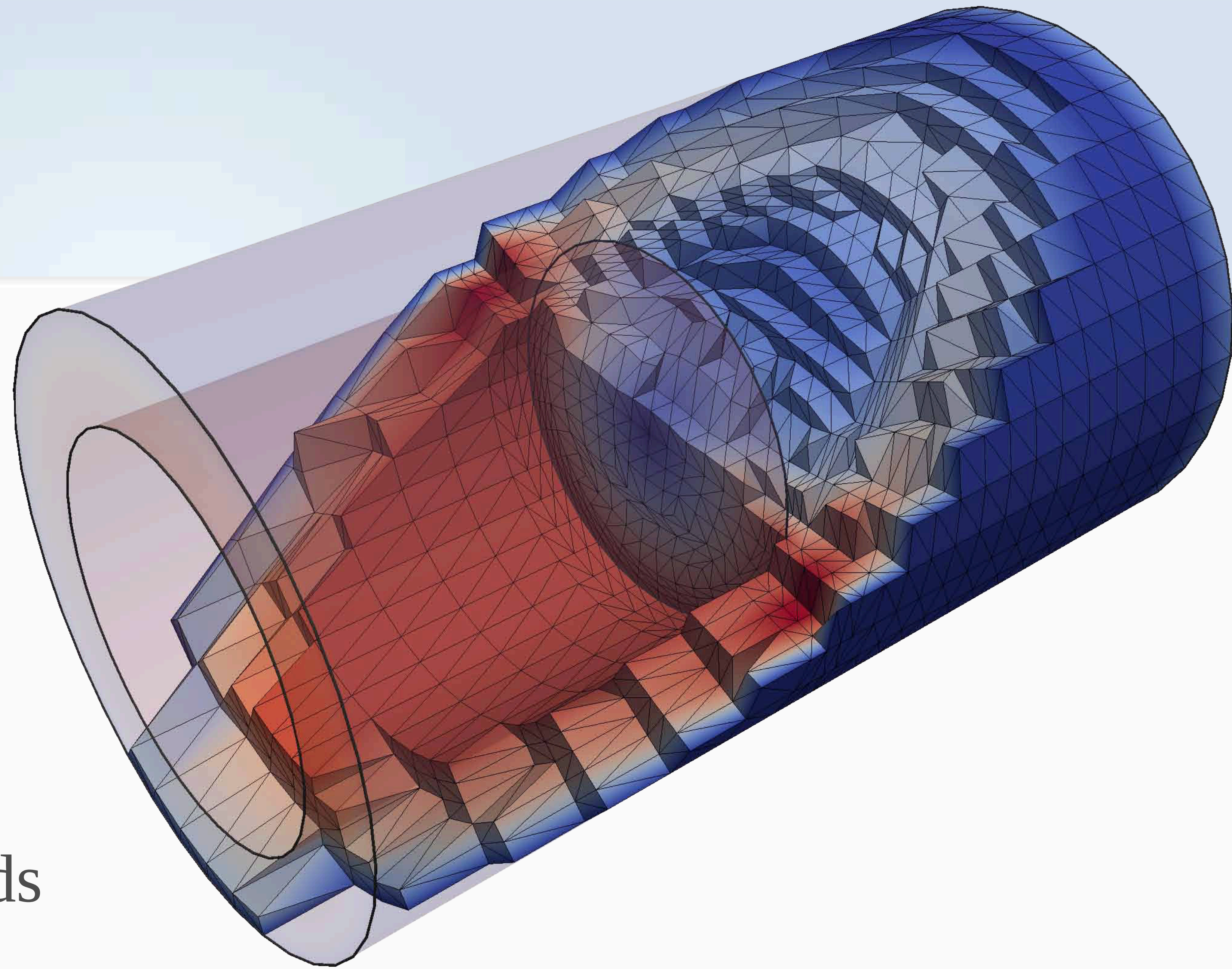
Geometry Analysis

Summary

- Overview of Topological Data Analysis
 - Combinatorial critical point extraction
 - High-level structures (Reeb graph, Morse-Smale complex)
- Upcoming challenges
- Topological Analysis of Uncertain Scalar Fields

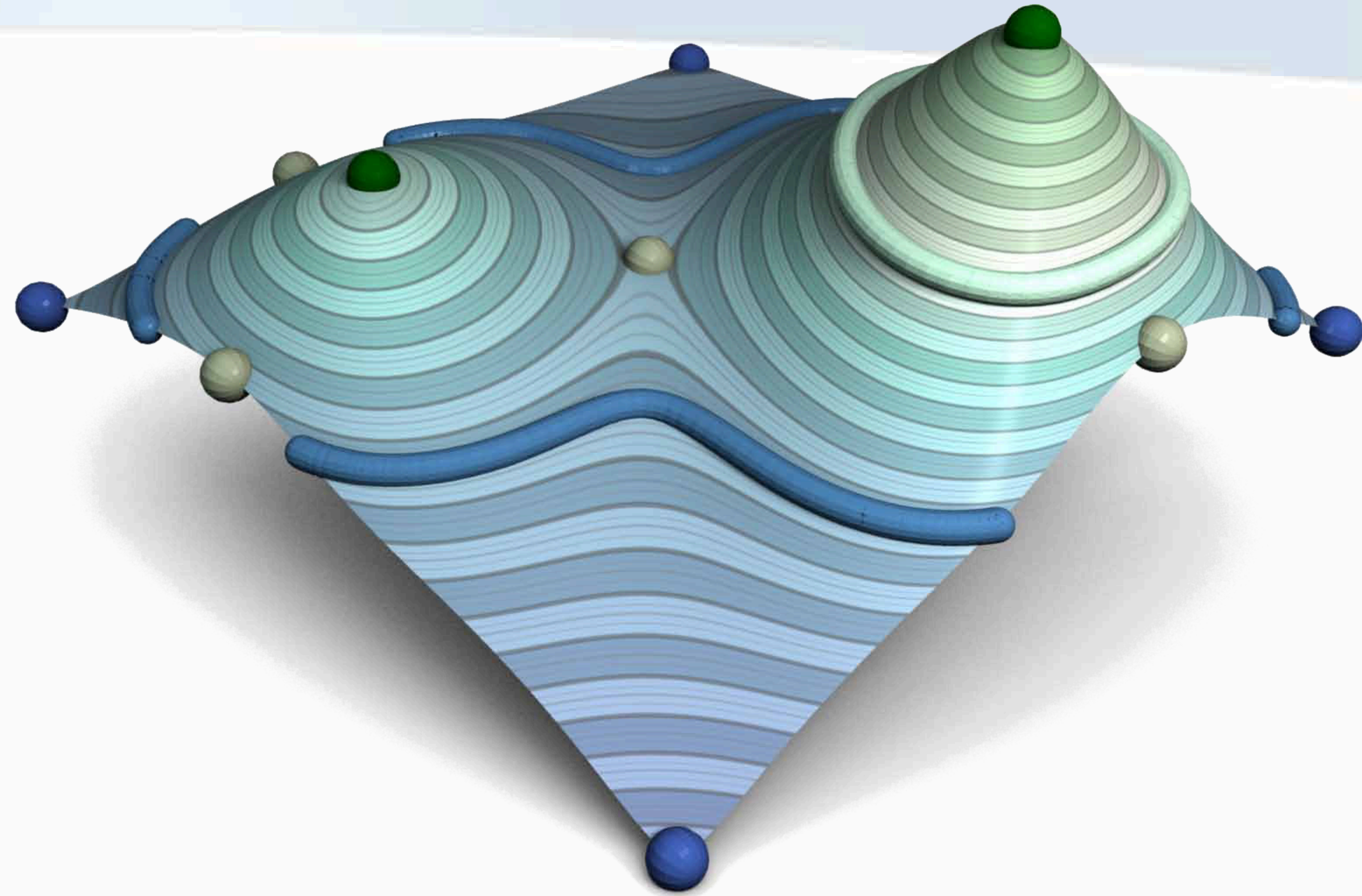
Input Data

- Domain representation
 - d-triangulation
 - Simplicial complex
- Input data
 - Piecewise linear (PL) scalar fields
 - $f : \mathcal{D} \rightarrow \mathbb{R}$
 - Linear interpolation with barycentric coordinates
 - Piecewise constant gradient



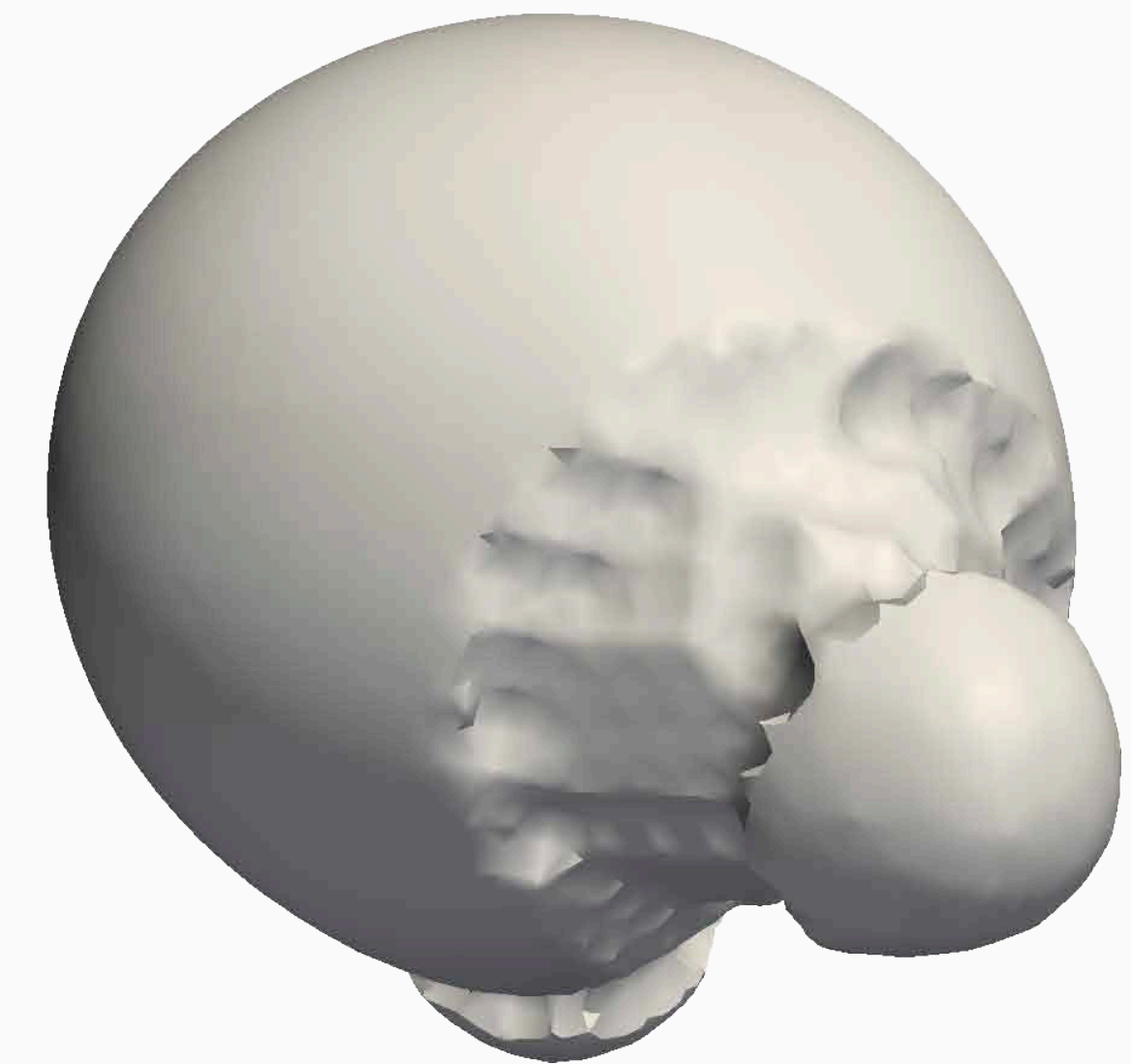
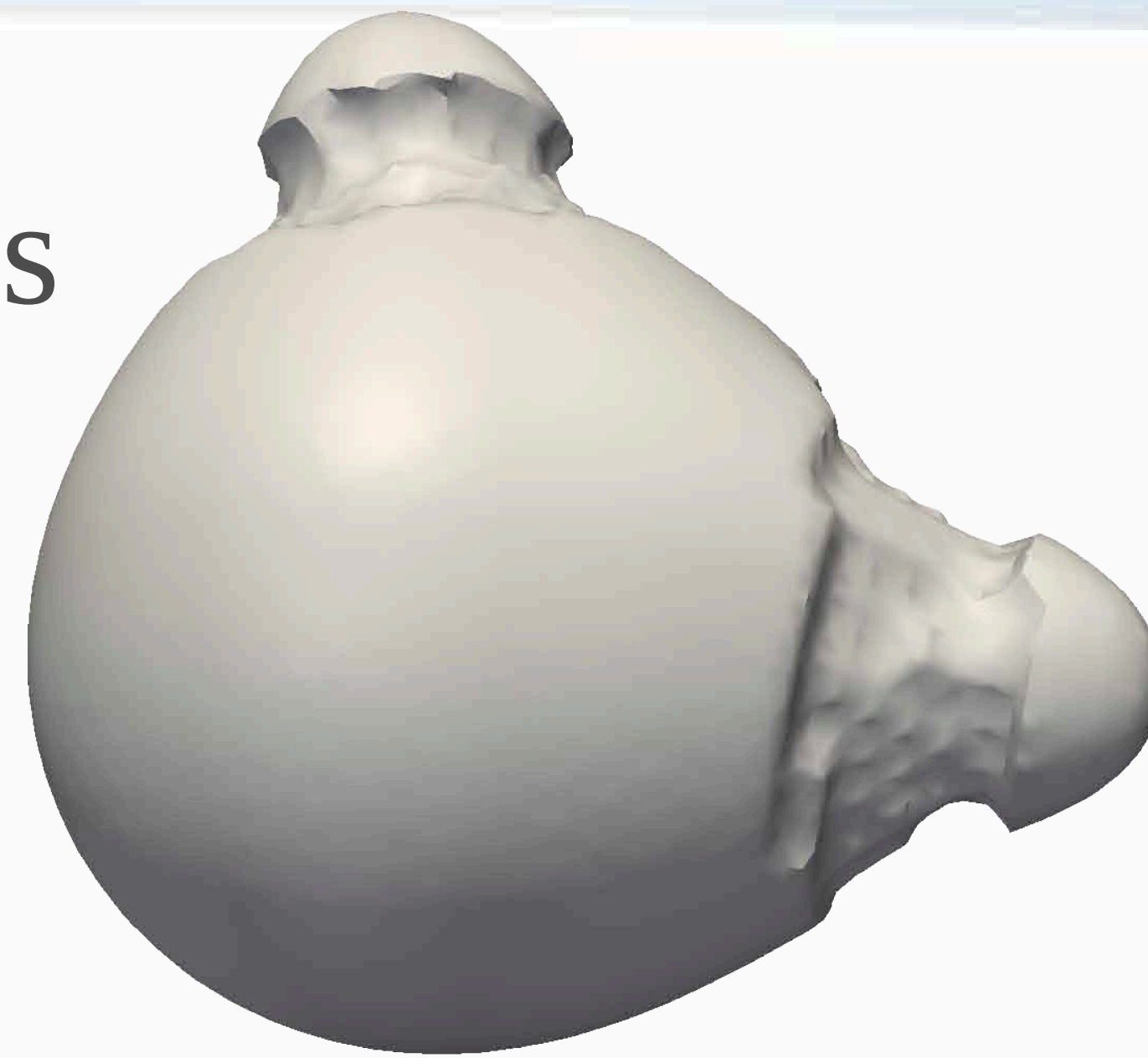
Critical Points of a Scalar Field

- Points where the gradient vanishes



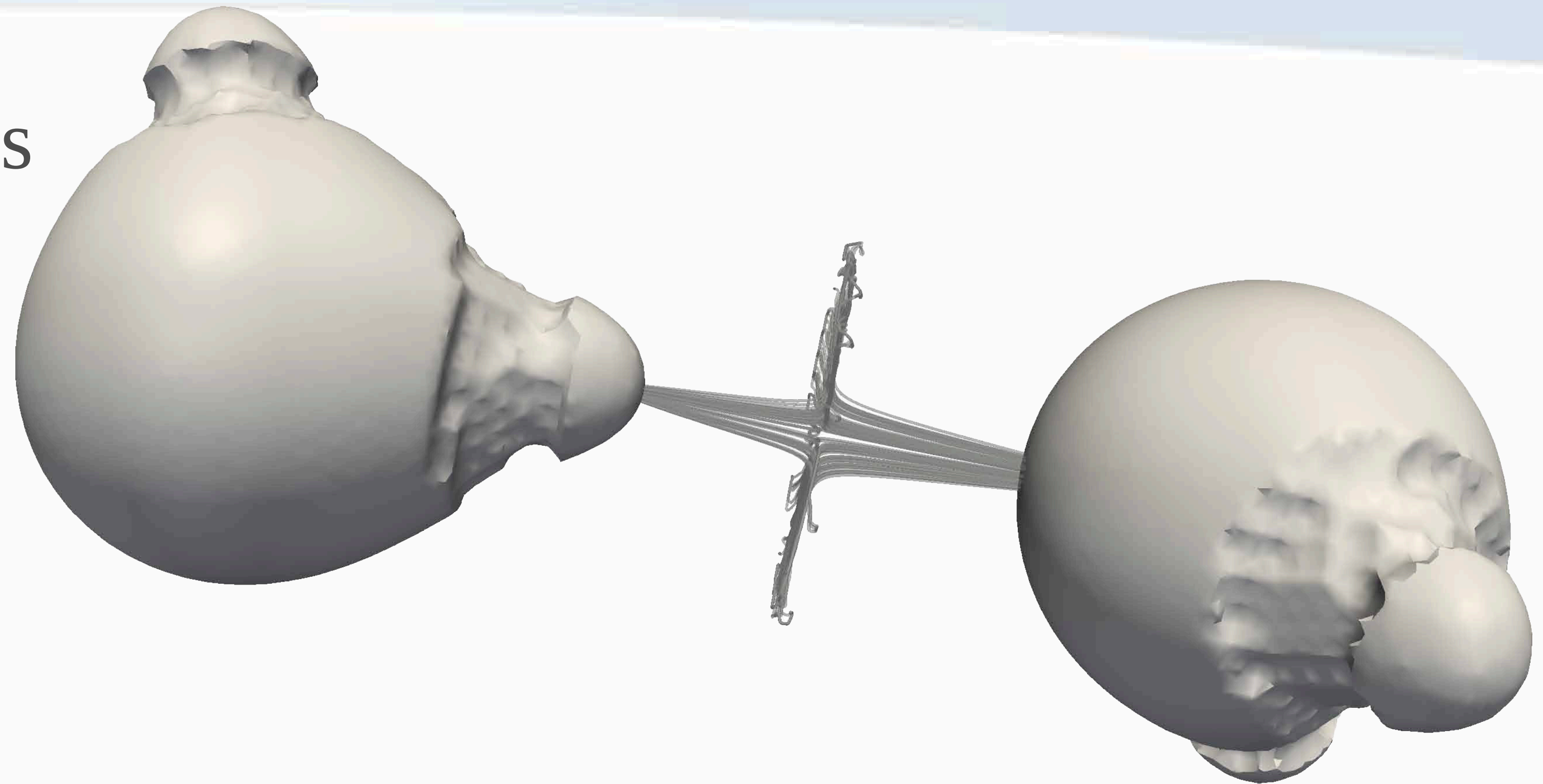
Critical Points of a Scalar Field

- Points where the gradient vanishes
- Applicative interest
 - Chemistry: centers of atoms



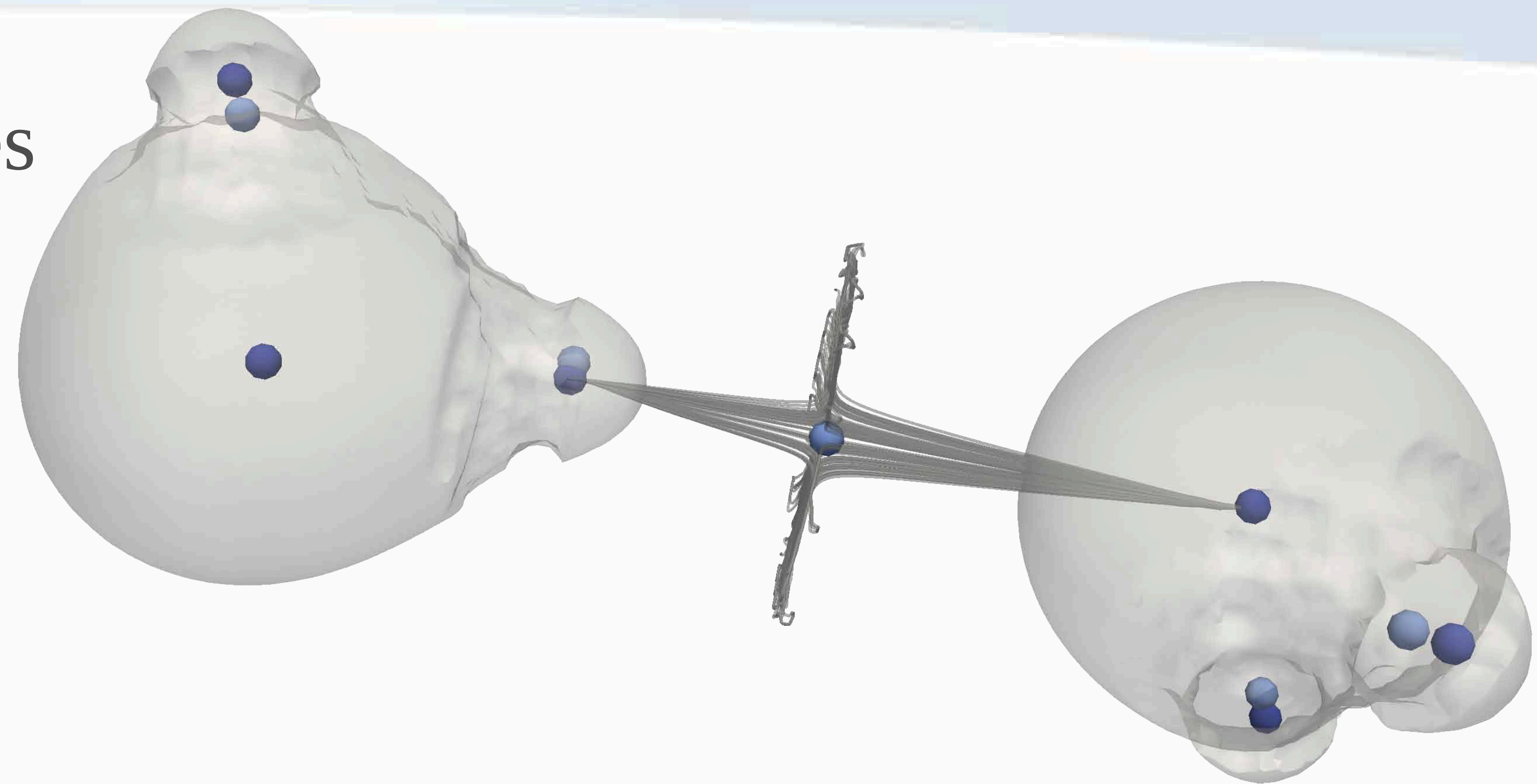
Critical Points of a Scalar Field

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- Applicative interest
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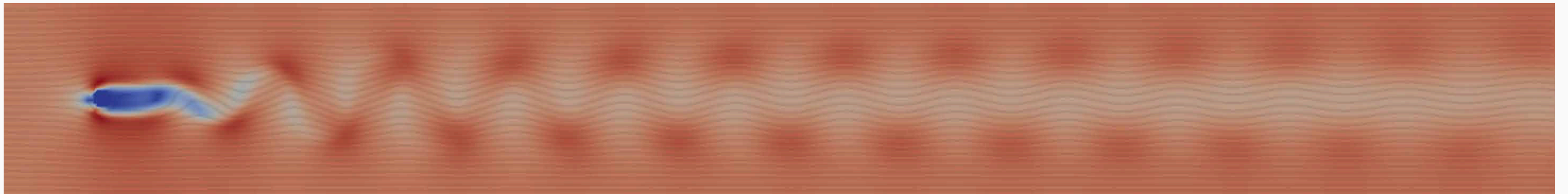
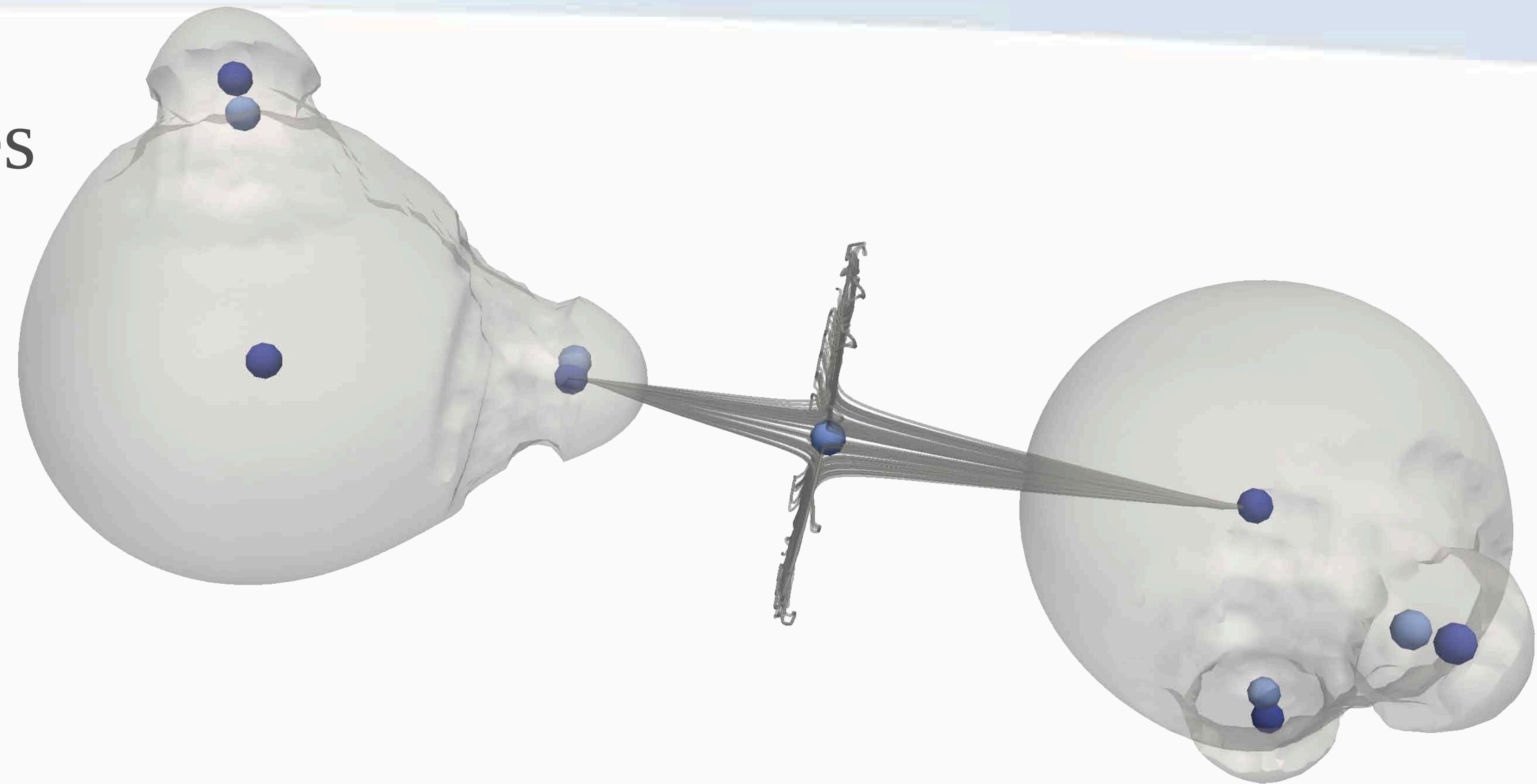
Critical Points of a Scalar Field

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 - Chemistry: centers of atoms



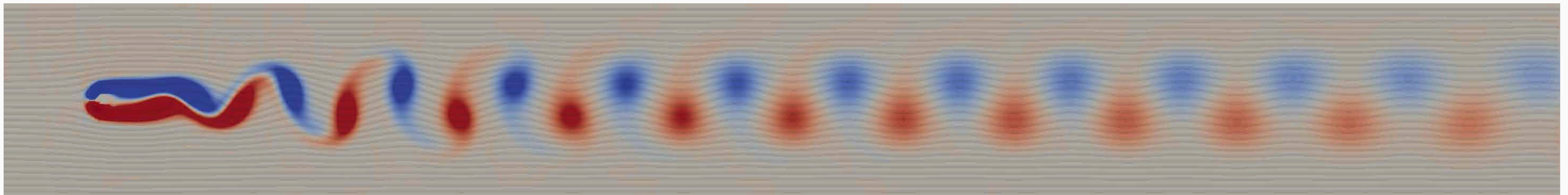
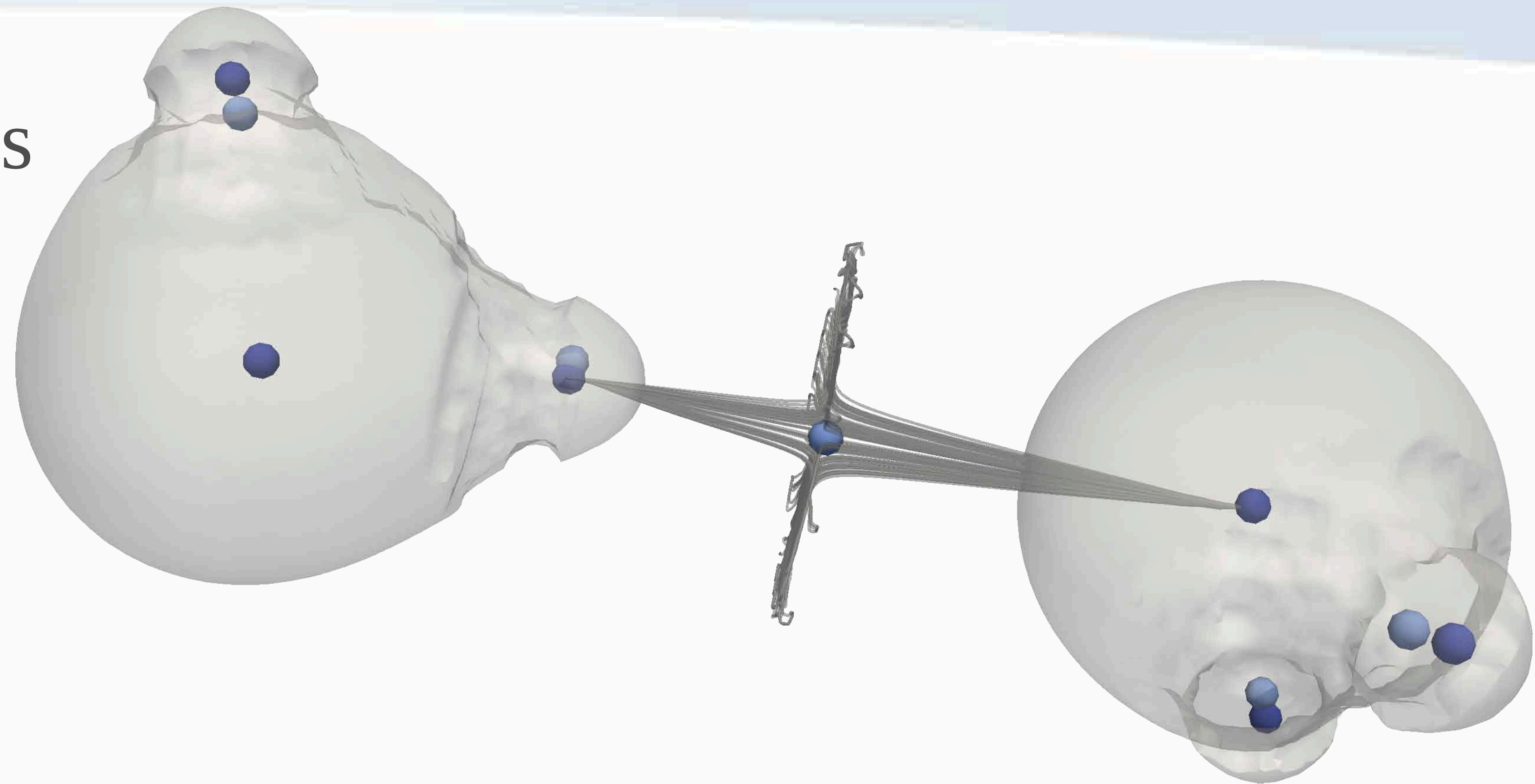
Critical Points of a Scalar Field

- Points where the gradient vanishes
- Applicative interest
 - Chemistry: centers of atoms
 - CFD: centers of vortices



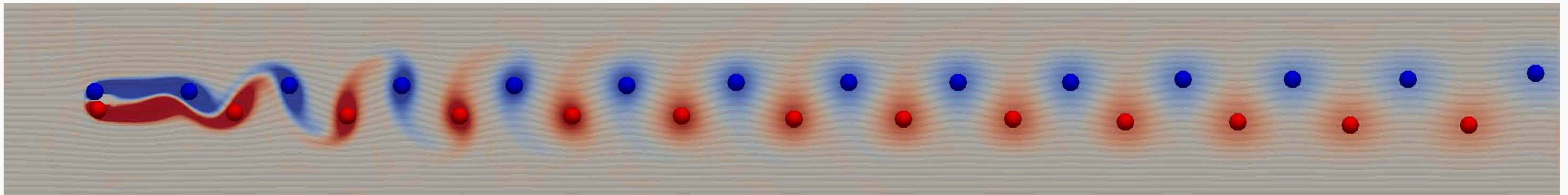
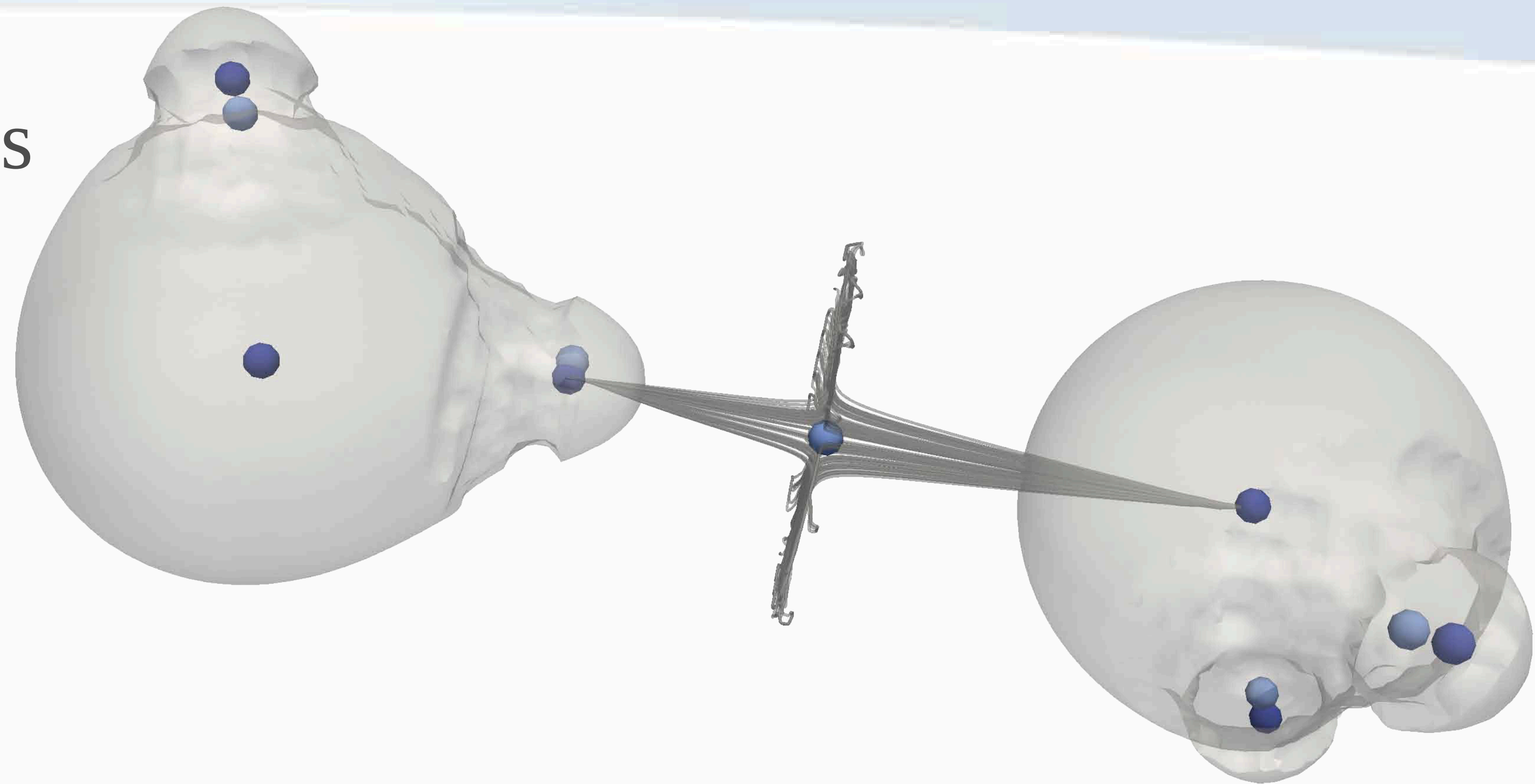
Critical Points of a Scalar Field

- Points where the gradient vanishes
- Applicative interest
 - Chemistry: centers of atoms
 - CFD: centers of vortices



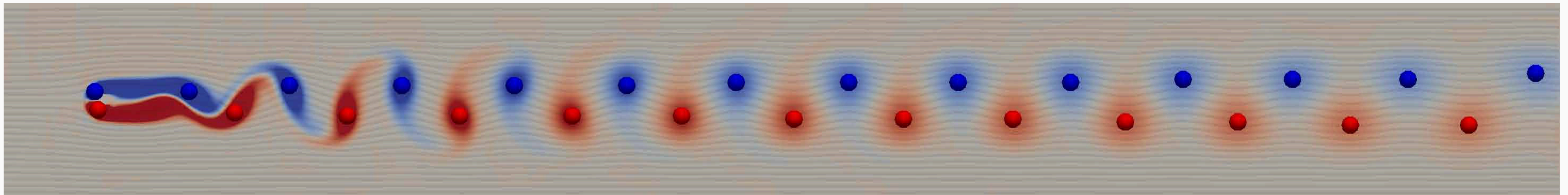
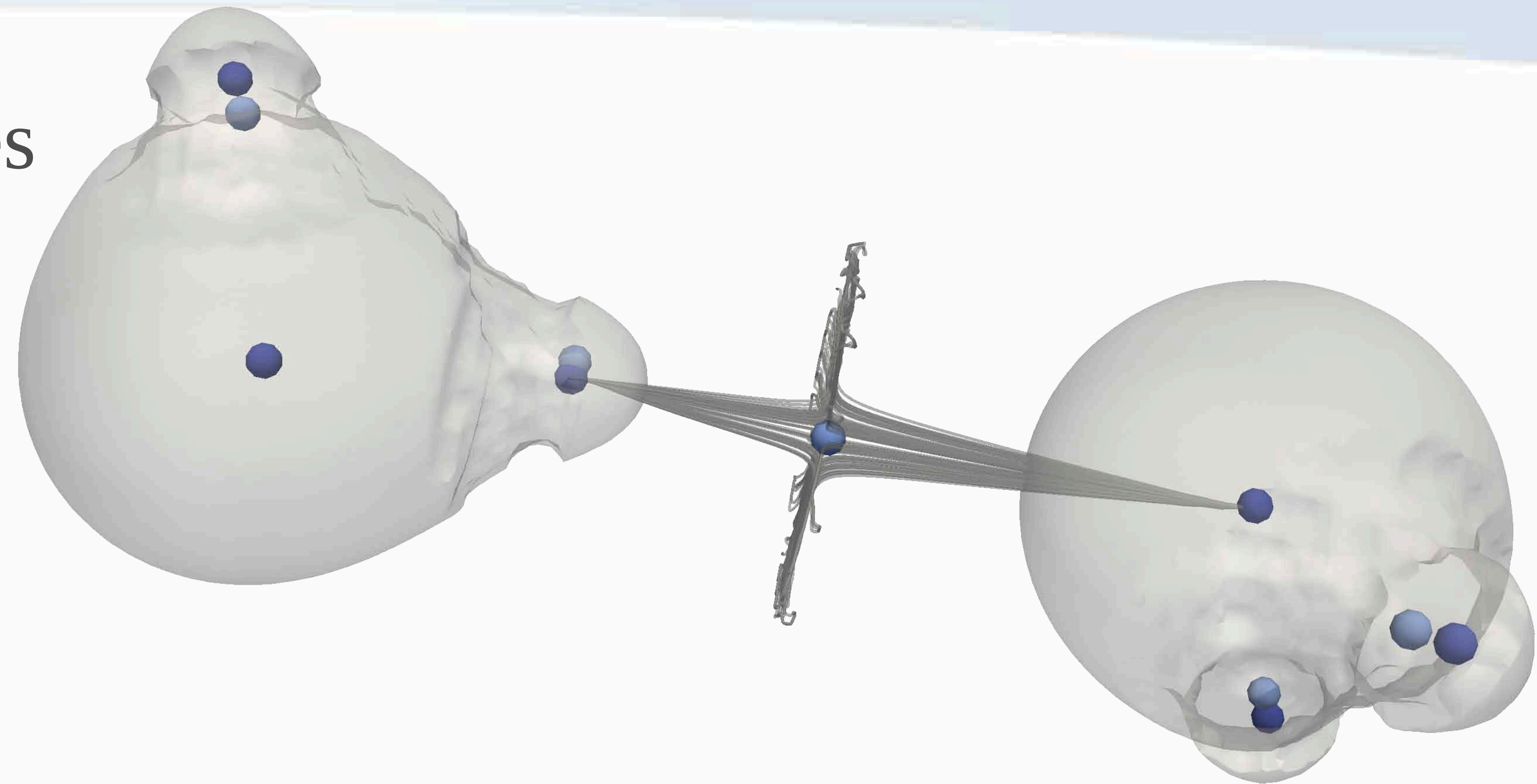
Critical Points of a Scalar Field

- Points where the gradient vanishes
- Applicative interest
 - Chemistry: centers of atoms
 - CFD: centers of vortices



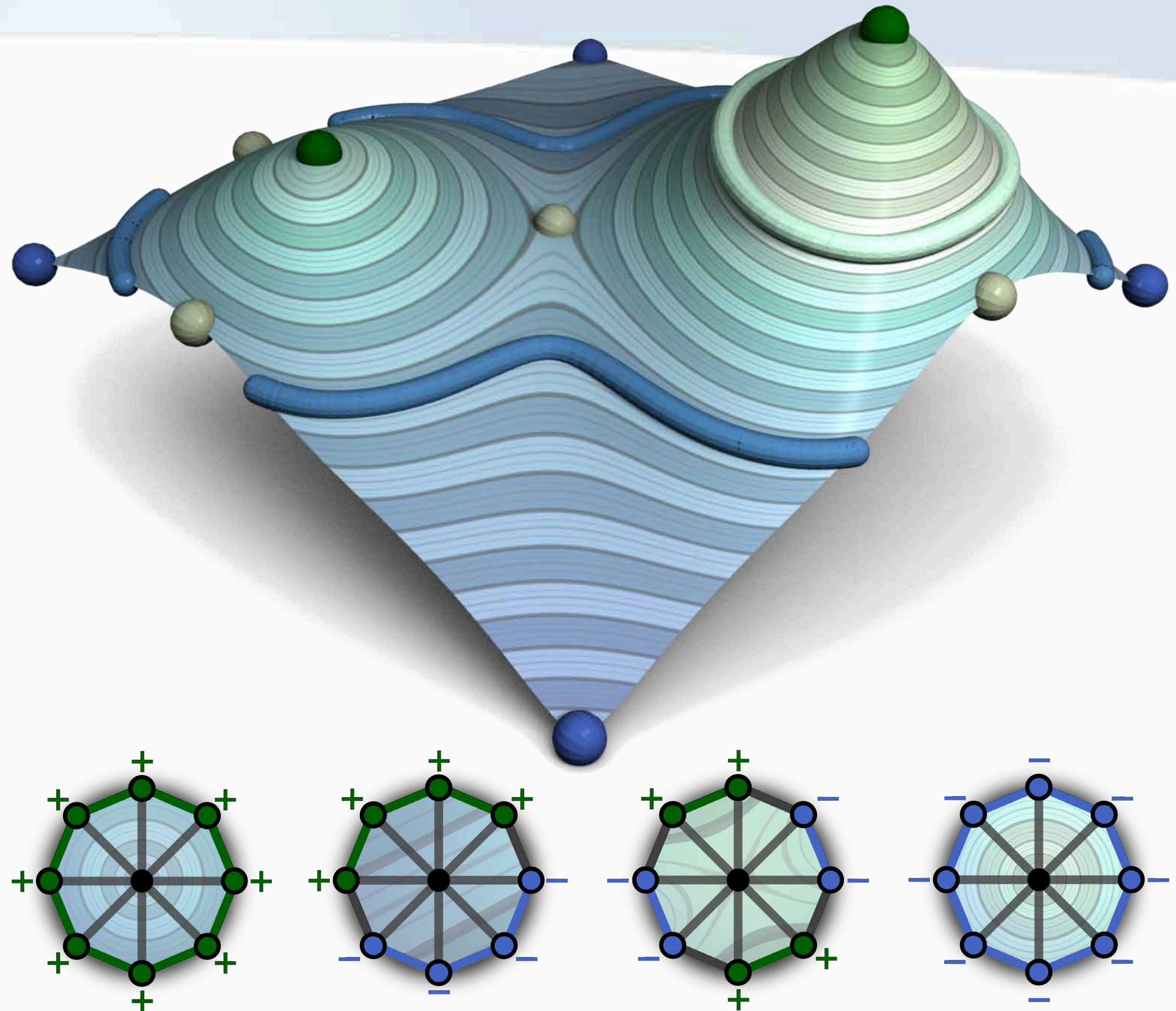
Critical Points of a Scalar Field

- Points where the gradient vanishes
- Applicative interest
 - Chemistry: centers of atoms
 - CFD: centers of vortices
 - Thermodynamics: hot spots
 - Etc.



Critical Point Extraction Algorithm

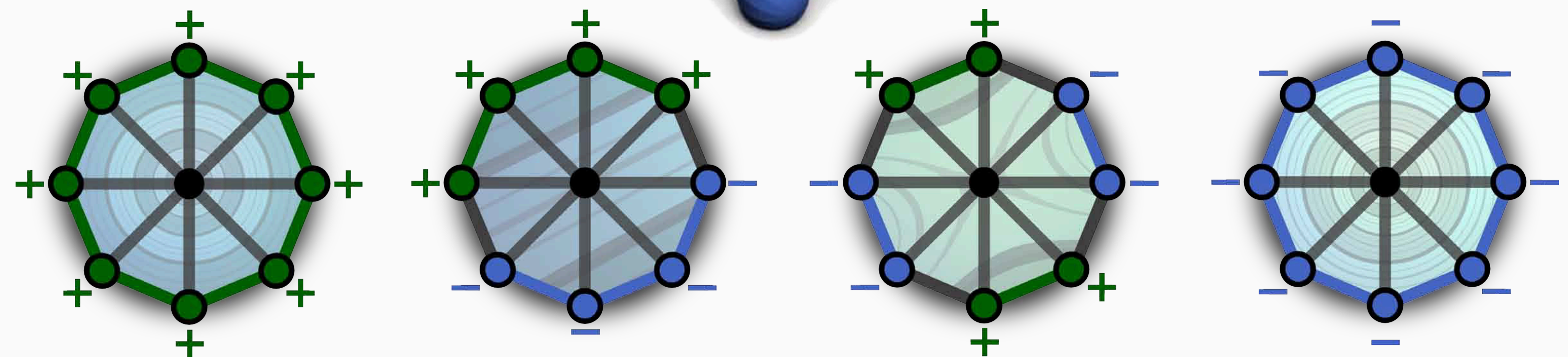
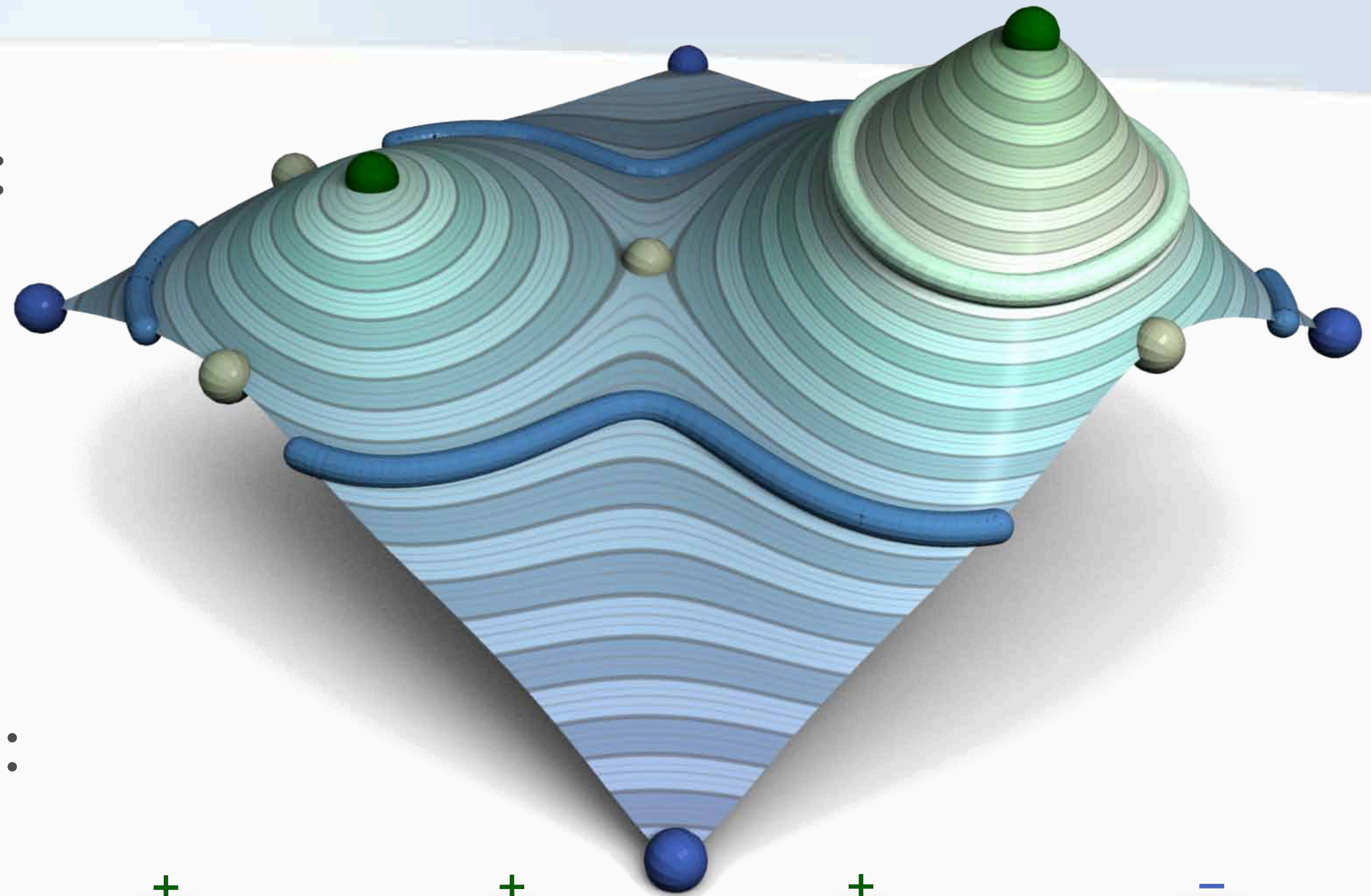
- Local, inexpensive operations:
 - Regular point:
 - $Lk^-(v)$ simply connected
 - $Lk^+(v)$ simply connected
 - Minimum:
 - $Lk^-(v) = \emptyset$
 - Maximum:
 - $Lk^+(v) = \emptyset$
 - Otherwise: Saddle point



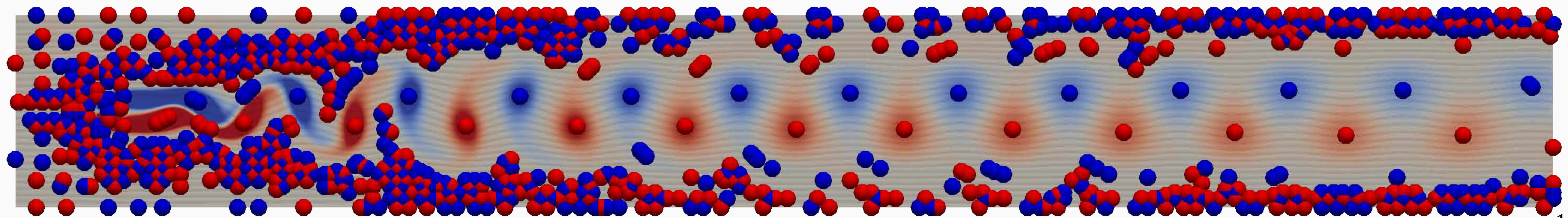
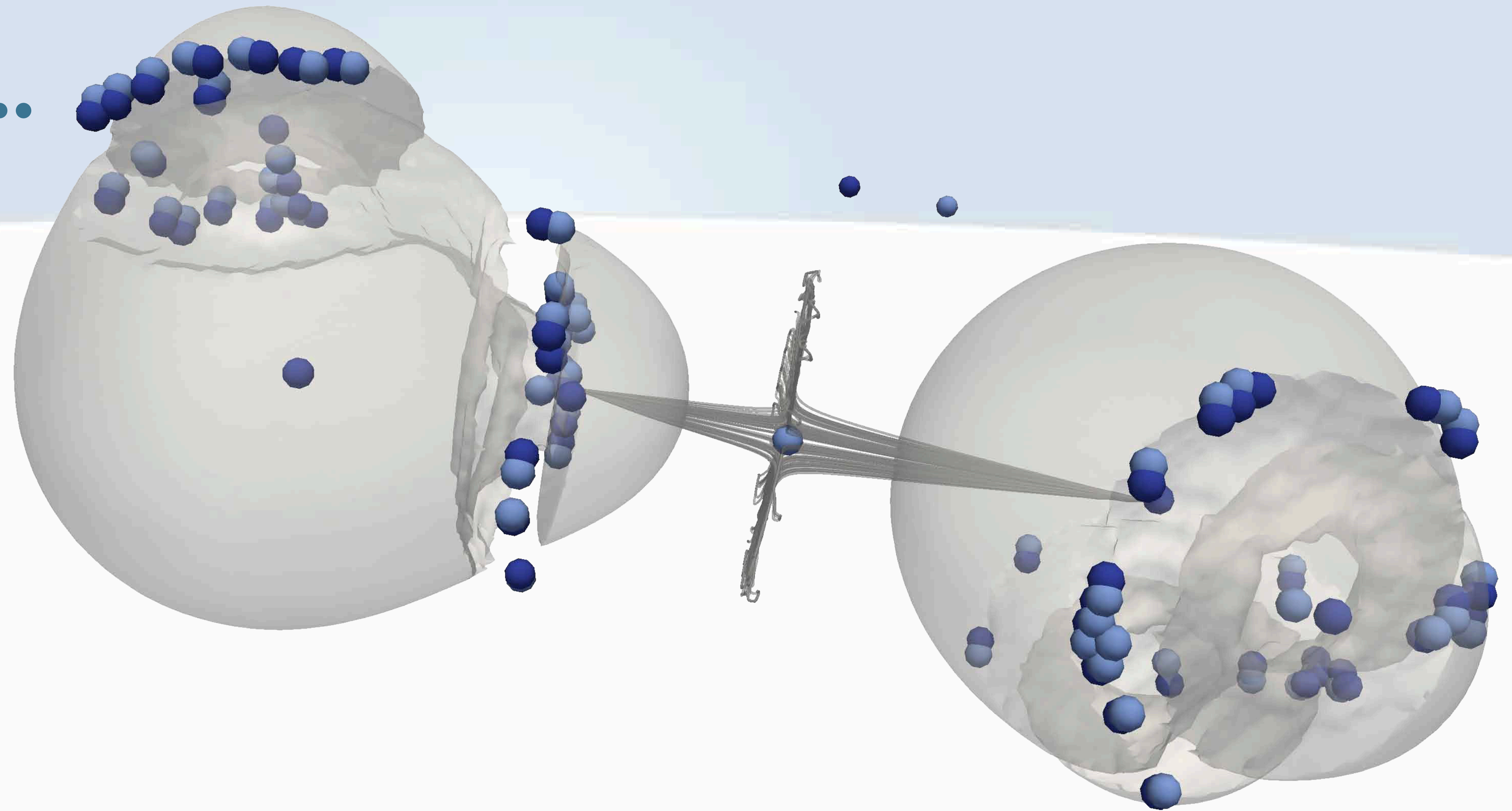
Critical Point Extraction Algorithm

- Interesting combinatorial properties:
 - Finite number of critical points
 - Under certain assumptions:
 - Only isolated critical points
 - No degenerate critical points
- Consistent with the smooth setting:

$$\chi_{\mathcal{D}} = \sum_{i=0}^{i=d} (-1)^i \mu_i(f)$$

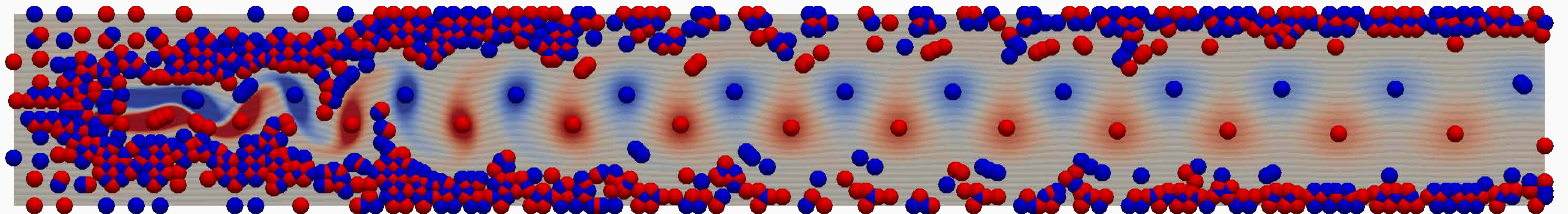
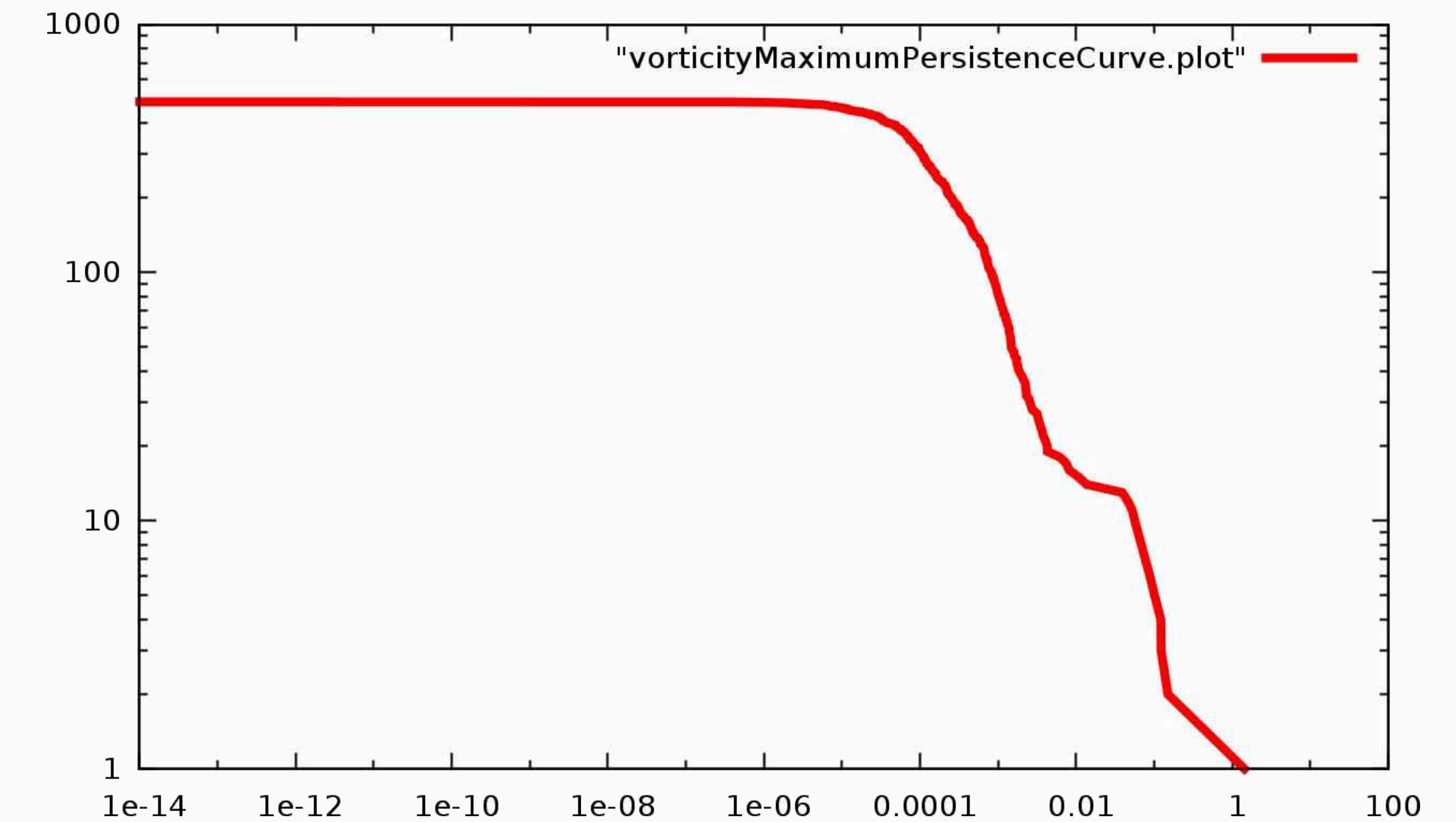
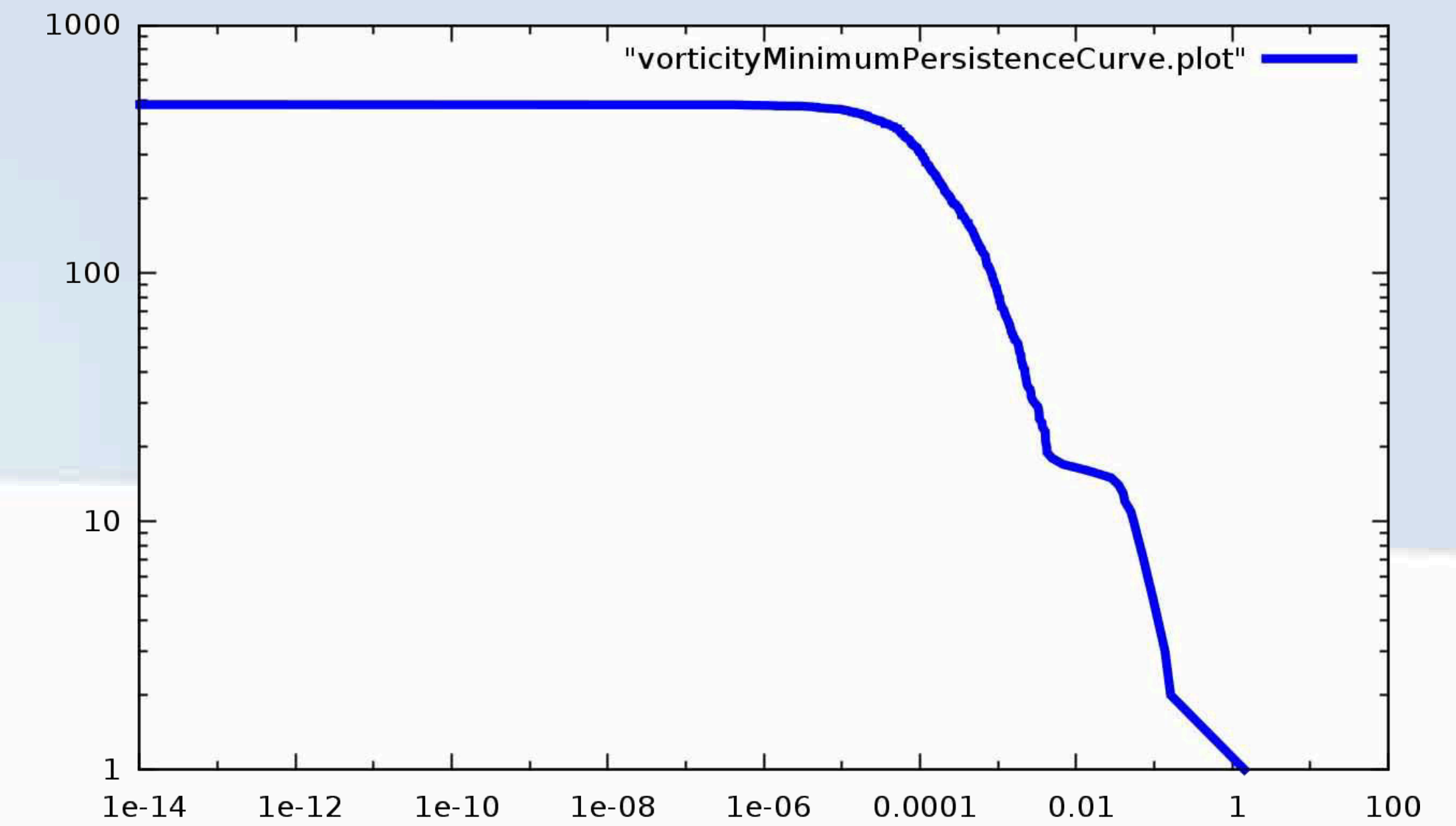


In practice...



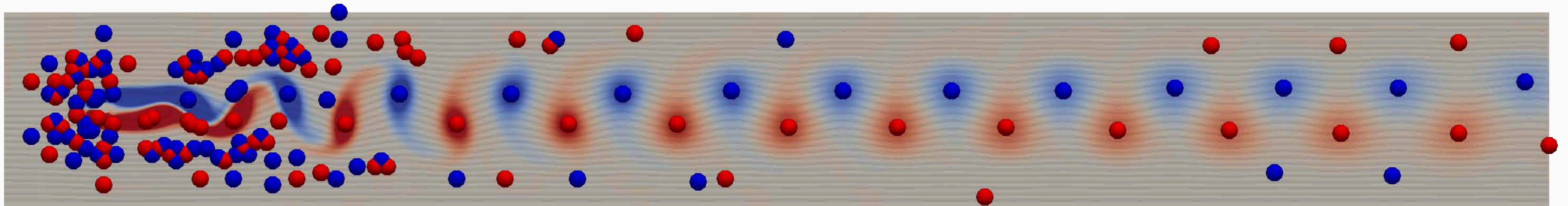
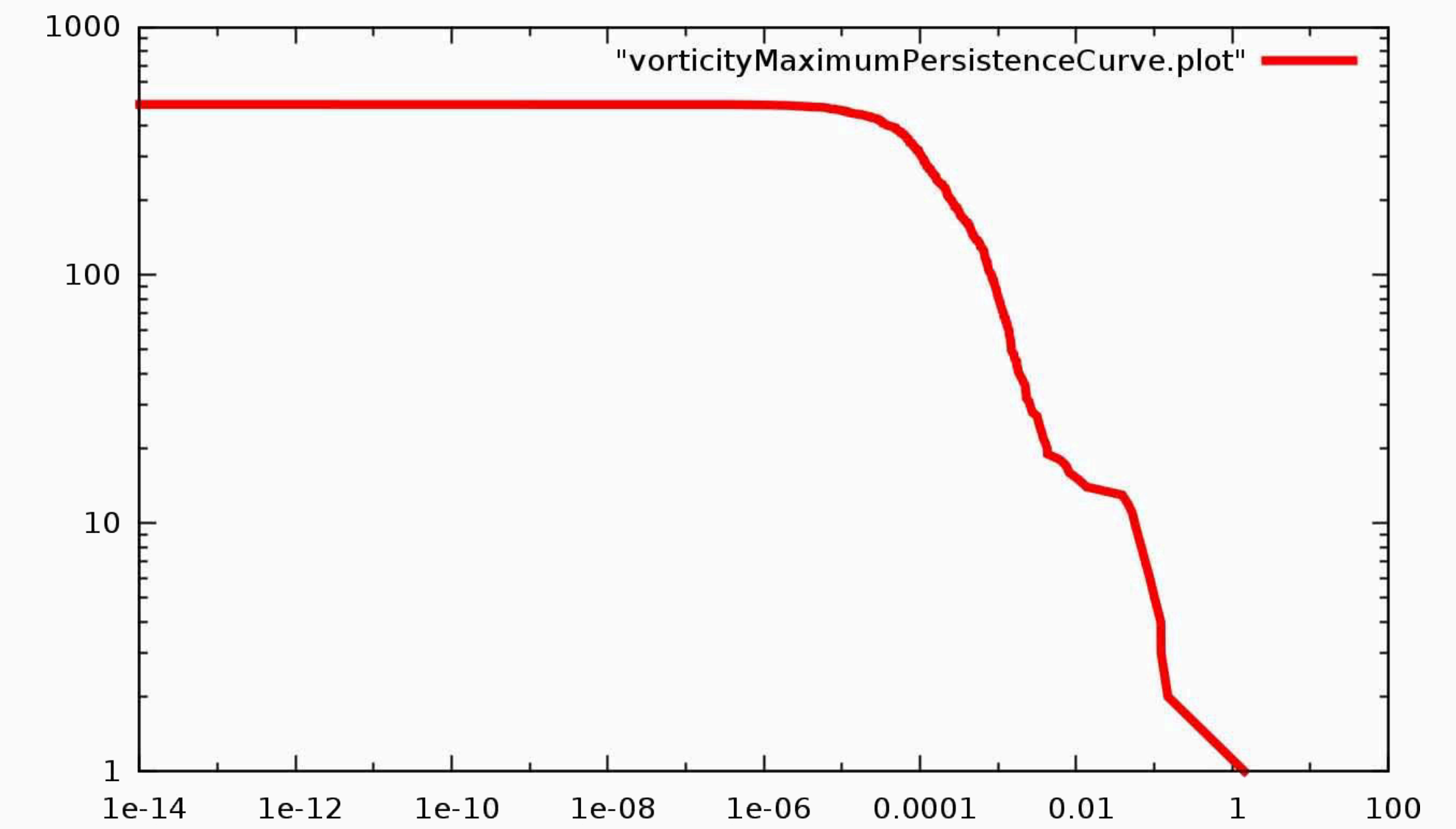
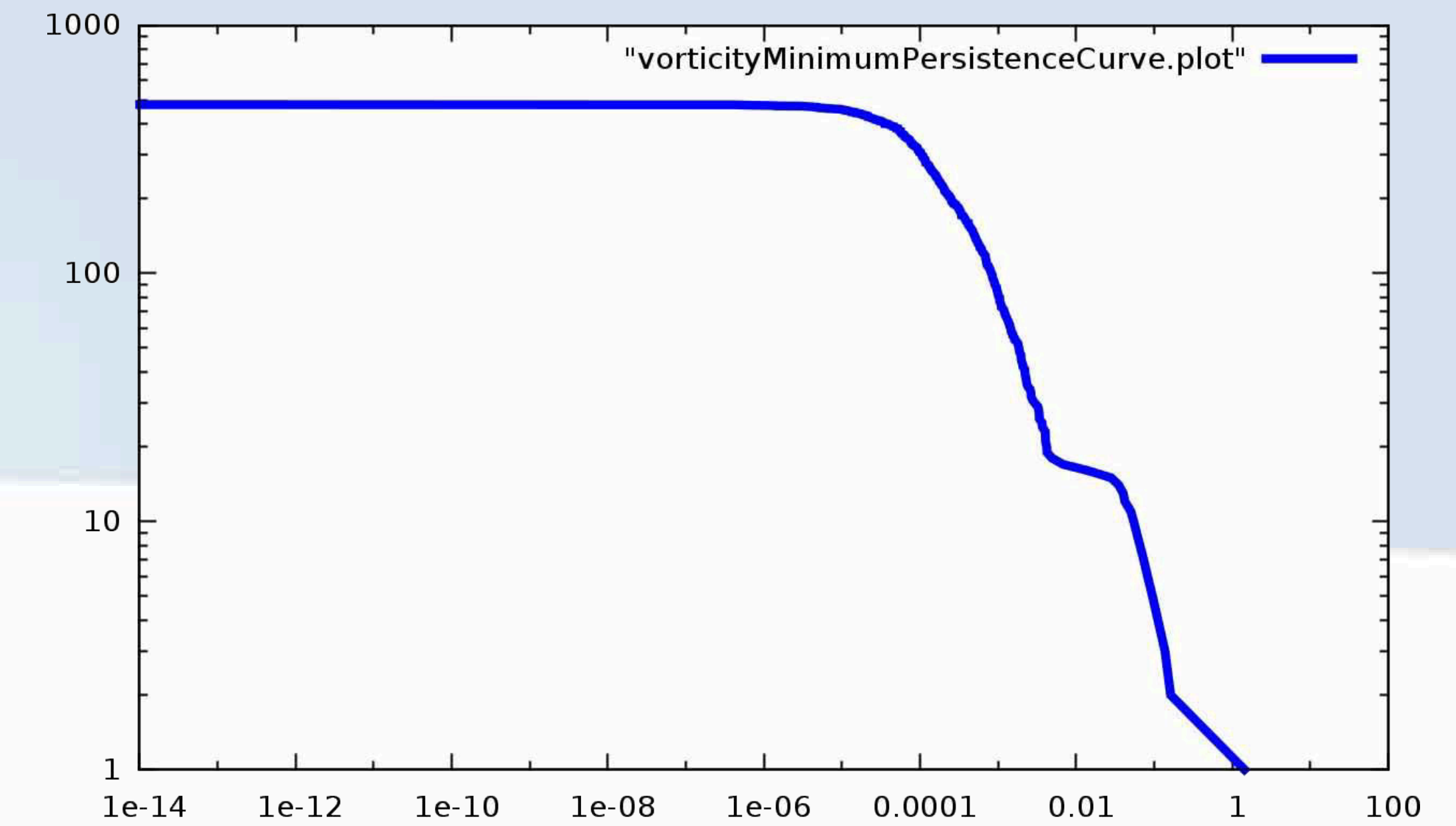
Persistence Curve

- Number of critical point pairs
 - As a function of their persistence



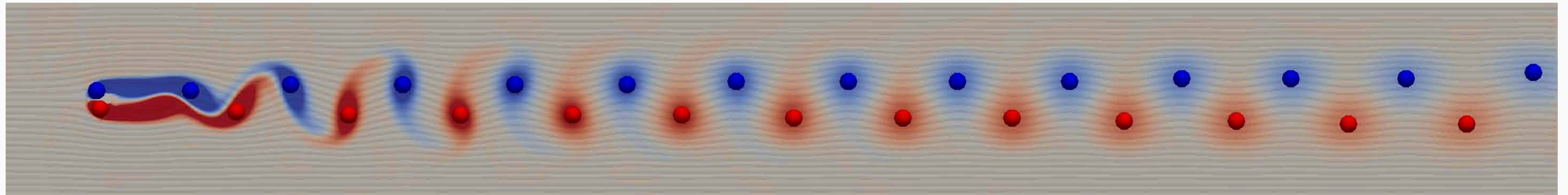
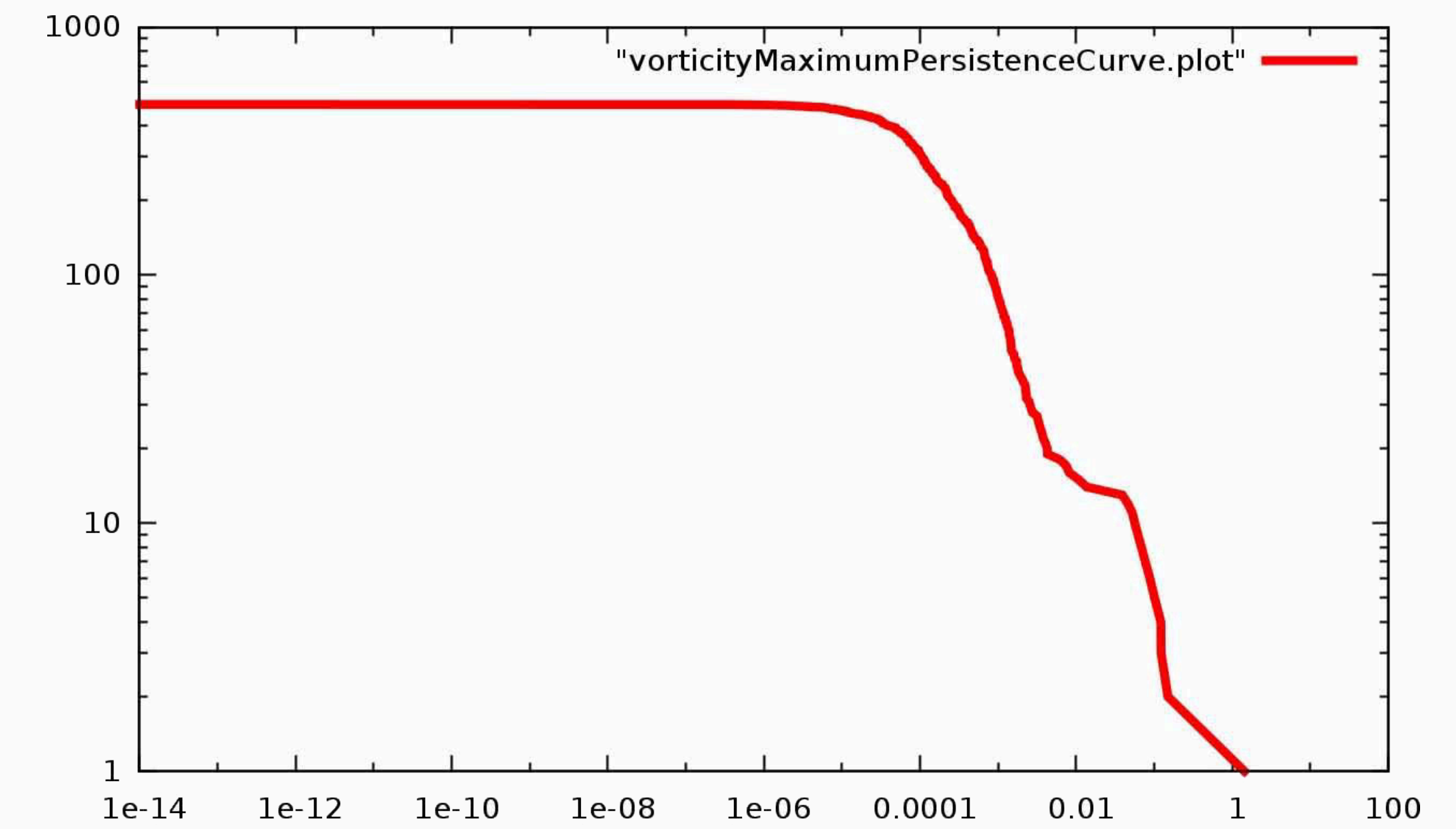
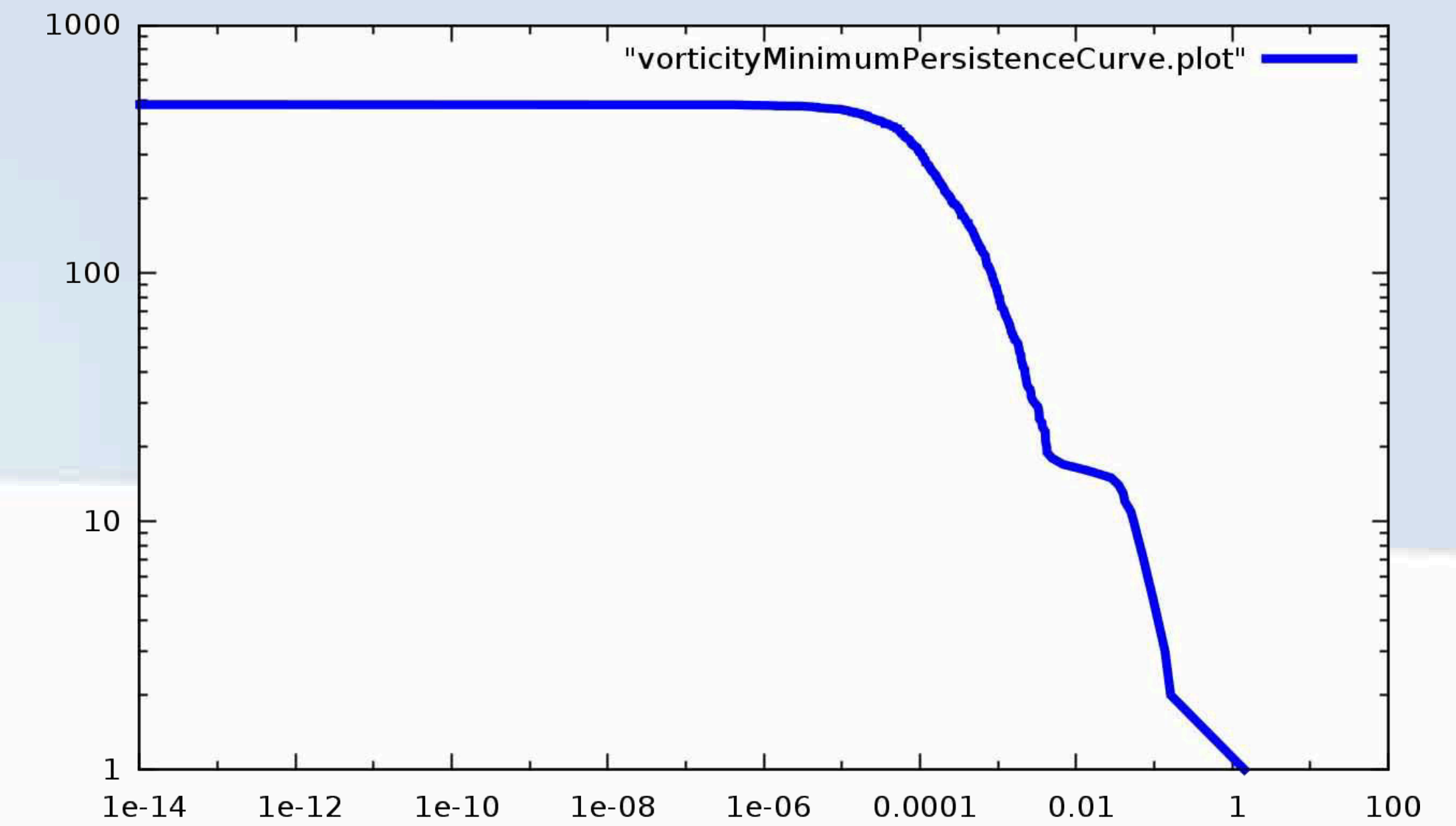
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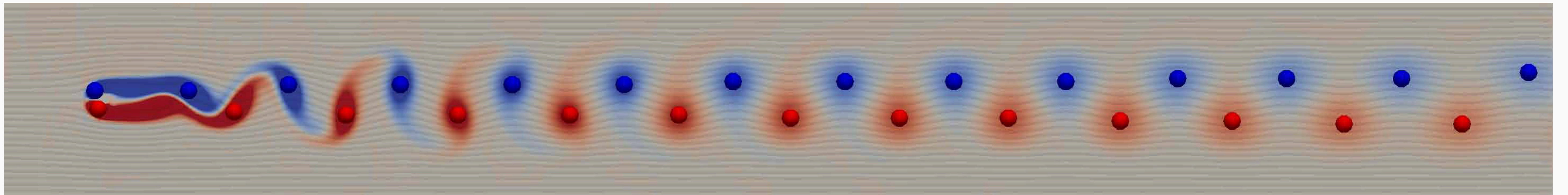
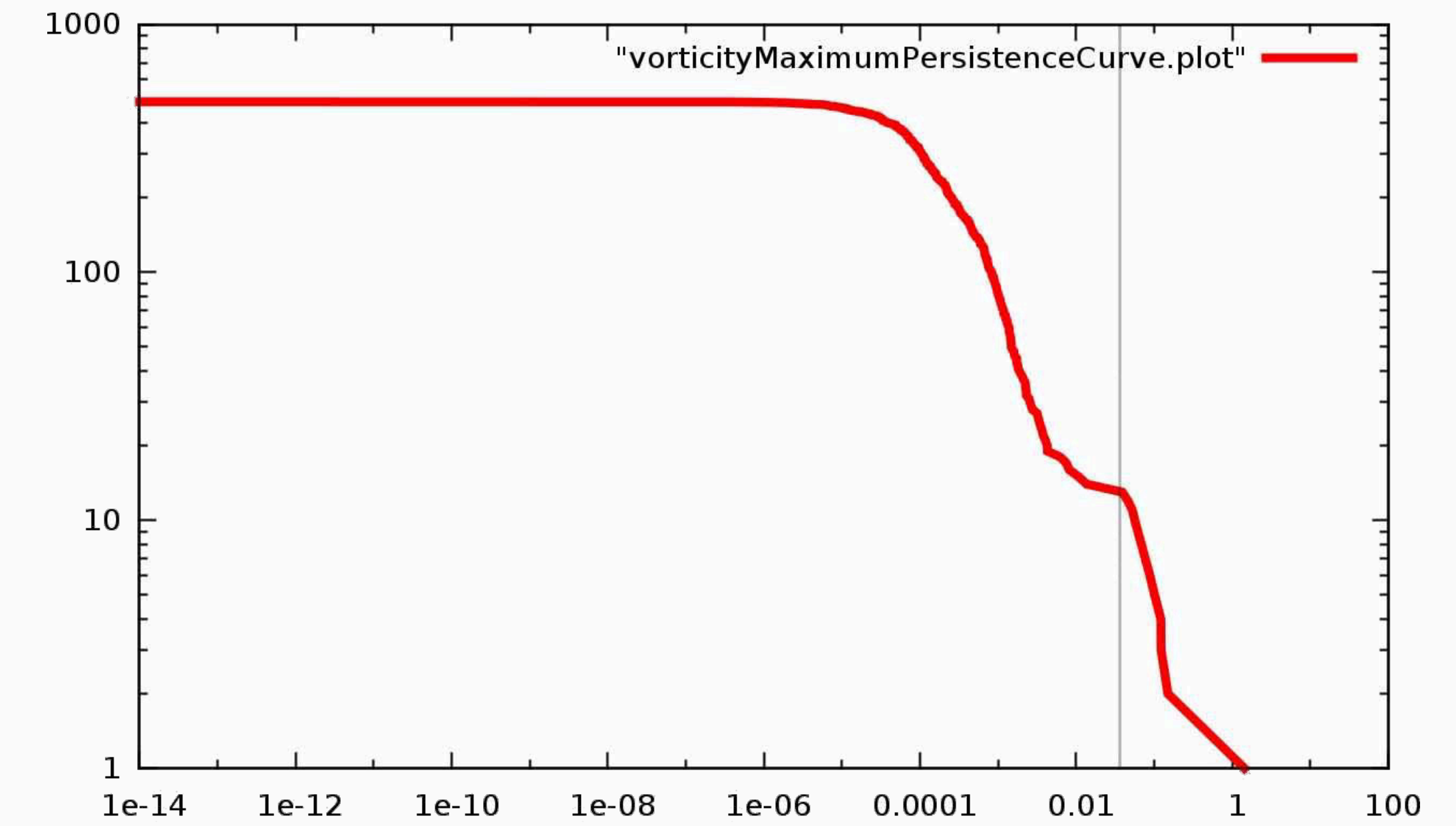
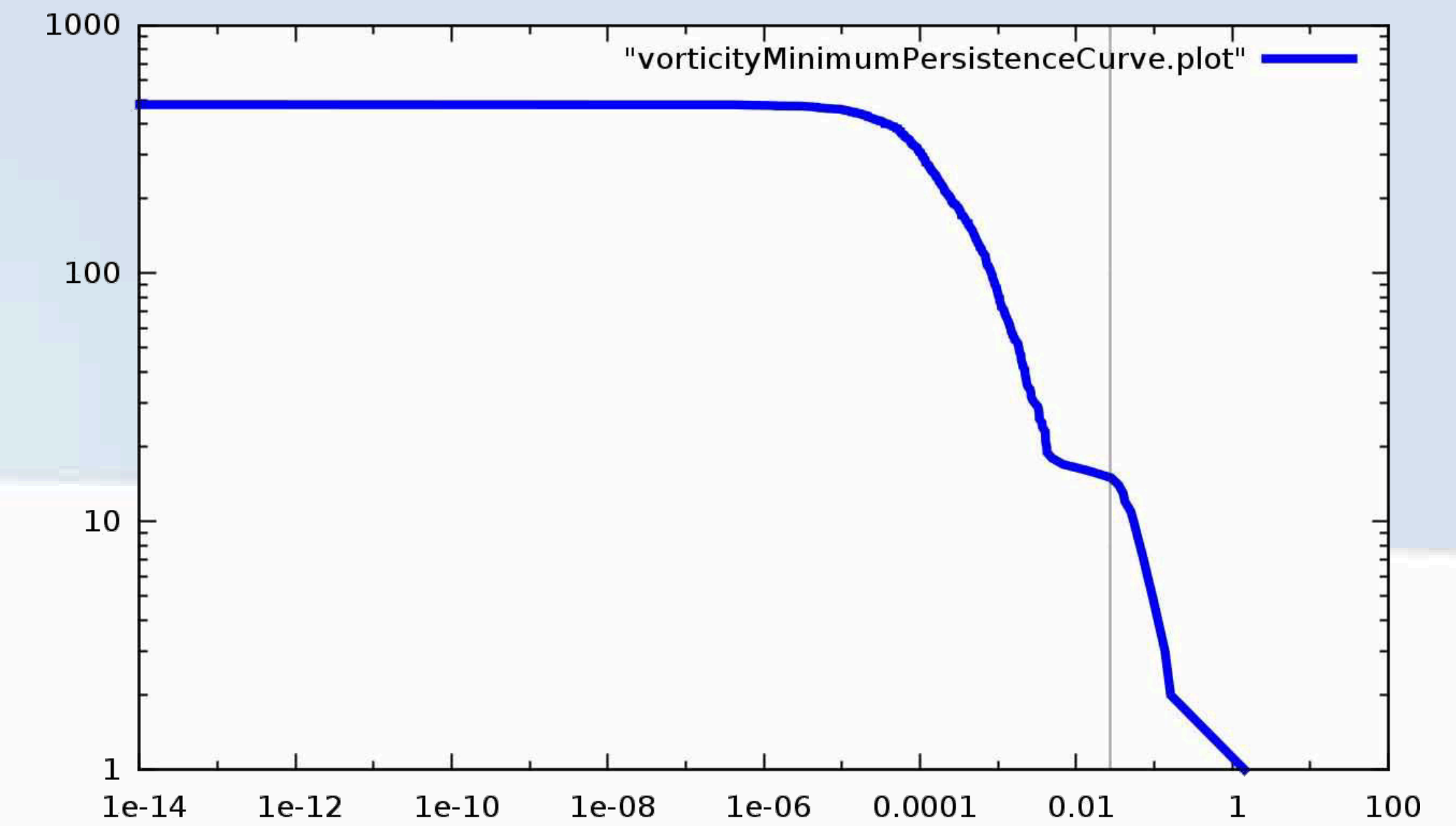
Persistence Curve

- Number of critical point pairs
 - As a function of their persistence

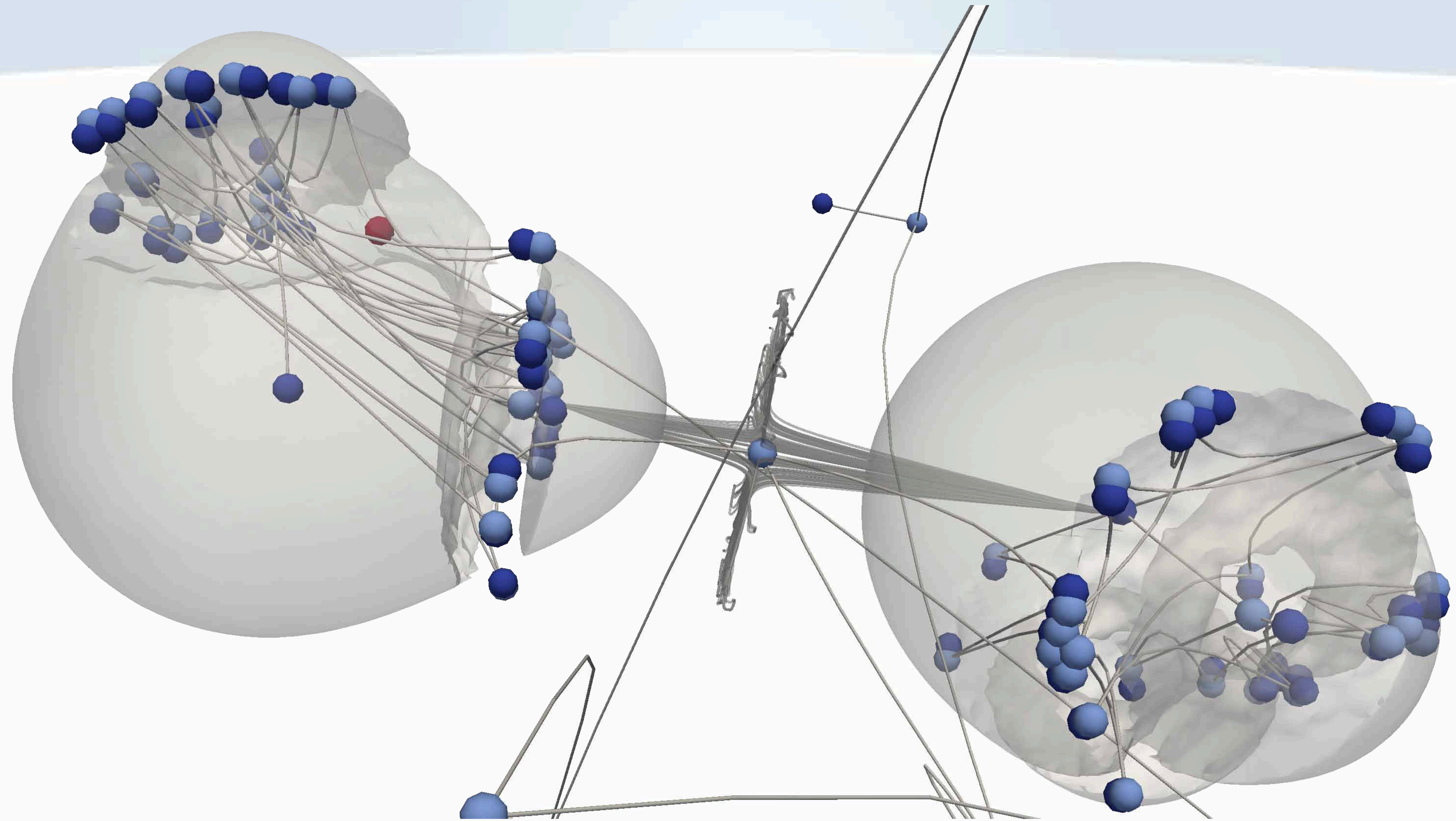


Persistence Curve

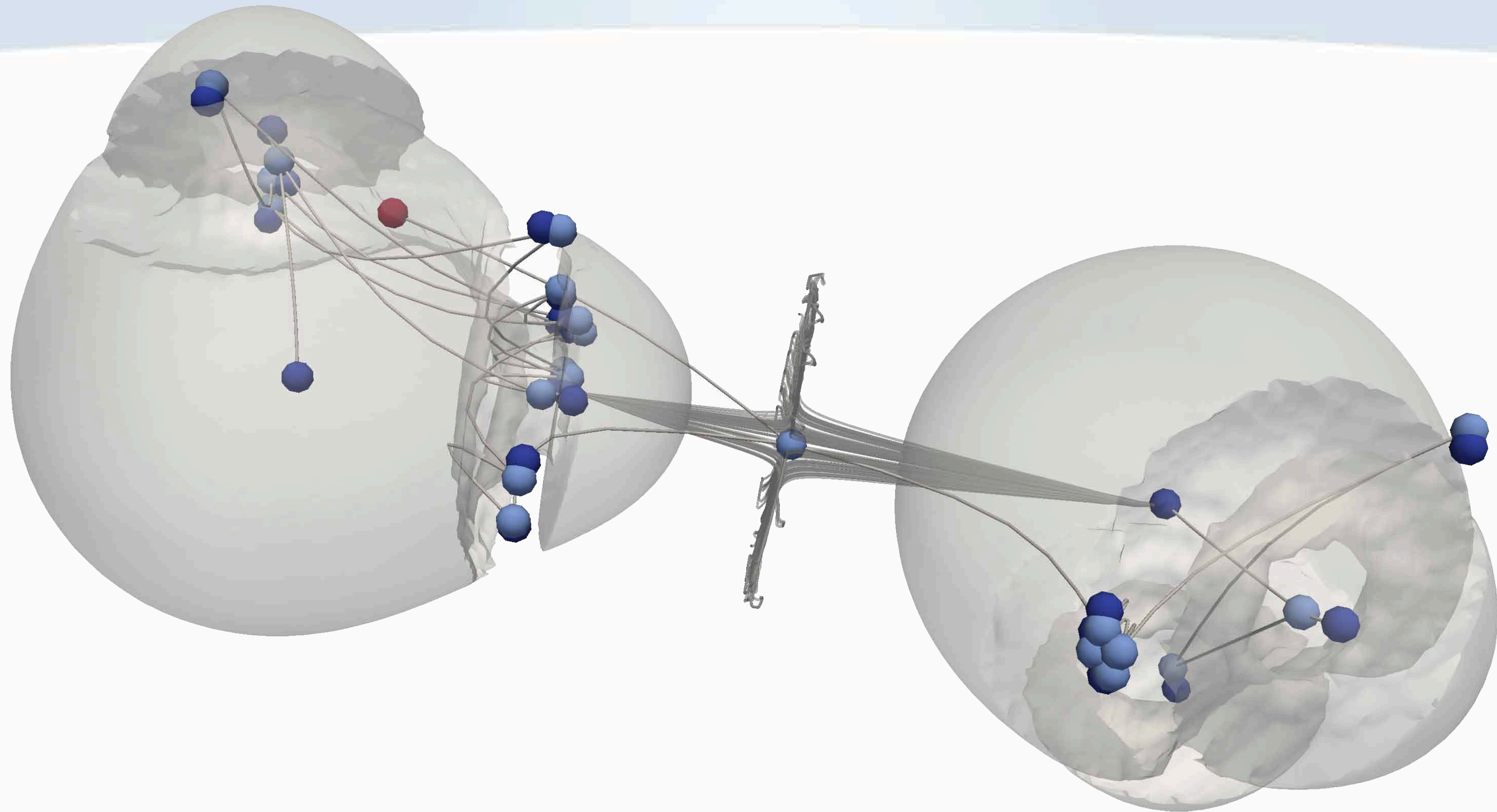
- Number of critical point pairs
 - As a function of their persistence



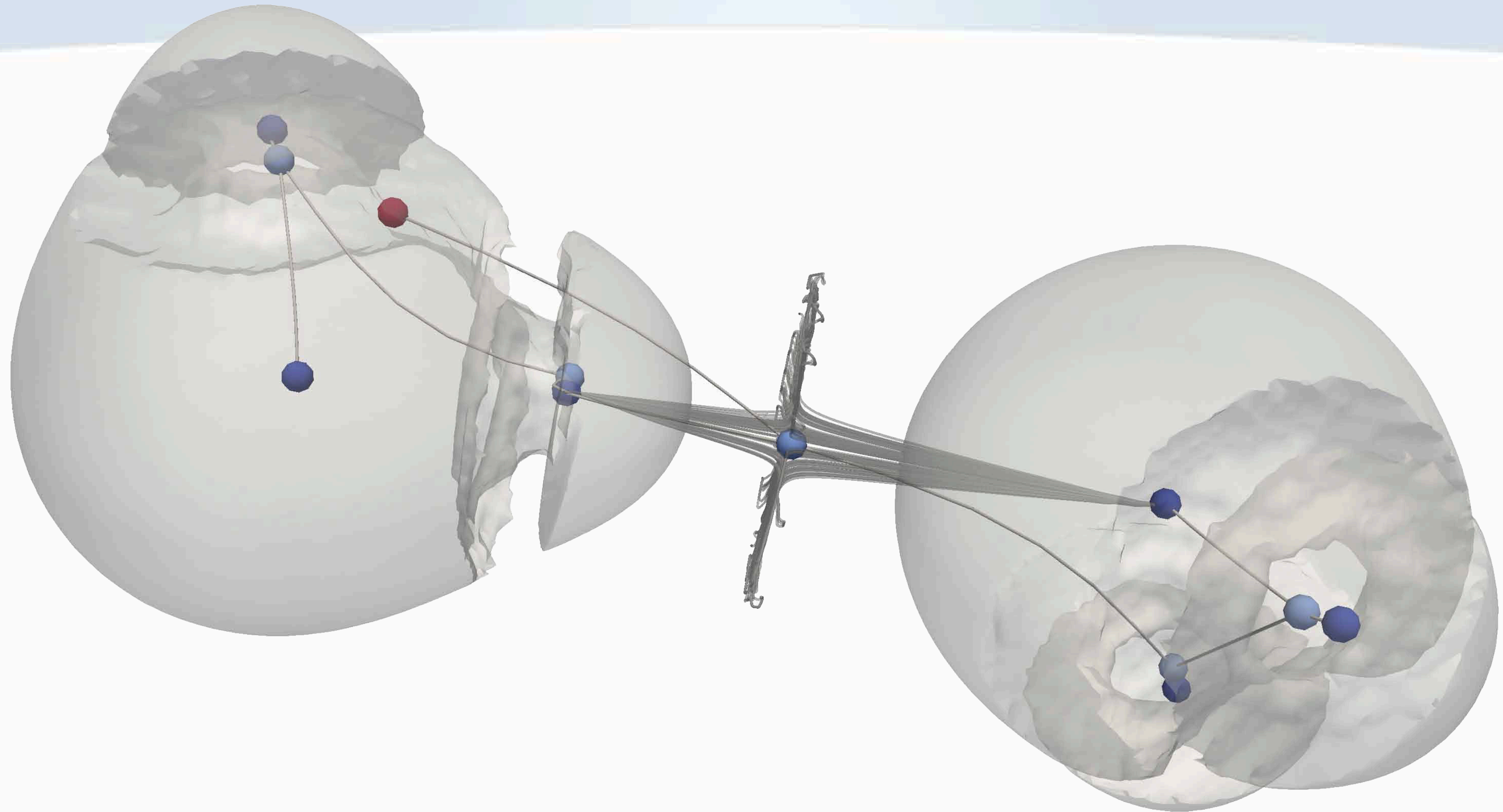
Persistence Curve



Persistence Curve



Persistence Curve



Structural Understanding

- “Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique”, Académie des Sciences, 1946



SÉANCE DU 8 AVRIL 1946. 847

Prenez pour (u_n) une surface $z = f(x, y)$, où f a des dérivées troisièmes, et pour $S(u)$ une sphère de rayon nul, \bar{Q}_n n'existant pour aucune valeur de u et le lieu de Q étant partout sans tangente. On peut prendre pour (x, y, z) et (u, v, w) des fonctions à dérivées premières continues. Nous avons $H = \sqrt{M_n^2} = \sqrt{M_n^2}$, donc H n'est pas dérivable par rapport à u , non plus que \bar{M}_n , et l'on montre aisément que ce fait ne dépend pas du remplacement de u par $u_1 = \varphi(u)$. L'existence des dérivées secondes de $u(x, y, z)$ entraînerait celle de \bar{M}_n . On peut donc conclure : *Il existe des systèmes triples-orthogonaux (u, v, w) pour lesquels le jeu des dérivées secondes est incomplet.*

D'ailleurs aucune famille de ces systèmes n'est formée de sphères, car dans ce dernier cas, on obtient immédiatement des exemples de disparition totale des courbures, vu la possibilité de choisir sur l'une des sphères un réseau orthogonal servant de base au système triple sur cette sphère, les courbes d'une famille étant des courbes uniparalingentes quelconques.

TOPOLOGIE. — Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique (*). Note (**) de M. GEORGES REEB, présentée par M. Elie Cartan.

I. FORMES DE PFAFF COMPLÈTEMENT INTÉGRABLES. — Propriété 1. — Soit $\omega = a_i x^i dx^i$ une forme de Pfaff complètement intégrable du R^n (a_i sont des constantes). Si la matrice (a_{ij}) des coefficients est de rang $p \geq 3$, la forme ω se réduit à une différentielle exacte.

En effet, $D\omega$ est à coefficients constants. Si $D\omega \neq 0$, la condition de complète intégrabilité entraîne que $D\omega = \omega_1 \wedge \omega_2$ (les formes ω_1 et ω_2 étant à coefficients constants et ω étant une combinaison de ω_1 et ω_2). D'où l'on conclut que, par changement de base, on peut écrire $\omega = \lambda dX^1 + \mu dX^2$, où λ et μ sont des formes linéaires de X^1 et X^2 . Ceci montre que le rang de la matrice (a_{ij}) est au plus égal à 2. Donc $D\omega = 0$, d'où $a_{ij} = a_{ji}$ et $\omega = (1/2)d(a_{ij}x^i x^j)$.

Soit maintenant $\omega = a_i(x) dx^i$ une forme deux fois continûment différentiable, définie dans une boule de centre O du R^n et admettant le point O de coordonnées $x^i = 0$ comme point singulier. Désignons par a_{ij} les valeurs des dérivées partielles des a_i calculées au point O .

Propriété 2. — Si $\omega = a_i dx^i$ est complètement intégrable, il en est de même de la forme $\omega^* = a_{ij} x^i dx^j$.

Si la forme quadratique $a_{ij} x^i x^j$ est définie positive ou négative, la forme

(*) Voir Ch. EHRESMANN et G. REEB, *Comptes rendus*, 218, 1944, p. 995; 220, 1945, p. 237.
 (**) Séance du 25 mars 1946.

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$\omega = a_i(x) dx^i$ satisfera dans un voisinage de O à la condition C suivante : $[a_i X^i] / \sqrt{\sum a_i^2 \sum (x^i)^2} \geq \varepsilon > 0$. D'une façon générale, nous posons :

Définition. — Si le point O (défini par $x^i = 0$) est un point singulier isolé de la forme complètement intégrable $\omega = a_i(x) dx^i$ et si dans tout un voisinage O la condition C est satisfaite, le point O sera appelé centre.

Propriété 3. — Si O est un centre pour ω , le système des variétés intégrales, dans un voisinage convenable de O , peut être transformé par un homéomorphisme h en un système de sphères concentriques du R^n (si $n \geq 3$).

Propriété 4. — Si ω est une forme complètement intégrable définie dans une variété V_n compacte et si les seuls points singuliers de ω sont des centres, alors V_n est une sphère S_n , les variétés intégrales sont des sphères S_{n-1} et les points singuliers sont au nombre de deux. De plus, ω admet un facteur intégrant dans tout V_n .

II. FONCTIONS NUMÉRIQUES. — Soient V_n une variété deux fois continûment différentiable et compacte, et f une fonction numérique deux fois continûment différentiable sur V_n admettant des points singuliers isolés. Supposons qu'en ces points singuliers la matrice (a_{ij}) des dérivées partielles secondes a_{ij} de f , soit de rang n .

Le point singulier sera caractérisé par l'indice de la forme quadratique associée à (a_{ij}) .

Appelons variété de niveau l'ensemble des points x vérifiant $f(x) = K$. Une variété de niveau est une véritable variété si K n'est pas une valeur singulière de f .

Désignons par variété connexe de niveau, une composante connexe d'une variété de niveau. Supposons de plus qu'à des points singuliers distincts correspondent des valeurs distinctes de f .

THÉORÈME 1. — *Le quotient de V_n par la relation d'équivalence « x et y appartiennent à la même variété connexe de niveau » est un complexe simplicial fini et connexe K_n de dimension 1. L'image réciproque $\tau^{-1}(o)$ d'un sommet O de K_n est une variété connexe de niveau singulière; l'image réciproque de l'intérieur de chaque simplexe est homéomorphe au produit topologique d'une variété connexe de niveau par R^1 .*

Appelons ordre d'un sommet de K_n le nombre de simplexes de dimension 1 de K_n ayant ce sommet pour sommet. L'indice d'un sommet de K_n sera l'indice du point singulier dont il est image.

THÉORÈME 2. — *L'ordre des sommets de K_n d'indice 0 (ou n) est 1 et réciproquement l'indice d'un sommet d'ordre 1 est 0 ou n .*

THÉORÈME 3. — *Si $n \geq 3$ l'ordre des sommets d'indice 1 (ou $n-1$) est 2 ou 3. Si $n=2$ l'ordre des sommets d'indice 1 est 2, 3 ou 4. L'ordre des sommets d'indice distinct de 0, 1, $n-1$ ou n est 2.*

La démonstration de ce dernier théorème repose sur le lemme suivant : Soit $x \in V_n$ un point singulier d'indice p . Dans une boule ouverte B_n de centre x

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et assez petite, l'ensemble des points y tels que $f(y) < f(x)$ a une ou deux composantes connexes, selon que $p > 1$ ou $p = 1$.

On en conclut en particulier :

THÉORÈME 4. — *Si $n \geq 3$ et si M est le nombre de points singuliers d'indice 0 ou n , et m celui des points singuliers d'indice 1 ou $n-1$, on a l'inégalité suivante $M \leq m + 2$. Si $n=2$, l'inégalité s'écrit $M \leq 2(m+1)$.*

THÉORÈME 5. — *En particulier, si $m=0$ on a $M=2$. Le complexe K_n sera homéomorphe à un intervalle fermé, et toutes les variétés de niveau sont connexes.*

THÉORÈME 6. — *Le groupe de Poincaré de K_n est isomorphe à un groupe quotient du groupe de Poincaré de V_n .*

COROLLAIRE. — *Si V_n a un groupe de Poincaré fini et si $M=2$, le complexe K_n sera homéomorphe à un intervalle fermé et toutes les variétés de niveau seront connexes.*

ANALYSE MATHÉMATIQUE. — Sur le problème de Cauchy pour les équations aux dérivées partielles, totalement hyperboliques, d'ordre plus grand que 2. Note (*) de M. FLORENT BUREAU, présentée par M. Jacques Hadamard.

1. Le problème de Cauchy pour les équations aux dérivées partielles d'ordre supérieur à deux a fait l'objet des recherches de E.-E. Levi, I. Petrovsky, G. Herglotz et N. Zeilon. Dans des travaux bien connus, M. J. Hadamard a montré que, pour les équations aux dérivées partielles du second ordre, du type simplement hyperbolique, tous les résultats relatifs au problème de Cauchy et au principe de Huygens peuvent et doivent se déduire de la seule solution élémentaire. Nous voudrions montrer que l'on peut étendre ce résultat à des équations aux dérivées partielles totalement hyperboliques, d'ordre quatre et à trois variables indépendantes.

Considérons l'équation

$$D_\varphi D_\psi u = F(x),$$

avec

$$D_\varphi = a_2 \frac{\partial^2}{\partial x_2^2} - a_1 \frac{\partial^2}{\partial x_1^2} - a_3 \frac{\partial^2}{\partial x_3^2},$$

$$D_\psi = b_2 \frac{\partial^2}{\partial x_2^2} - b_1 \frac{\partial^2}{\partial x_1^2} - b_3 \frac{\partial^2}{\partial x_3^2},$$

les a et les b étant des constantes positives telles que $0 < a_2 < a_1 < a_3$, $0 < b_1 < b_2 < b_3$. Nous poserons

$$f = a_1 a_2 x_2^2 - a_2 a_1 x_1^2 - a_3 a_1 x_3^2,$$

$$g = b_1 b_2 x_2^2 - b_2 b_1 x_1^2 - b_3 b_1 x_3^2,$$

$$z h = (a_1 b_2 + a_2 b_1) x_2^2 - (a_2 b_1 + a_3 b_2) x_1^2 - (a_3 b_2 + a_1 b_3) x_3^2,$$

$$p_1 = a_1 b_2 - a_2 b_1, \quad p_2 = a_2 b_1 - a_3 b_2, \quad p_3 = a_3 b_2 - a_1 b_3,$$

$$c_1^2 = p_1 p_2, \quad c_2^2 = p_2 p_3, \quad c_3^2 = p_3 p_1,$$

$$c_1 = c_1', \quad c_2 = c_2', \quad c_3 = c_3',$$

(*) Séance du 25 mars 1946.

Reeb Graph

- $\mathcal{R}(f)$
- Quotient space on $\mathcal{D} \times \mathbb{R}$
- By the equivalence relation
$$(p_1, f(p_1)) \sim (p_2, f(p_2))$$

$$f(p_1) = f(p_2)$$

$$p_2 \in (f^{-1}(f(p_1)))_{p_1}$$

- Demo!

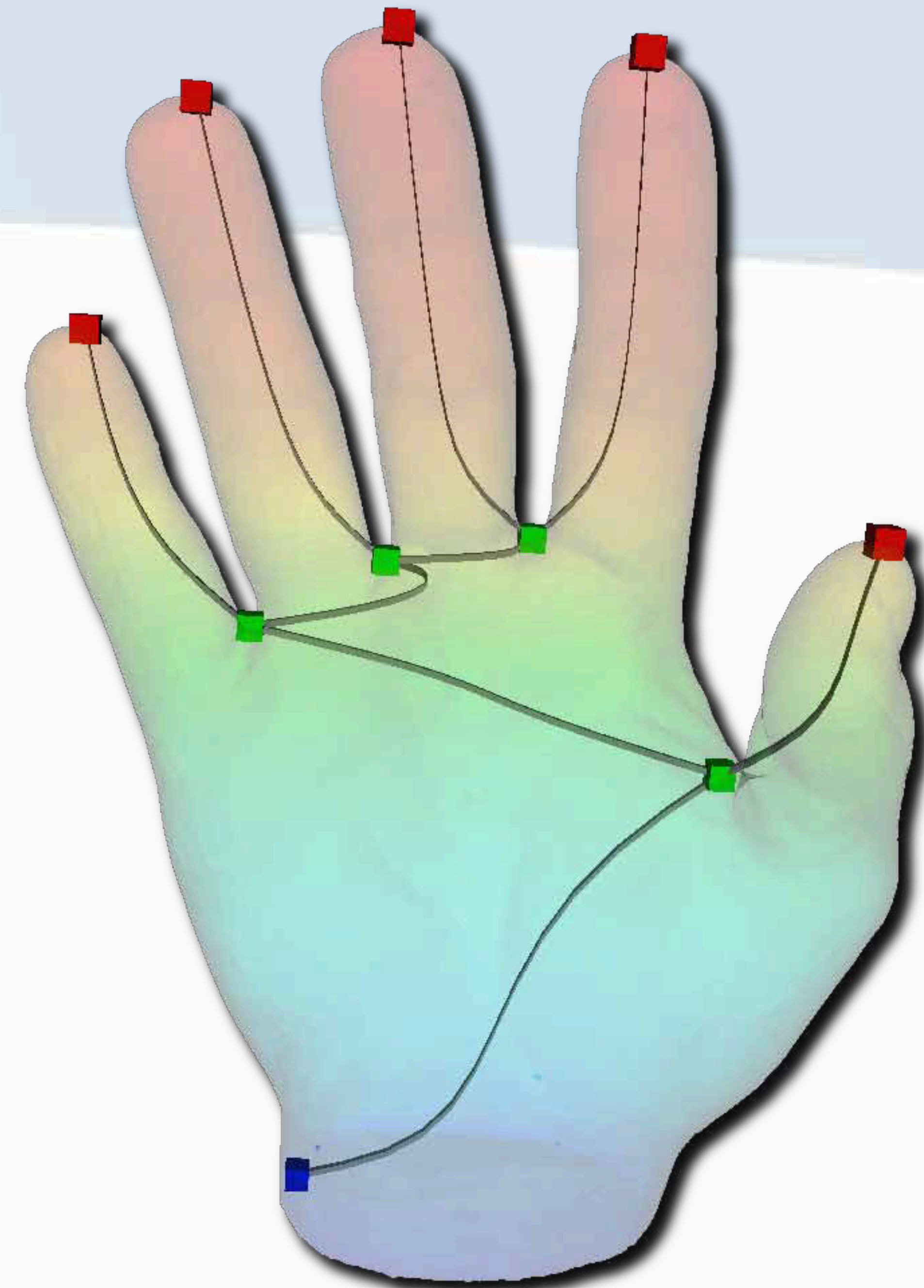


Reeb Graph in Practice

- Applicative interests
 - Persistence diagram computation (low dimension)

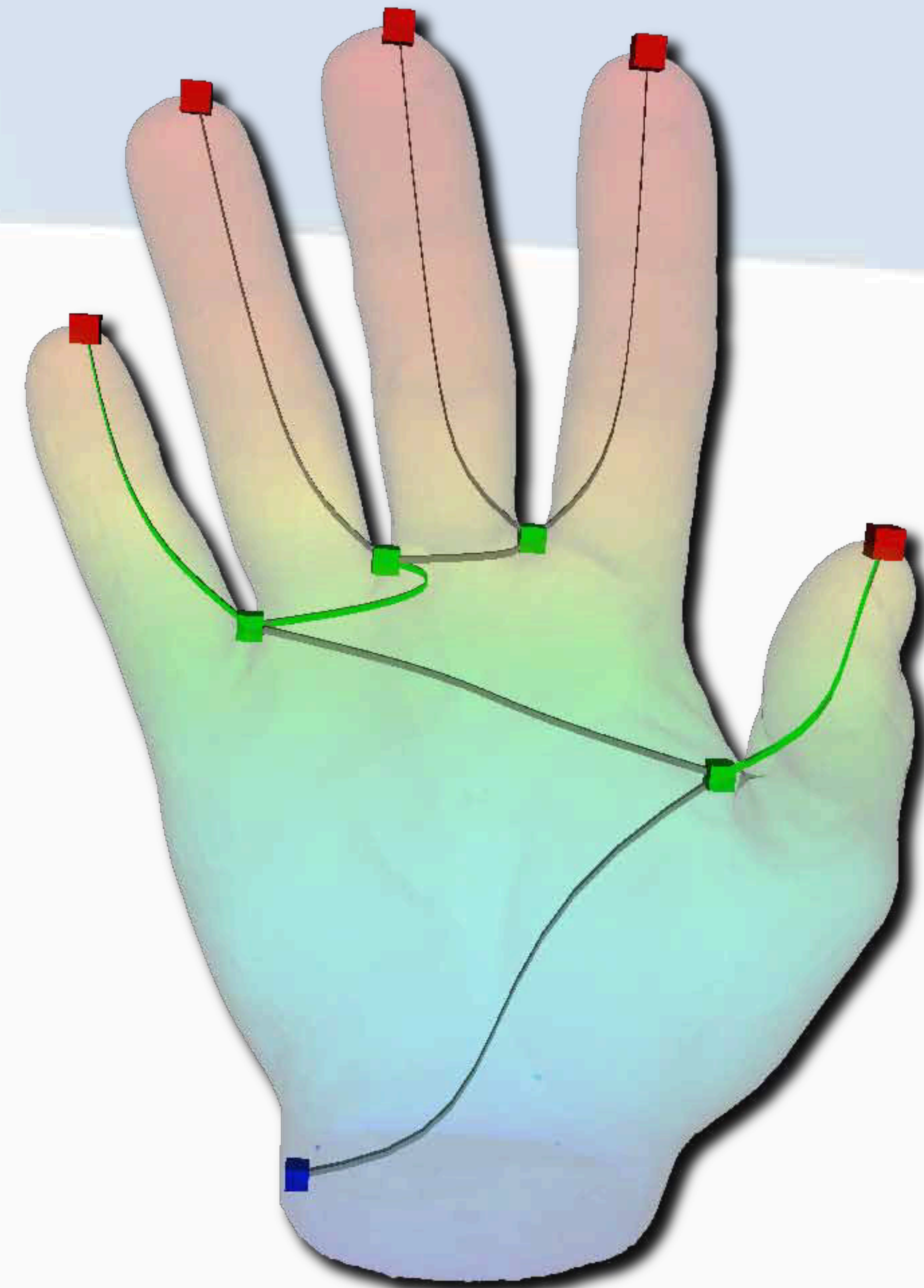
Reeb Graph in Practice

- Applicative interests
 - Persistence diagram computation (low dimension)
 - Optimal iso-surface extraction and simplification



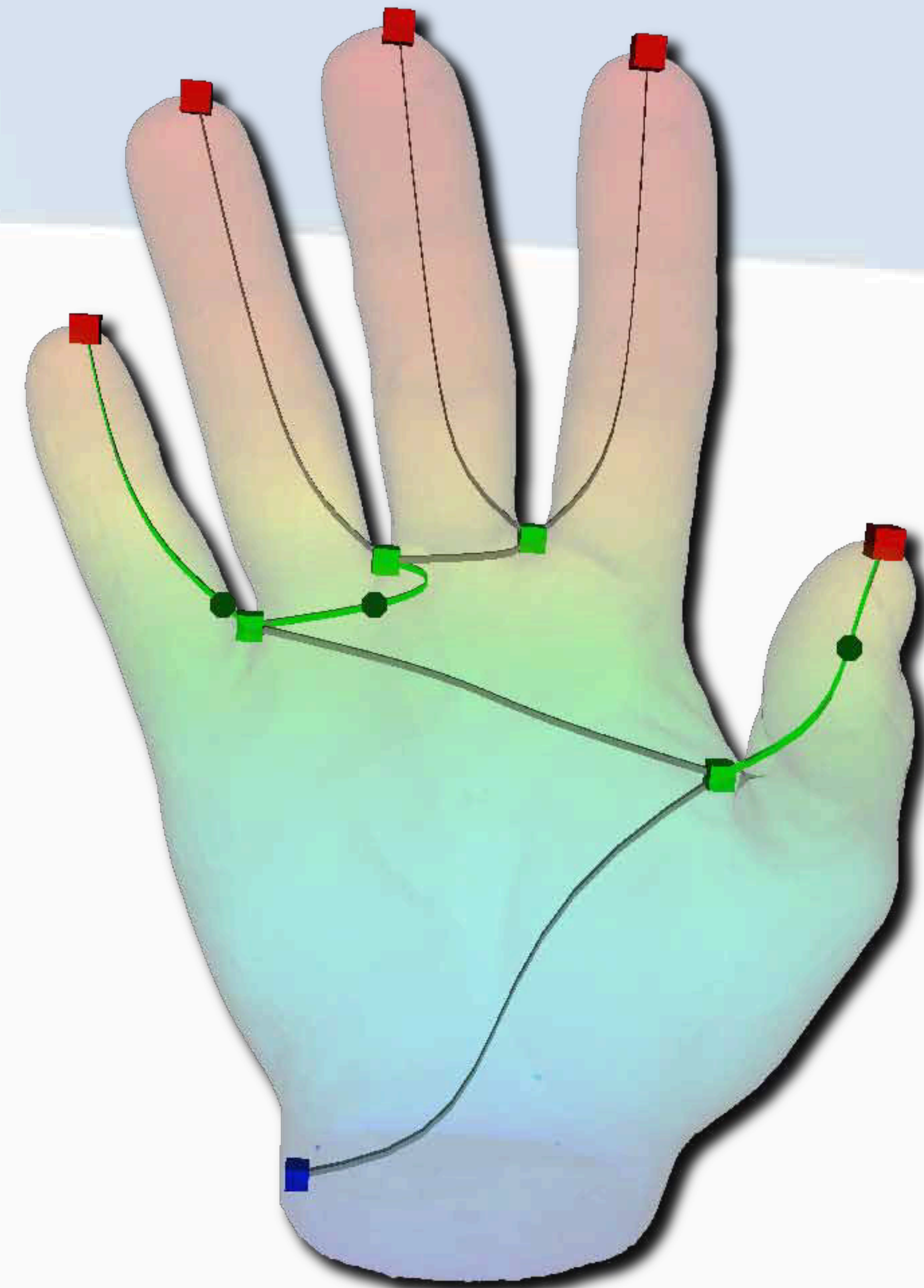
Reeb Graph in Practice

- Applicative interests
 - Persistence diagram computation (low dimension)
 - Optimal iso-surface extraction and simplification



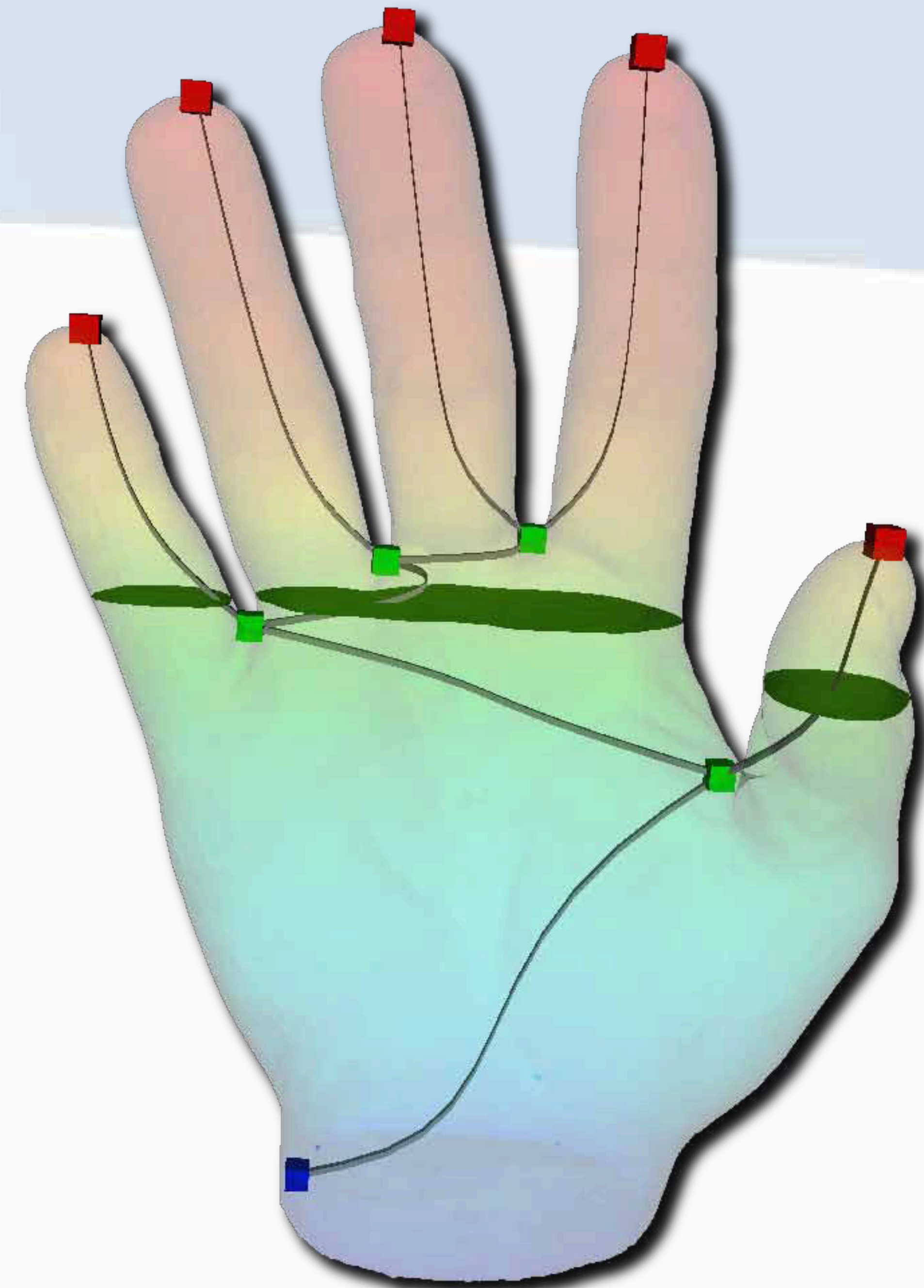
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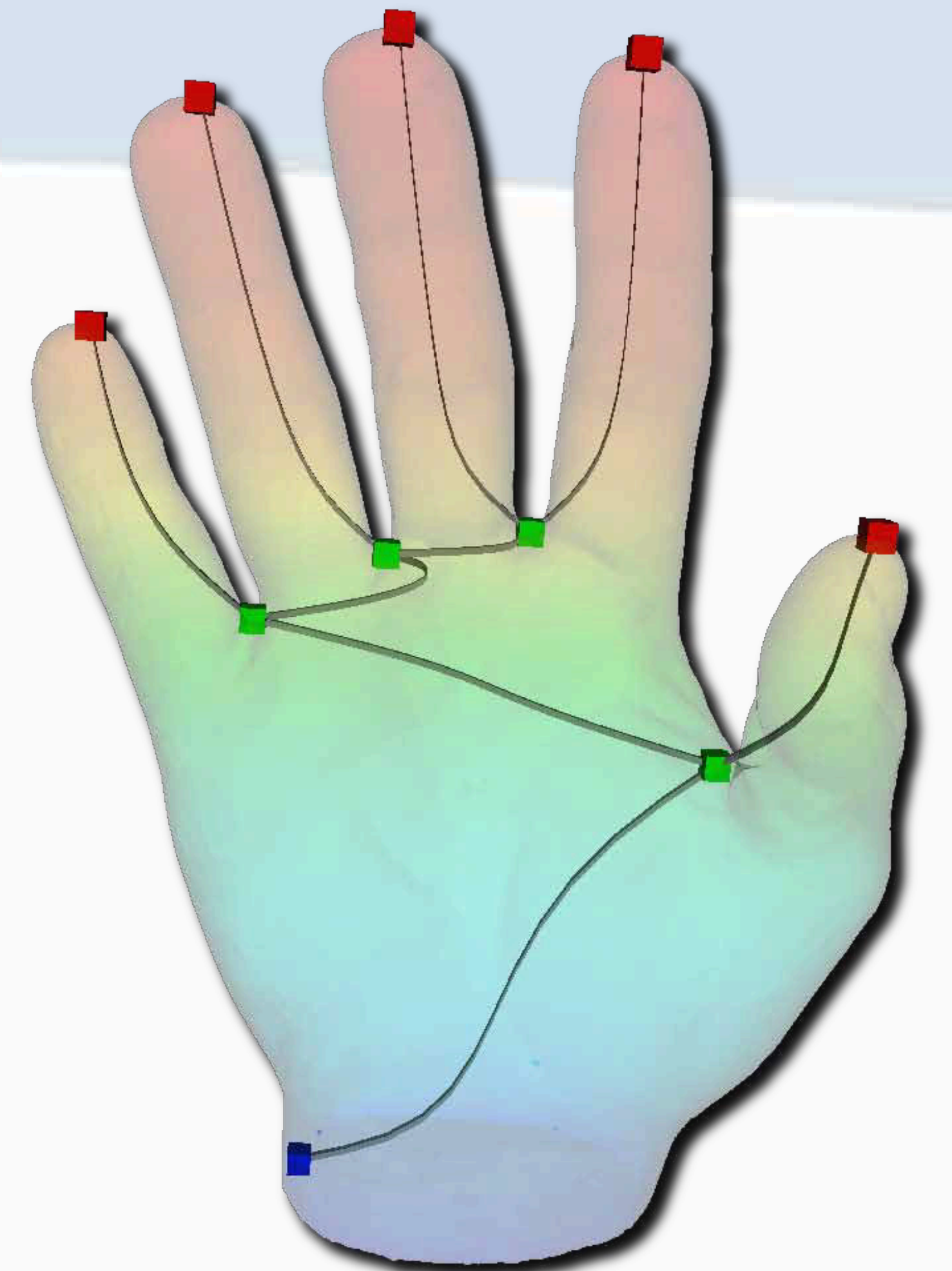
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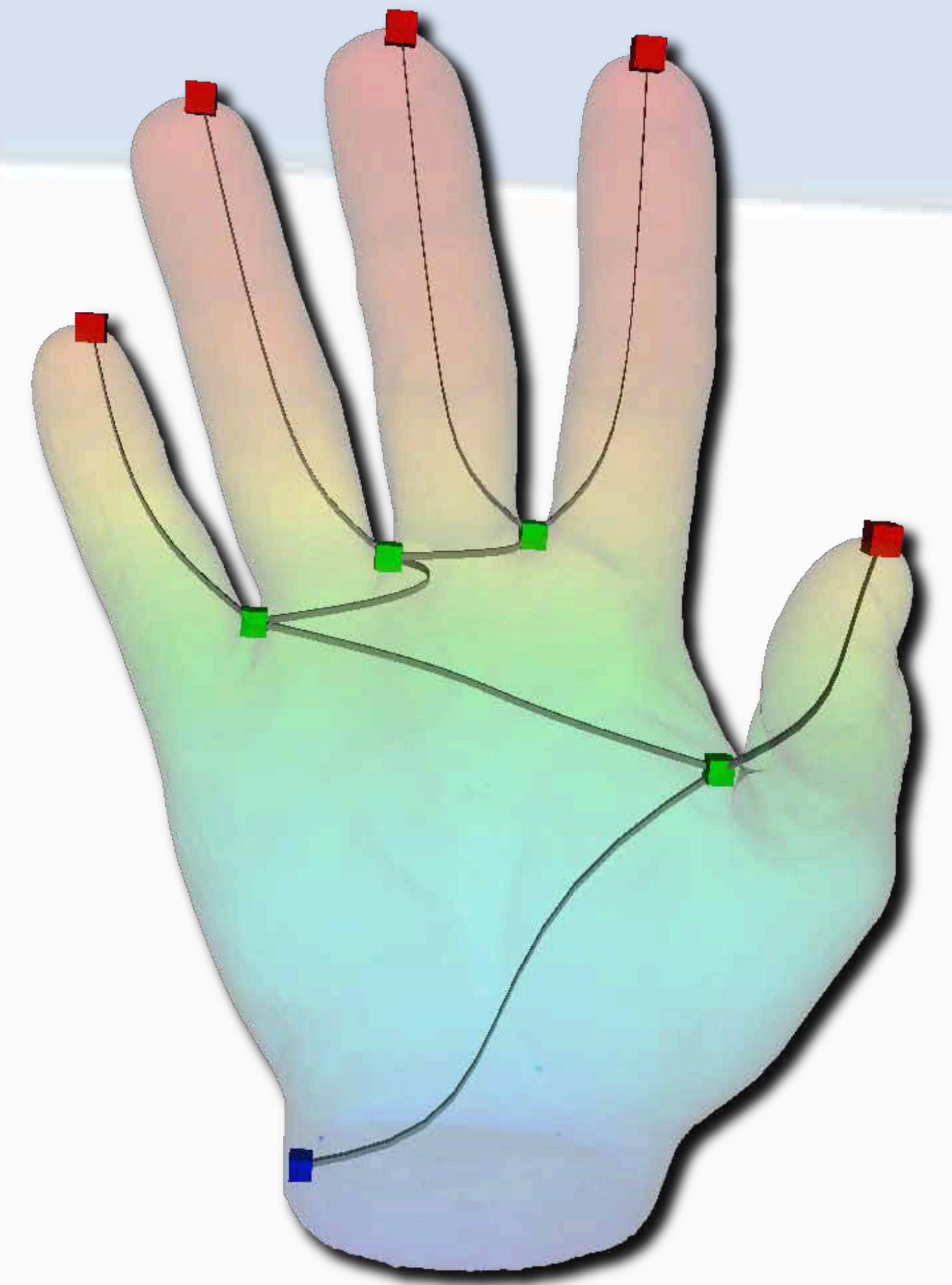
Reeb Graph in Practice

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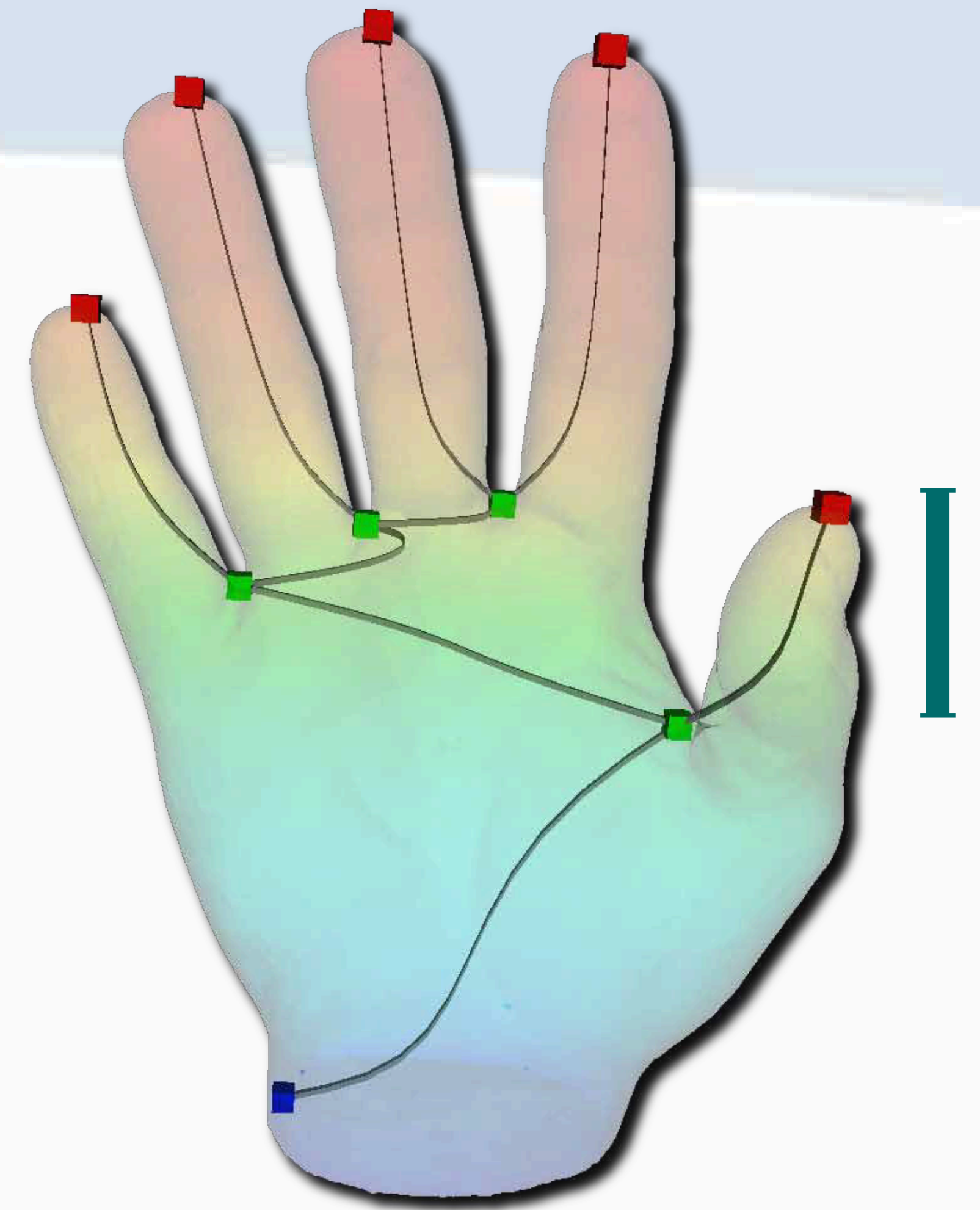
Reeb Graph in Practice

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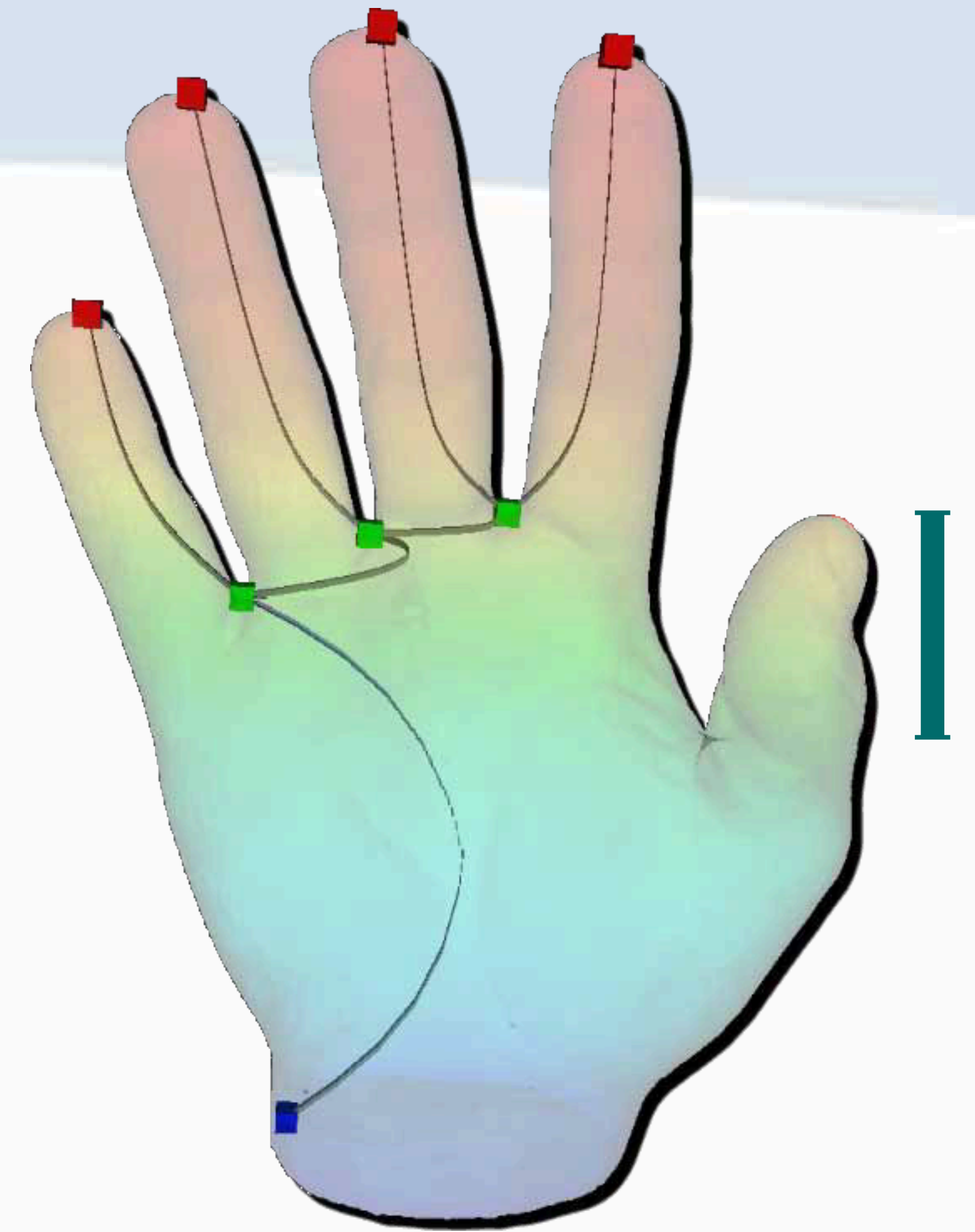
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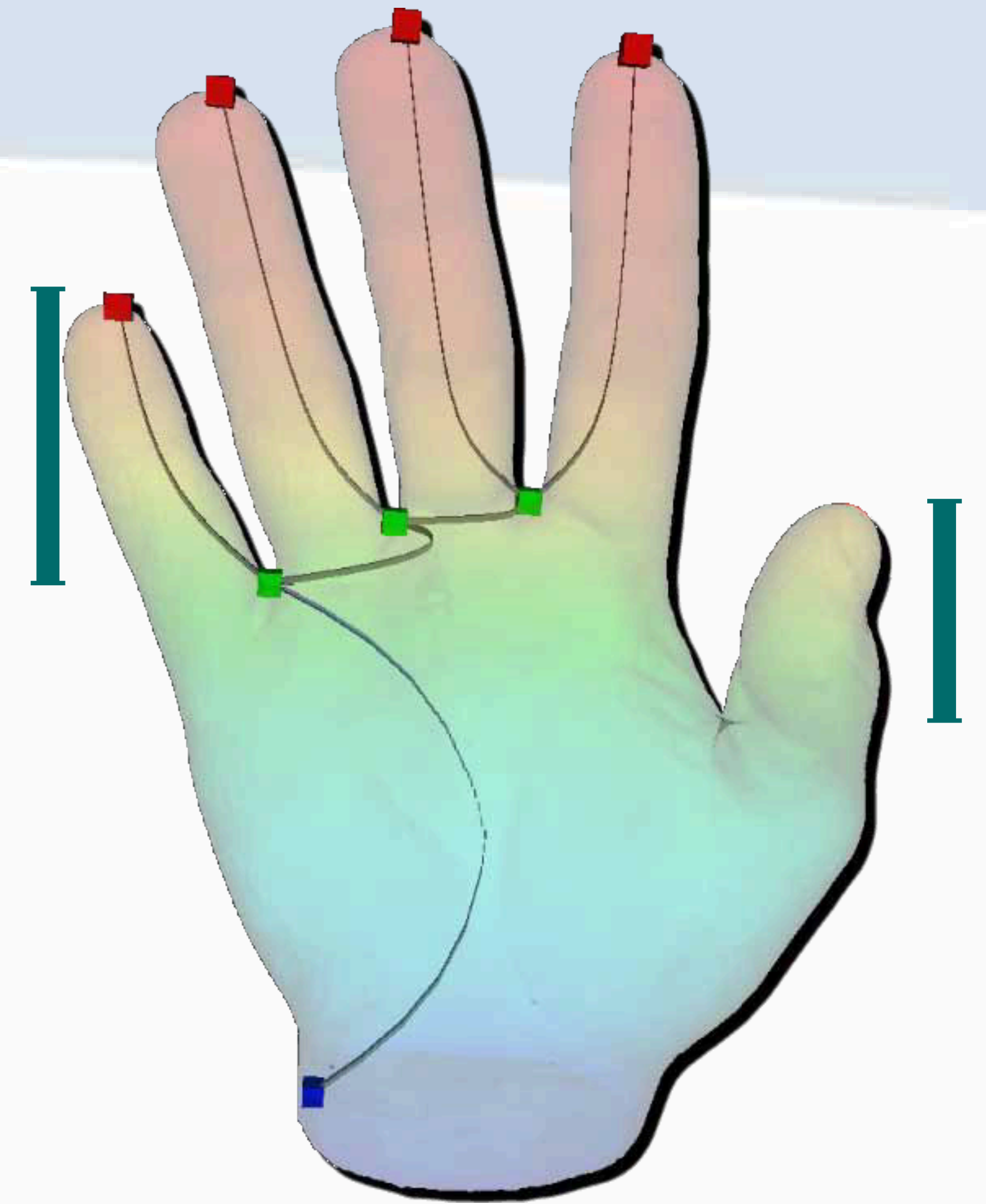
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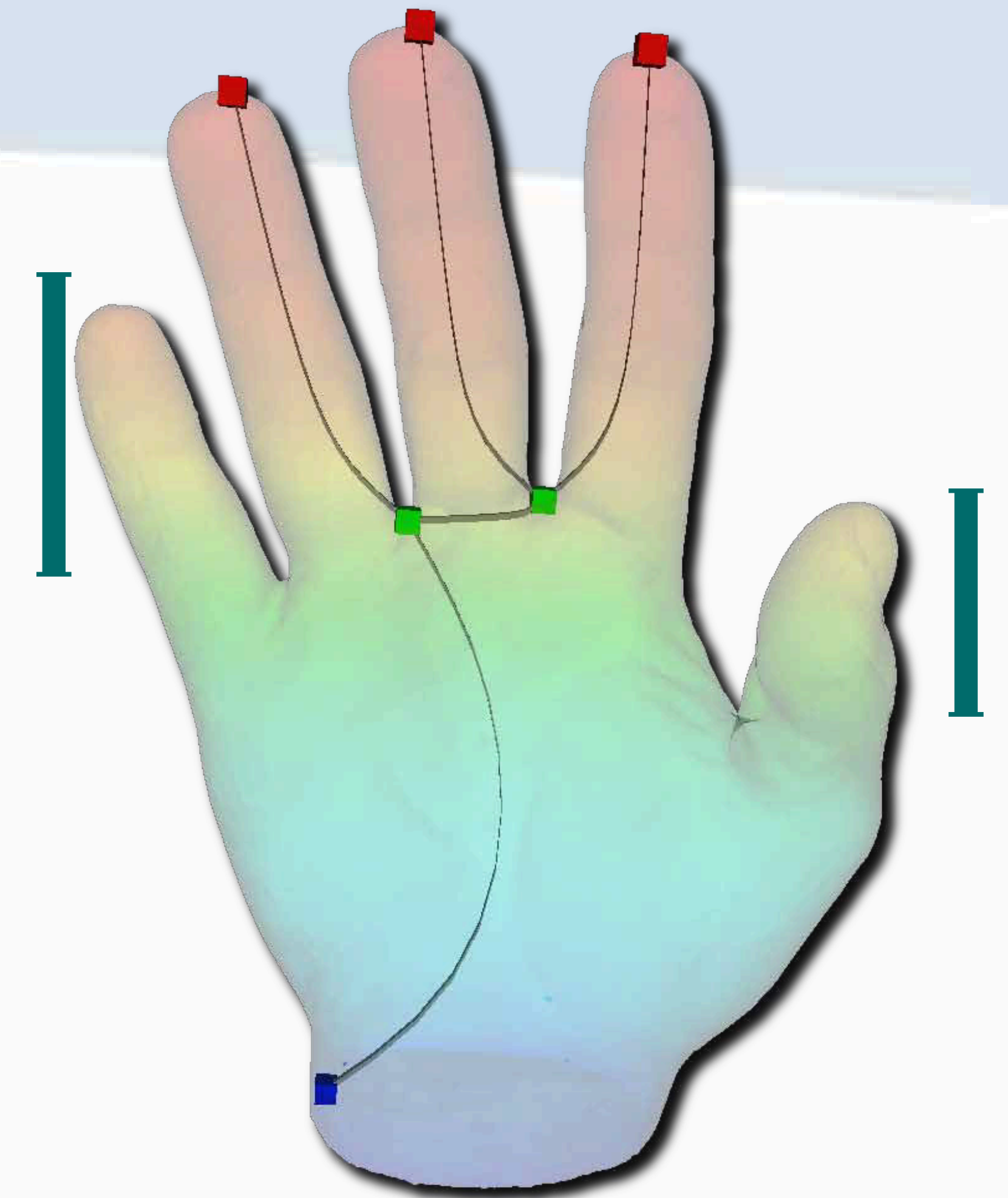
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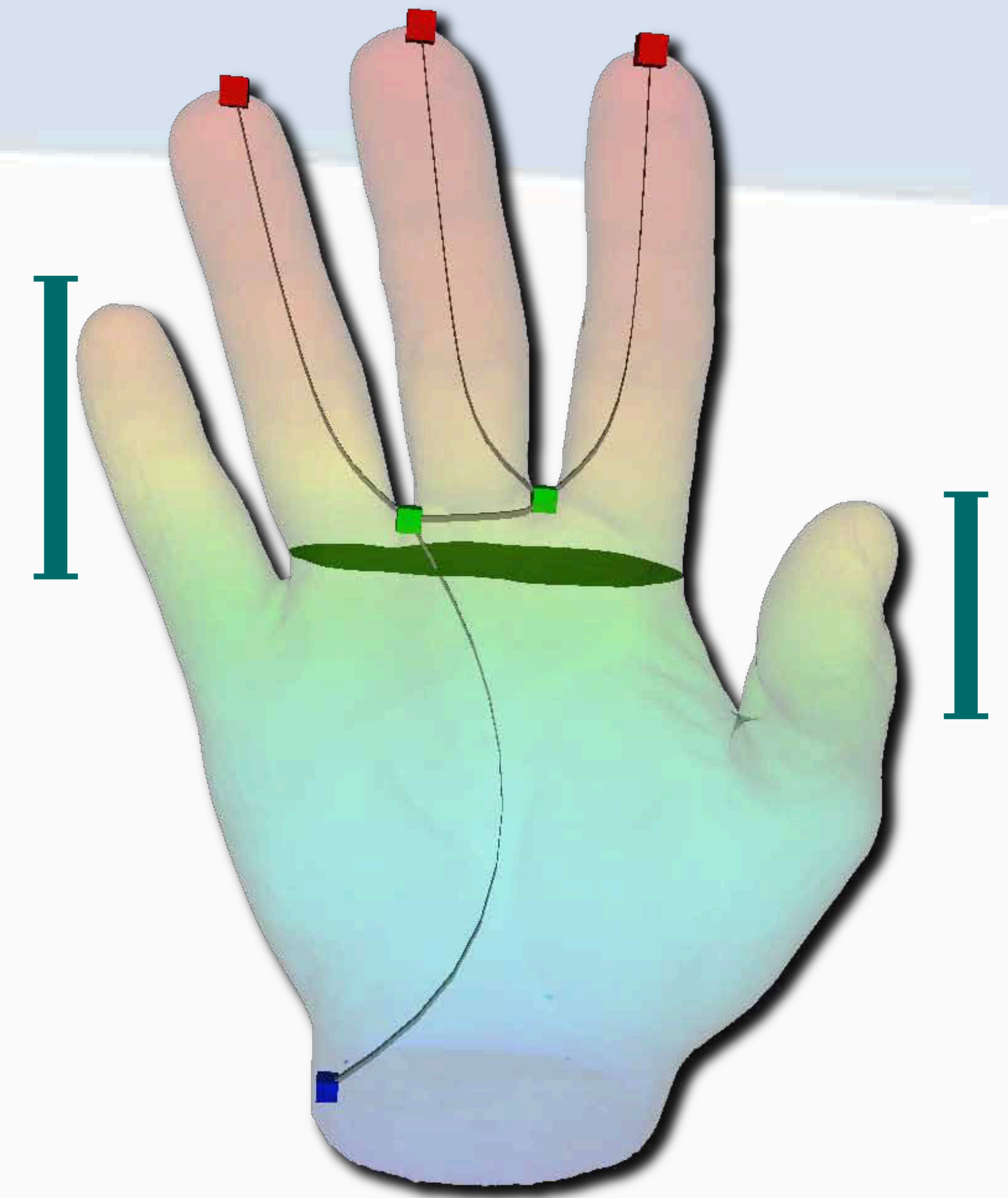
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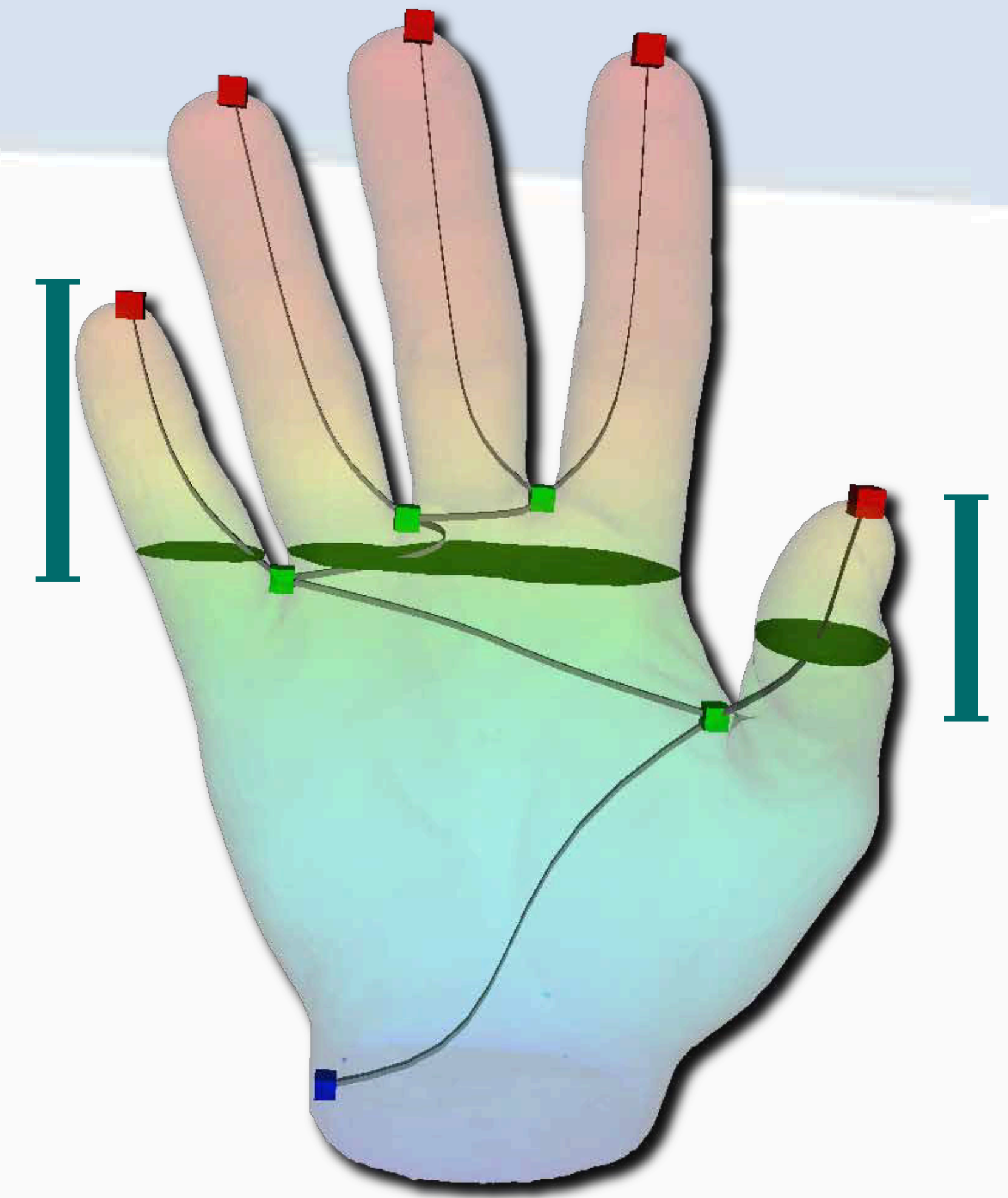
Reeb Graph in Practice

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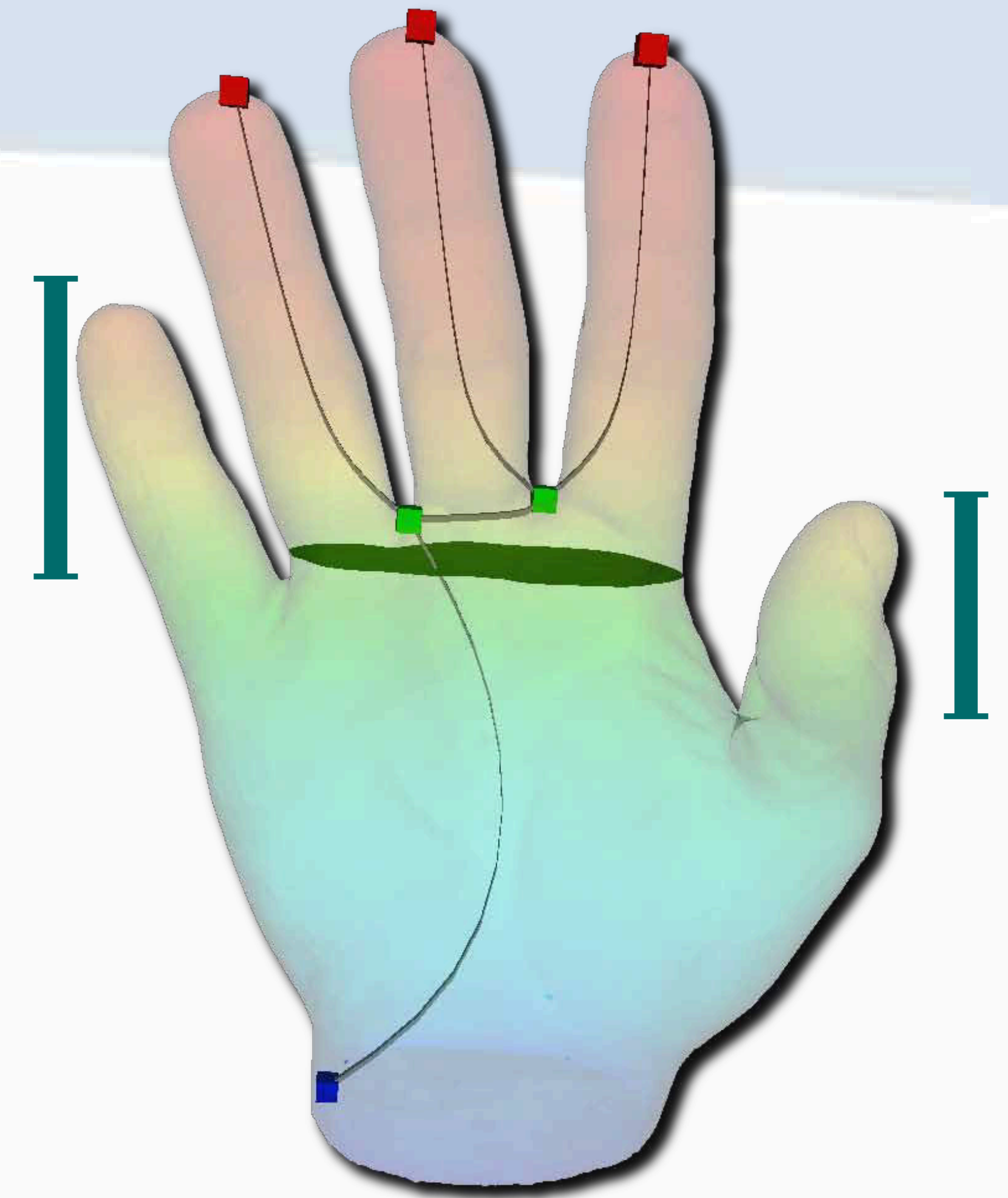
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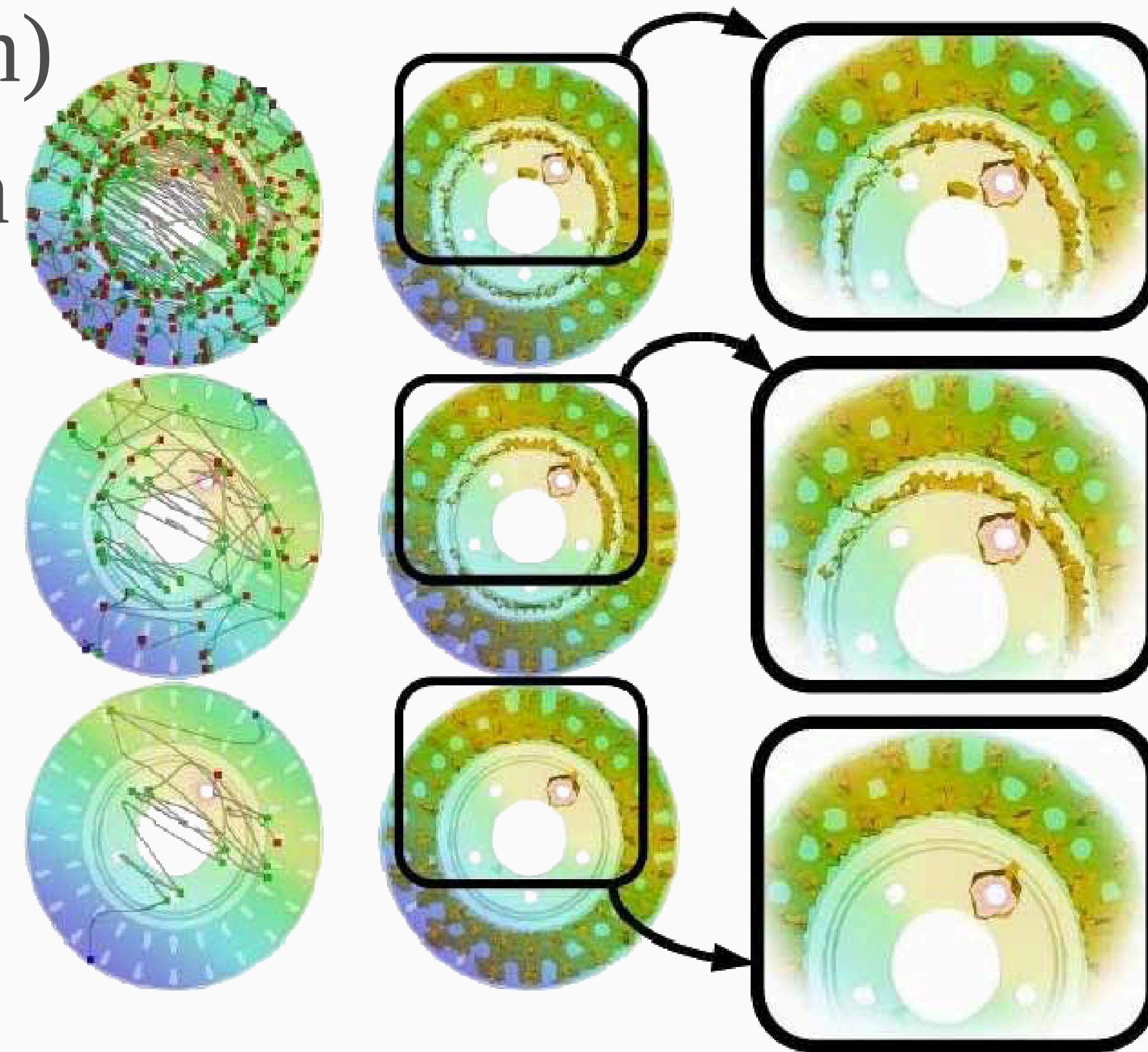
Reeb Graph in Practice

- Applicative interests
 - Persistence diagram computation (low dimension)
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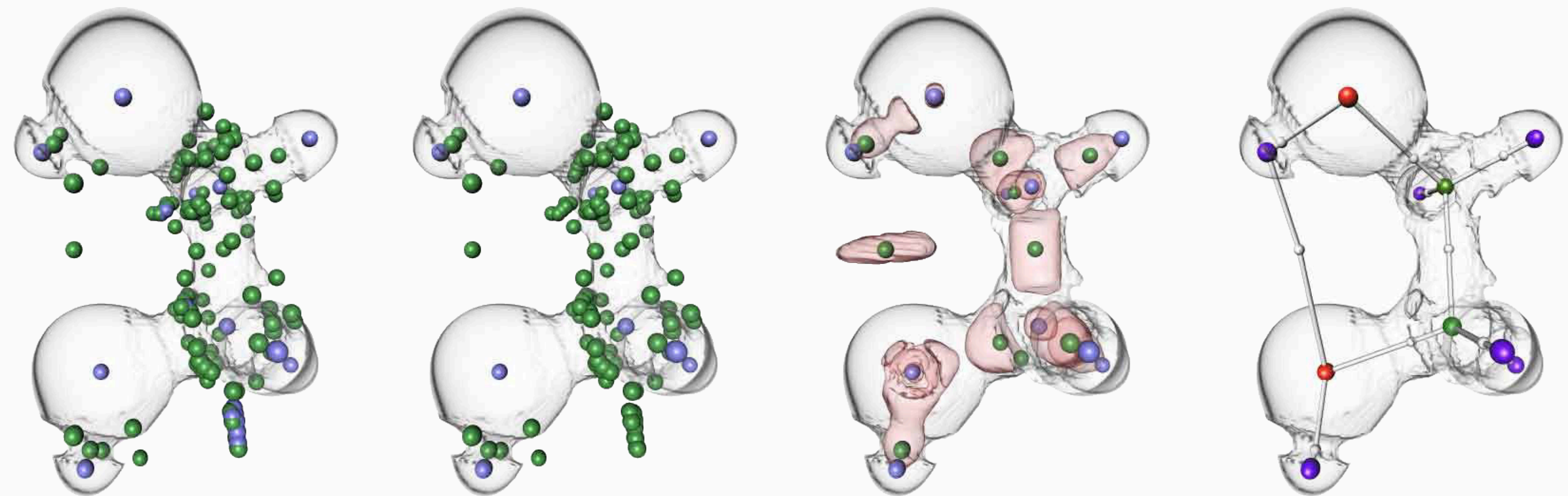
Reeb Graph in Practice

- Applicative interests
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 - Optimal iso-surface extraction and simplification



Reeb Graph in Practice

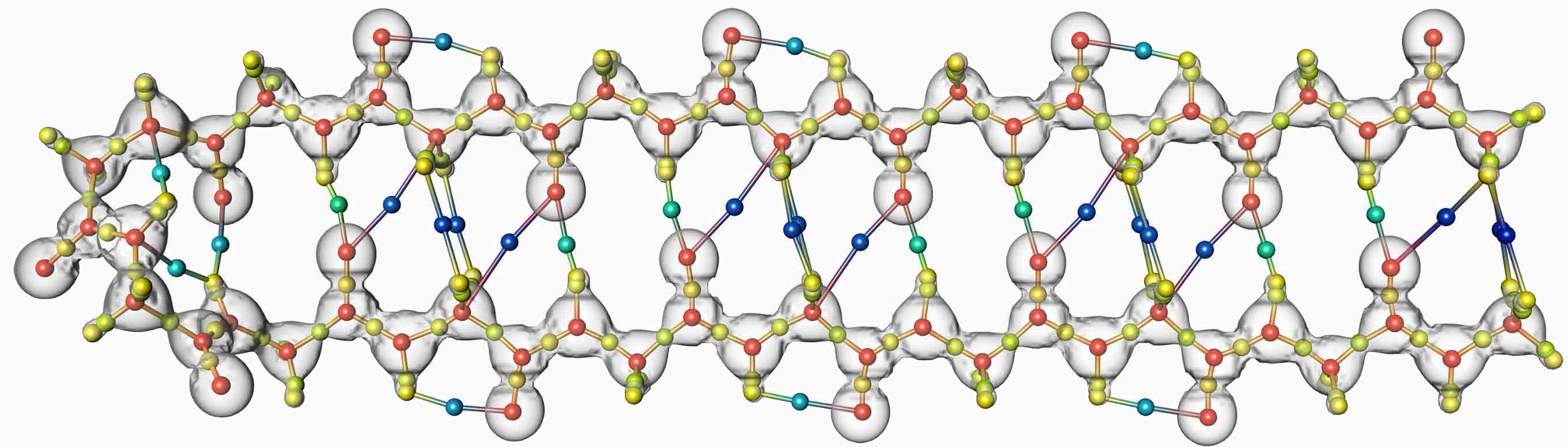
- Applicative interests
 - Persistence diagram computation (low dimension)
 - Optimal iso-surface extraction and simplification
 - Data Segmentation



[Guenther et al. VIS14]

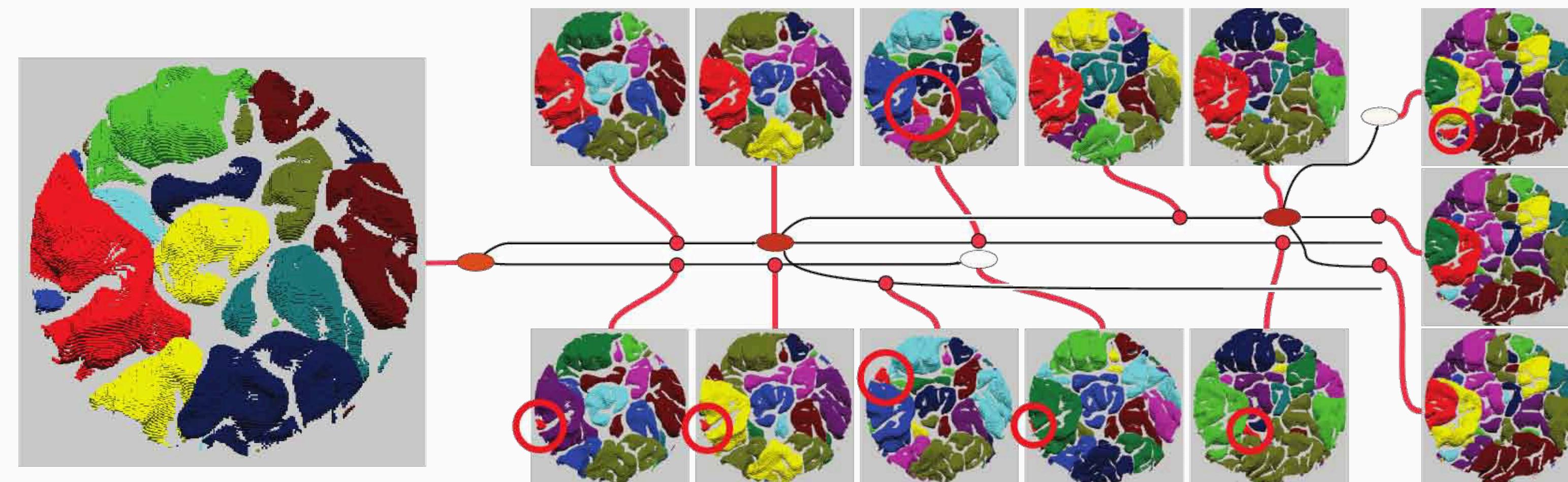
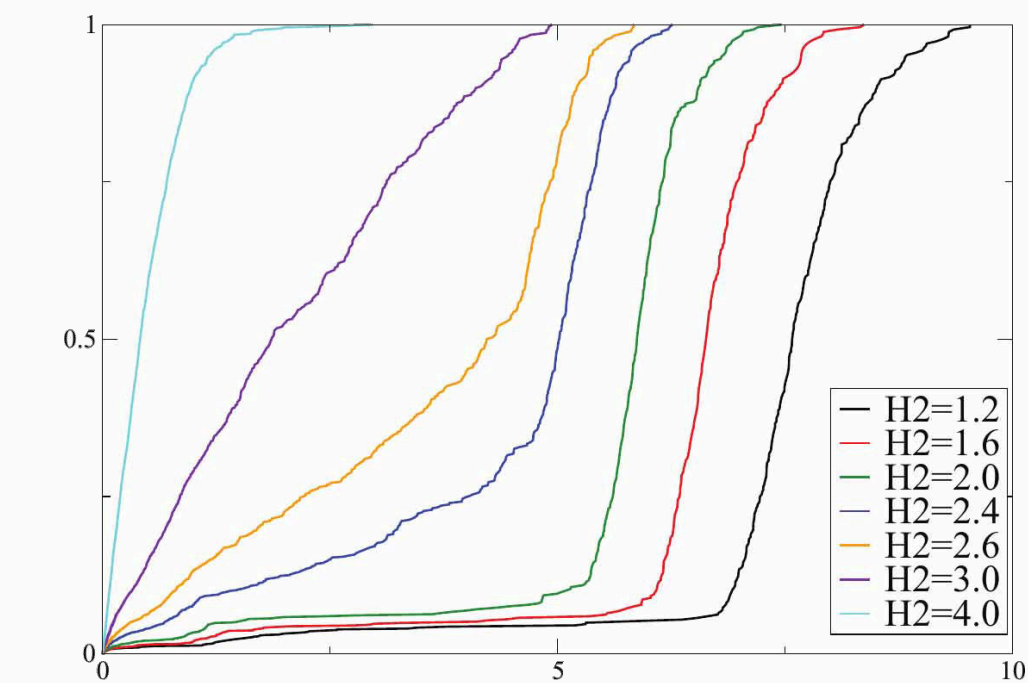
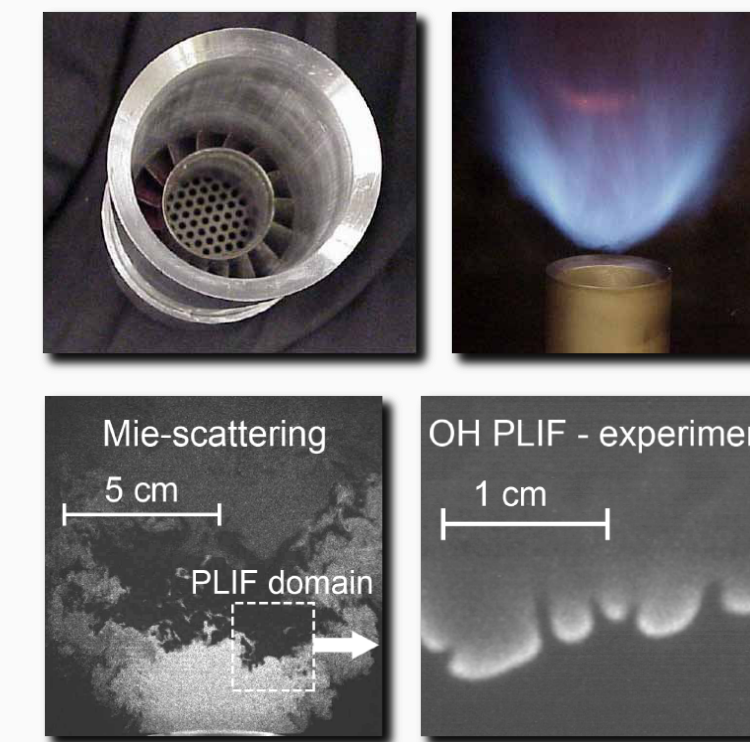
Reeb Graph in Practice

- Applicative interests
 - Persistence diagram computation (low dimension)
 - Optimal iso-surface extraction and simplification
 - Data Segmentation



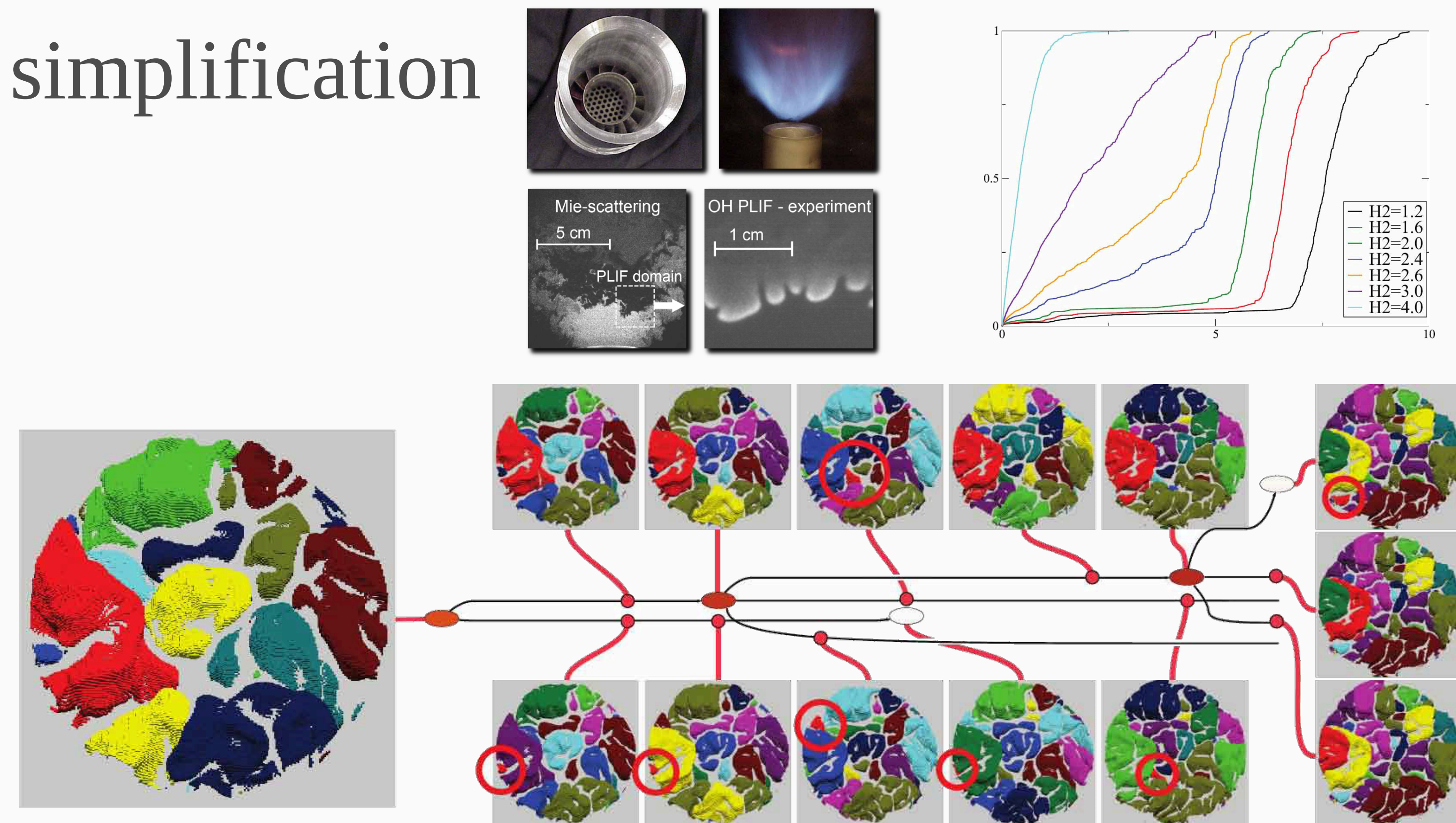
Reeb Graph in Practice

- Applicative interests
 - Persistence diagram computation (low dimension)
 - Optimal iso-surface extraction and simplification
 - Data Segmentation
 - Feature tracking



Reeb Graph in Practice

- Applicative interests
 - Persistence diagram computation (low dimension)
 - Optimal iso-surface extraction and simplification
 - Data Segmentation
 - Feature tracking
 - Comparison
 - Verification
 - Etc.

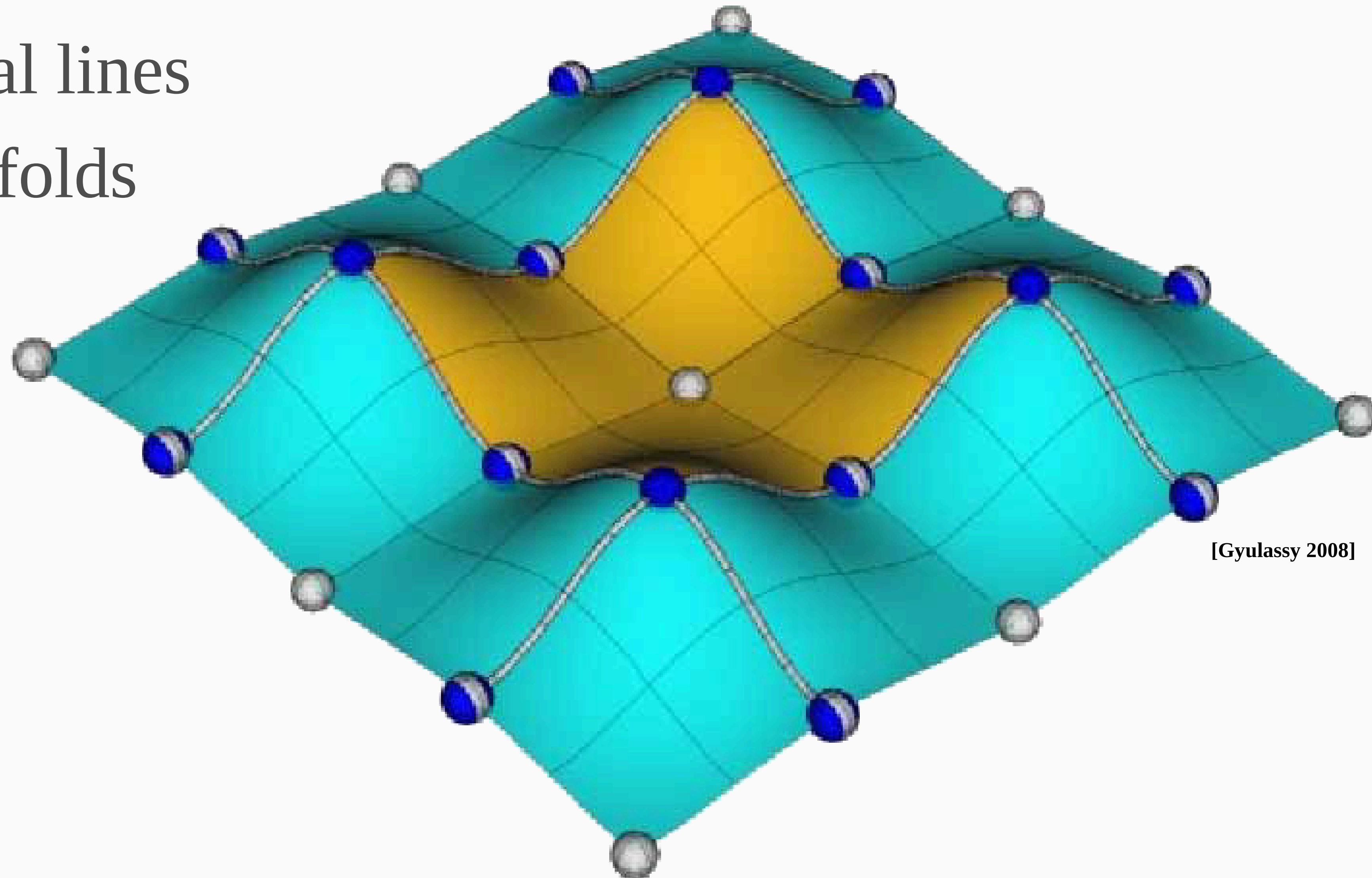


Morse-Smale Complex

- Equivalence relation on integral lines

Morse-Smale Complex

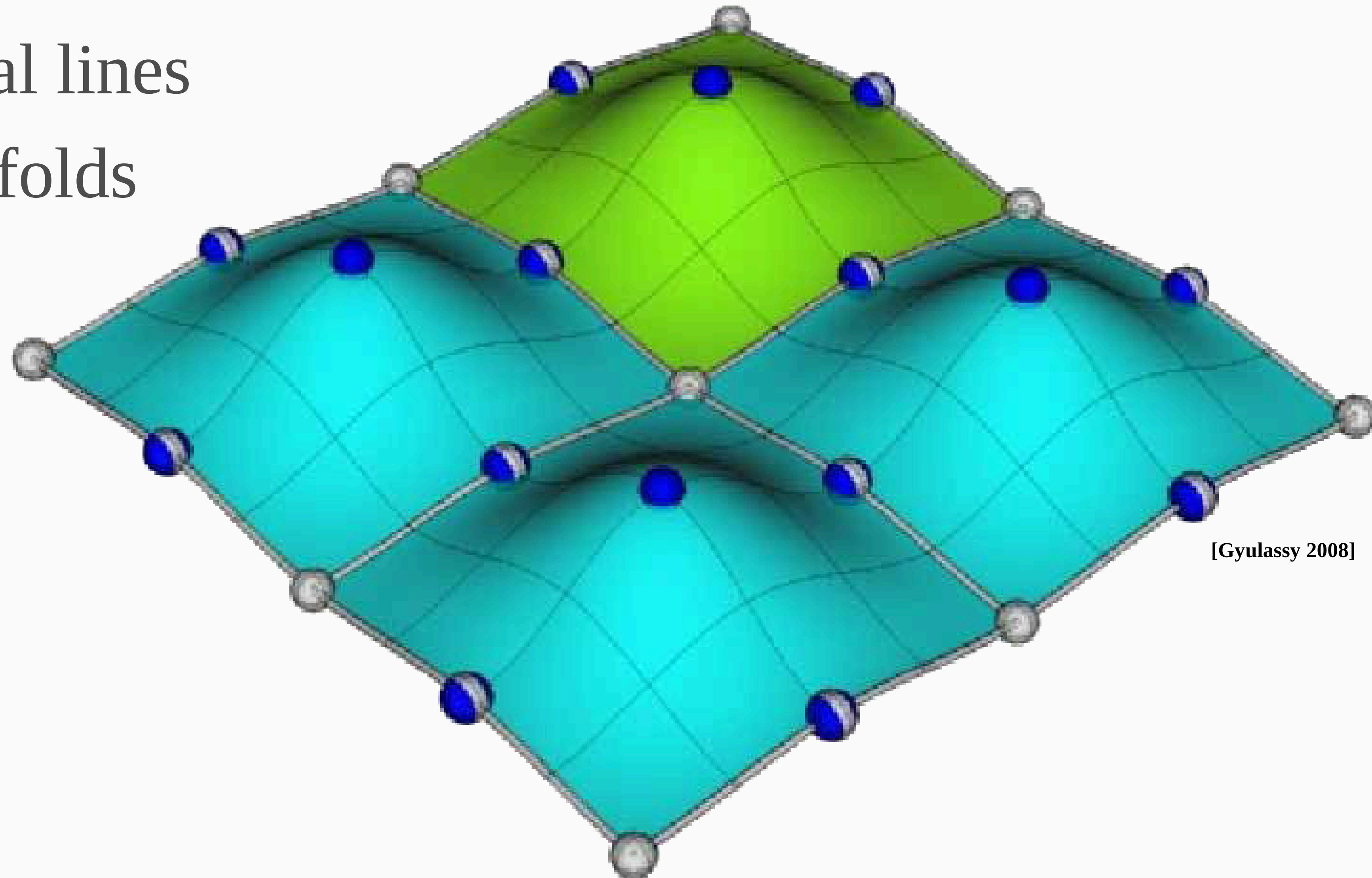
- Equivalence relation on integral lines
 - Ascending/Descending manifolds



[Gyulassy 2008]

Morse-Smale Complex

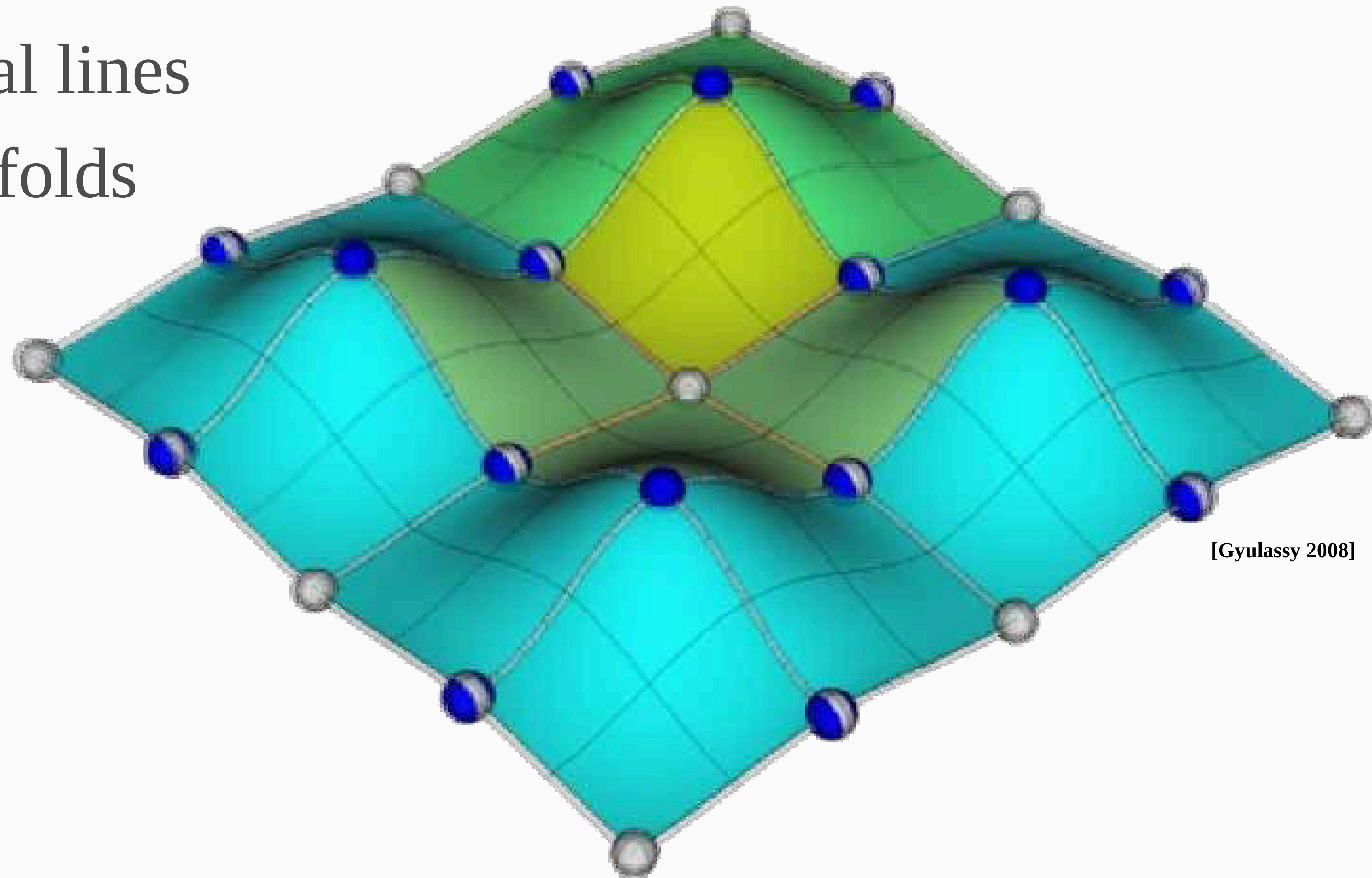
- Equivalence relation on integral lines
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[Gyulassy 2008]

Morse-Smale Complex

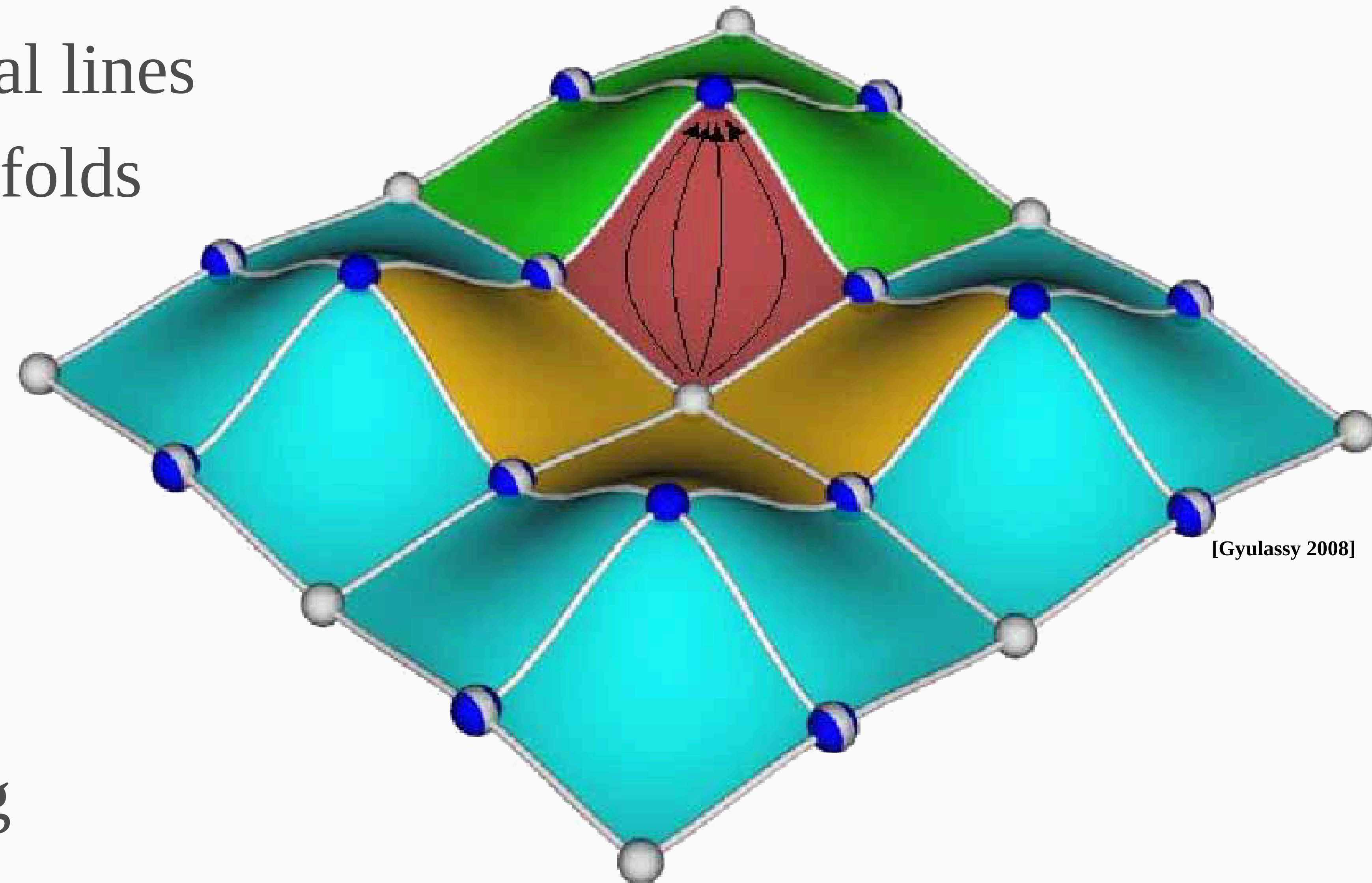
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[Gyulassy 2008]

Morse-Smale Complex

- Equivalence relation on integral lines
 - Ascending/Descending manifolds

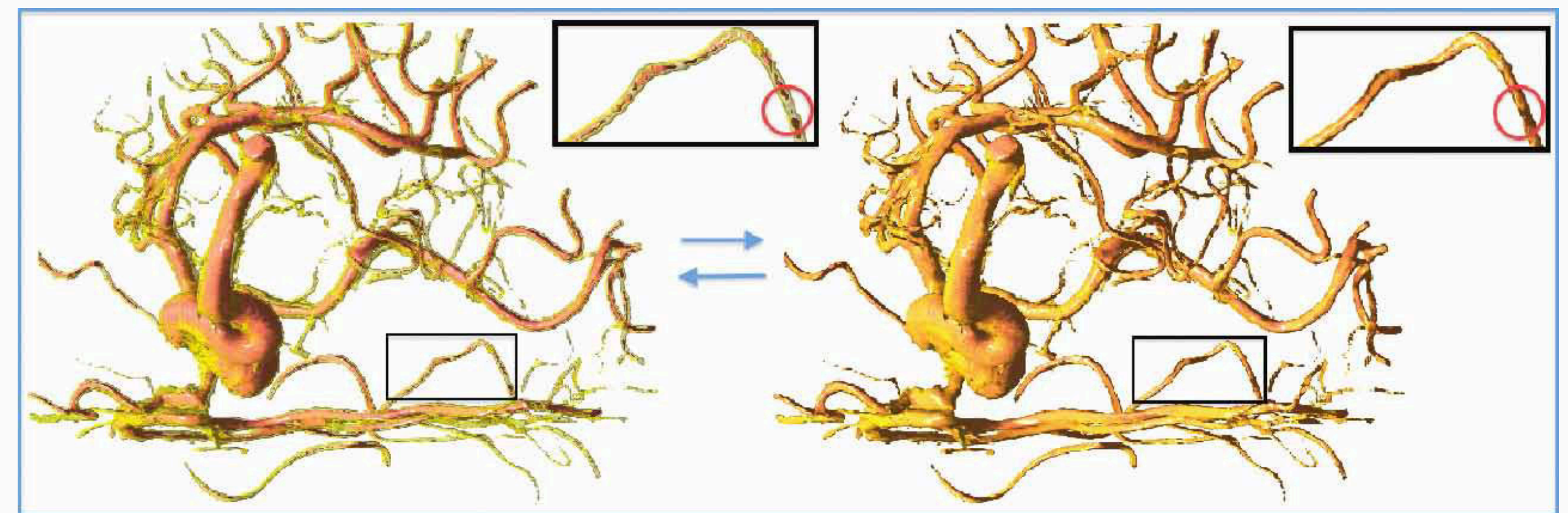
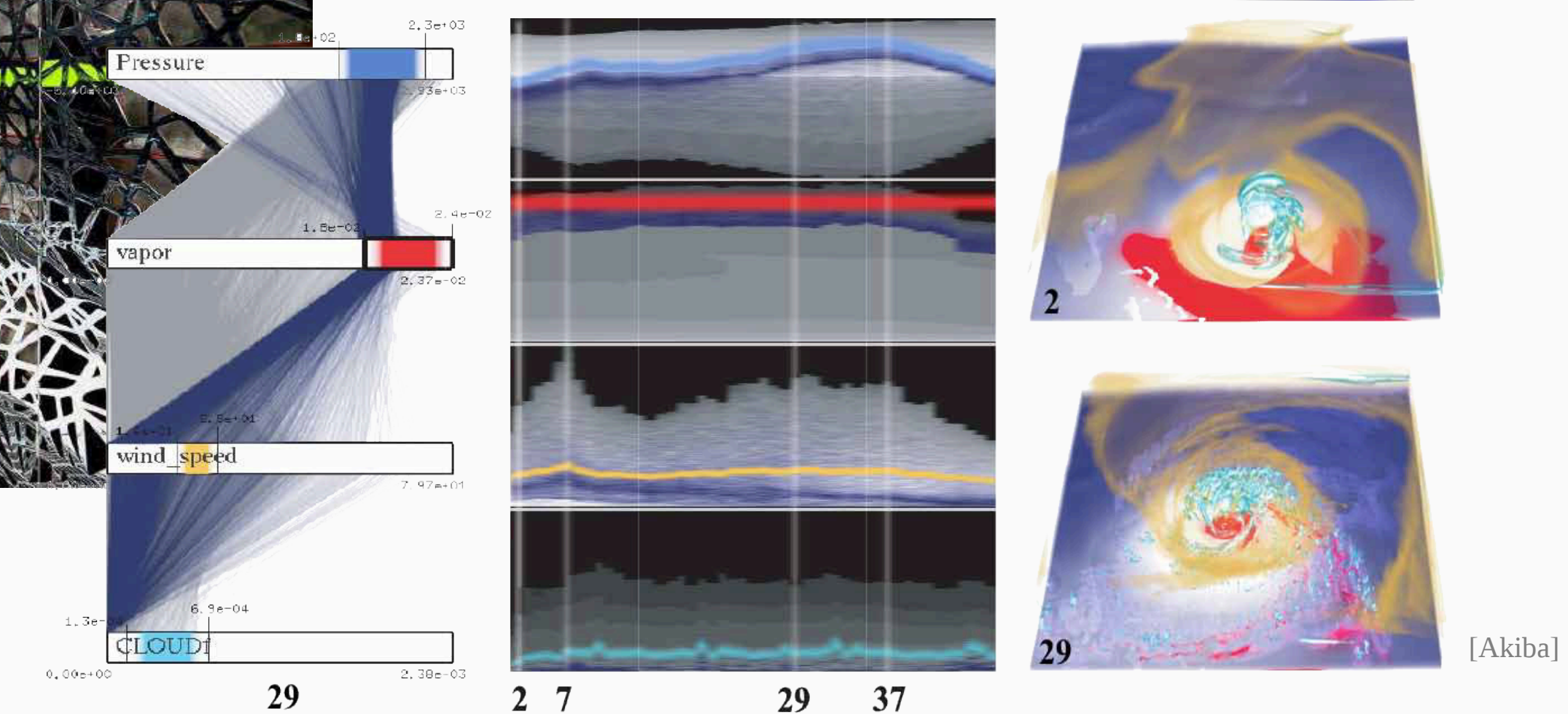
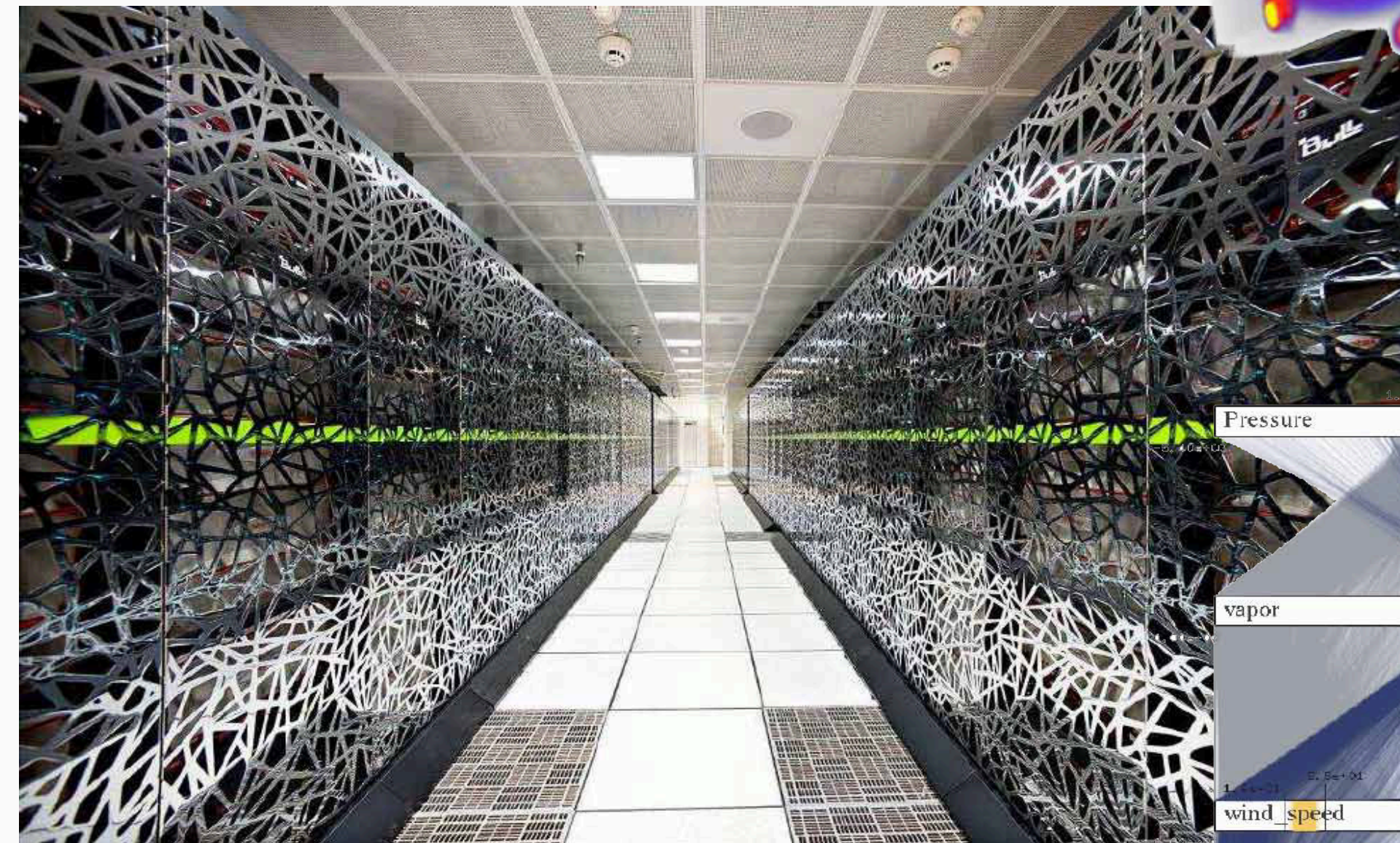
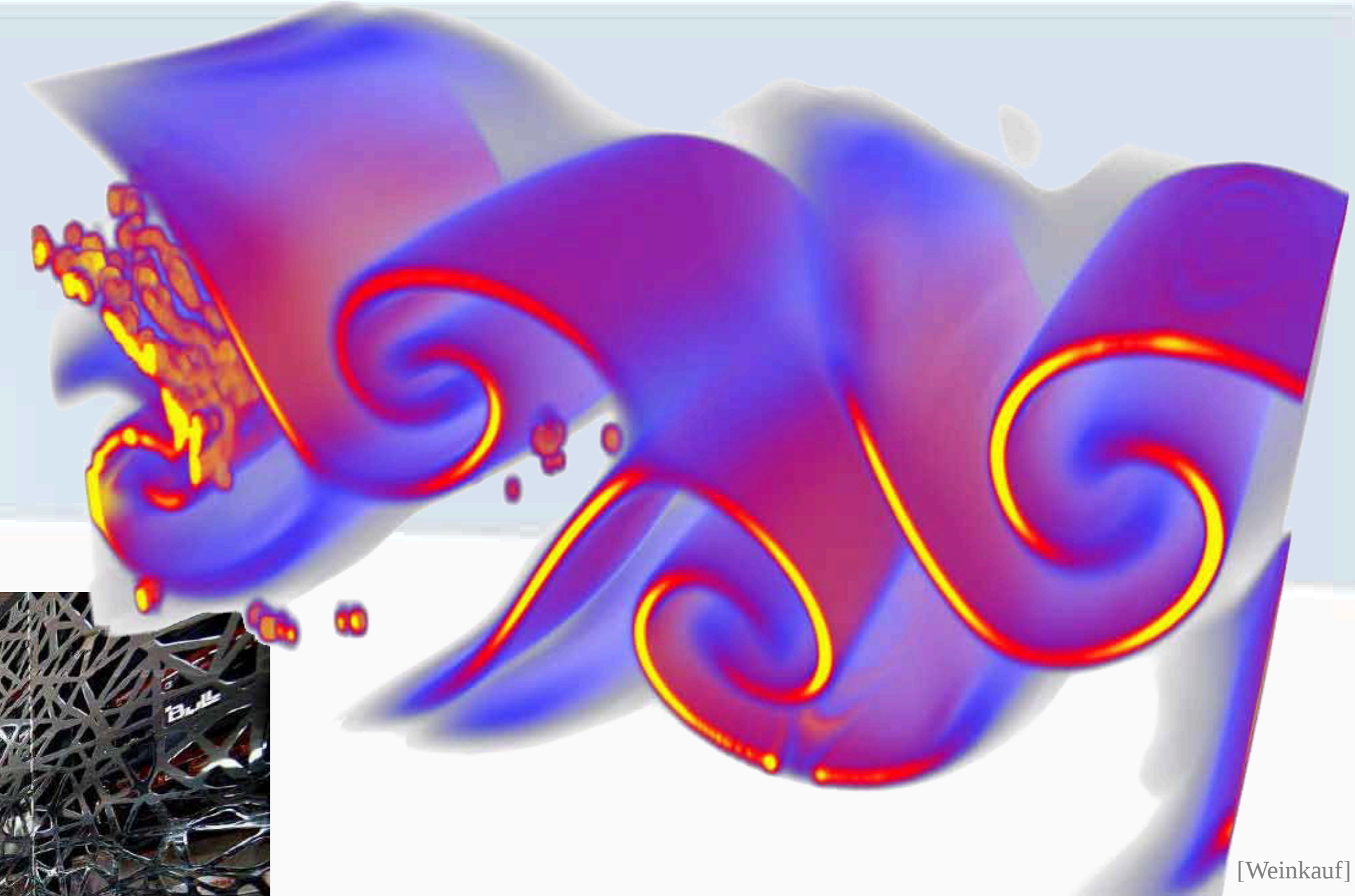


- Multi-scale representation
 - Structural abstraction
 - Data segmentation
 - Feature analysis and tracking

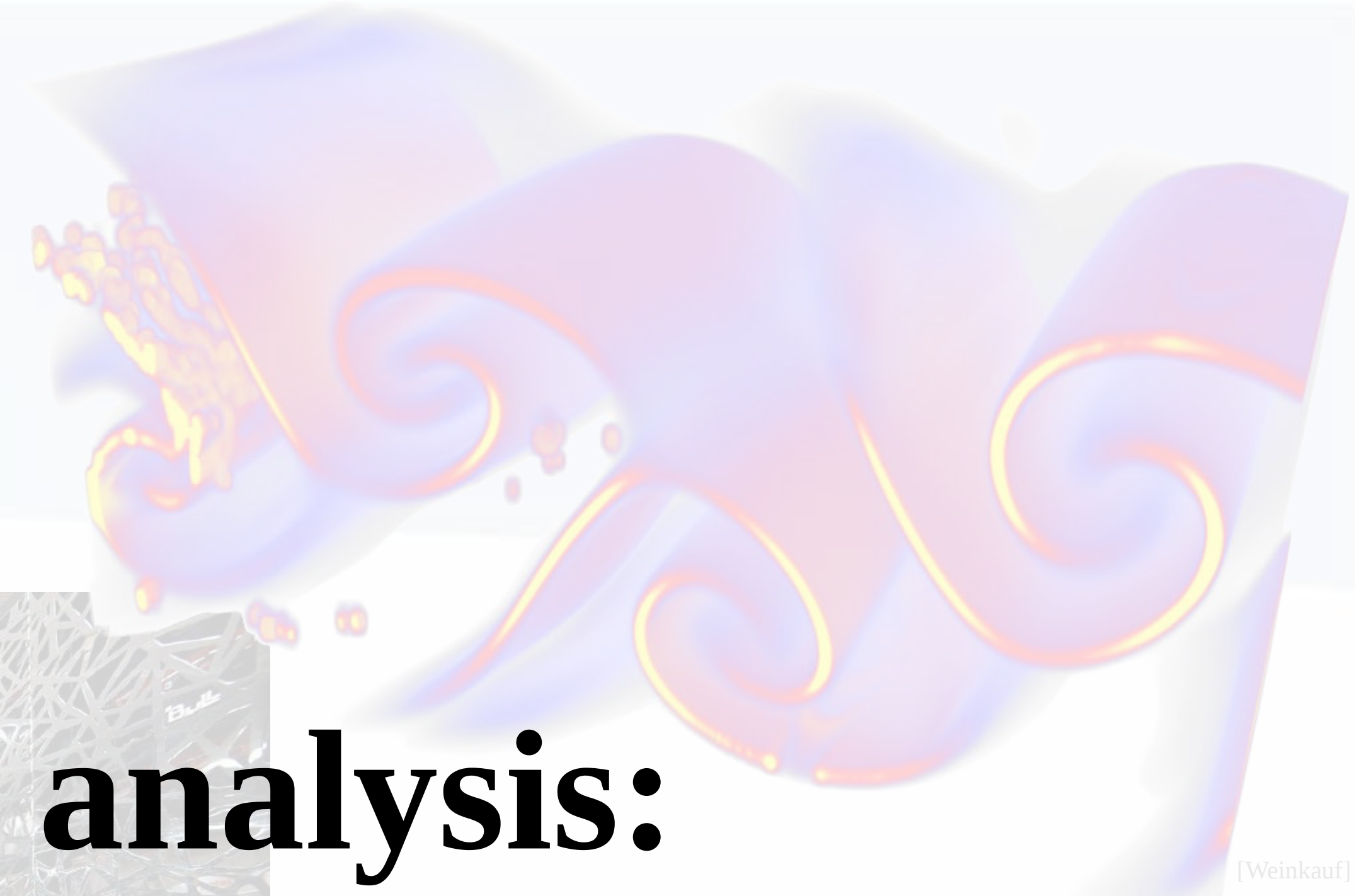
Major Scientific Challenges

- Large-scale data, in-situ Viz
- Time-varying data
- Multi-variate data
- **Uncertain data**

$$\cancel{f : \mathcal{D} \rightarrow \mathbb{R}}$$



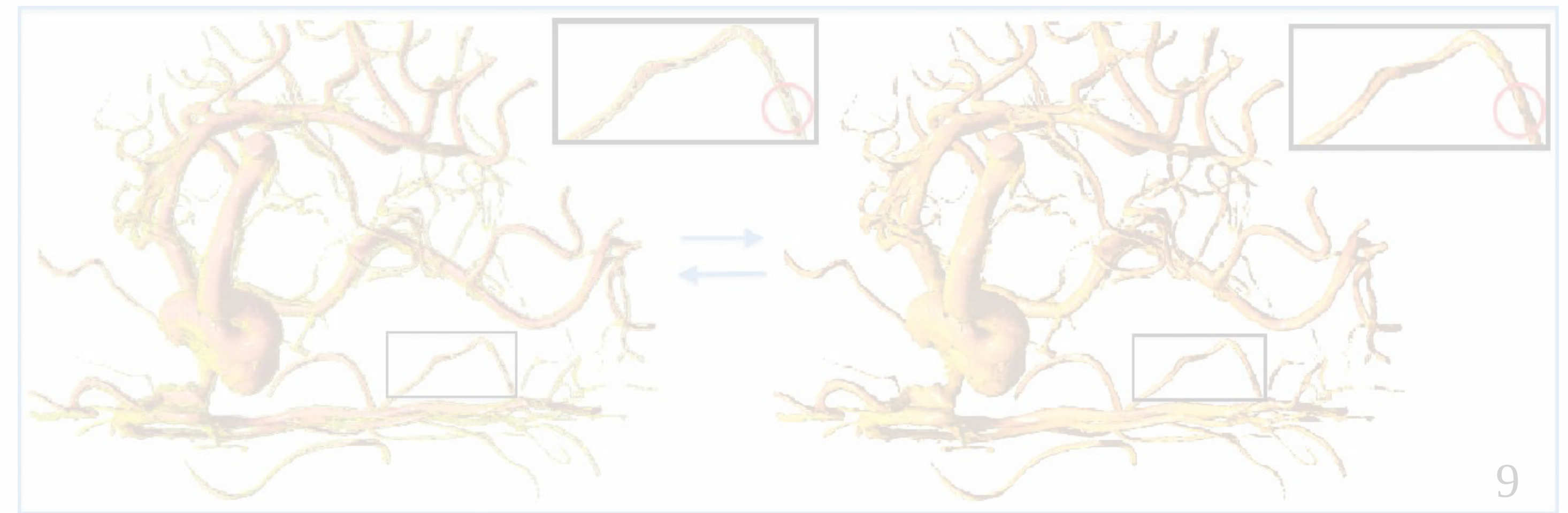
Major Scientific Challenges



**Overall, topological data analysis:
~ 20 years of research**

- Large-scale data, in situ VIZ
- Time-varying data
- Multi-variate data
- Uncertain data

... to reboot!



~~$f : \mathcal{D} \rightarrow \mathbb{R}$~~

Uncertainty in Scientific Data



- Omnipresent in Science
 - Acquisition imprecision
 - Numerical error propagation
 - Parameter studies

$$f_n : \mathcal{D} \rightarrow \mathbb{R}$$

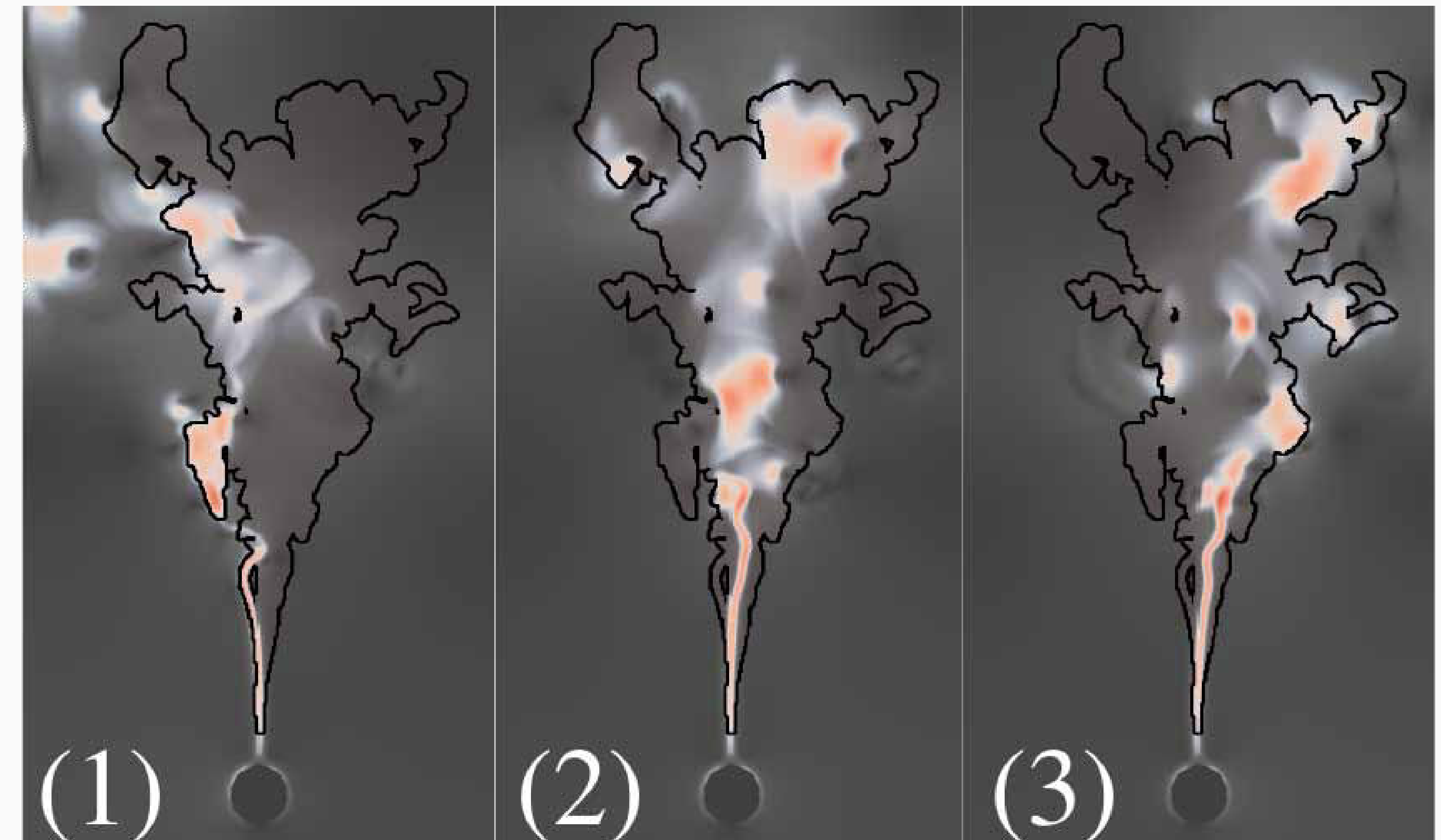
⋮

$$f_1 : \mathcal{D} \rightarrow \mathbb{R}$$

$$f_0 : \mathcal{D} \rightarrow \mathbb{R}$$

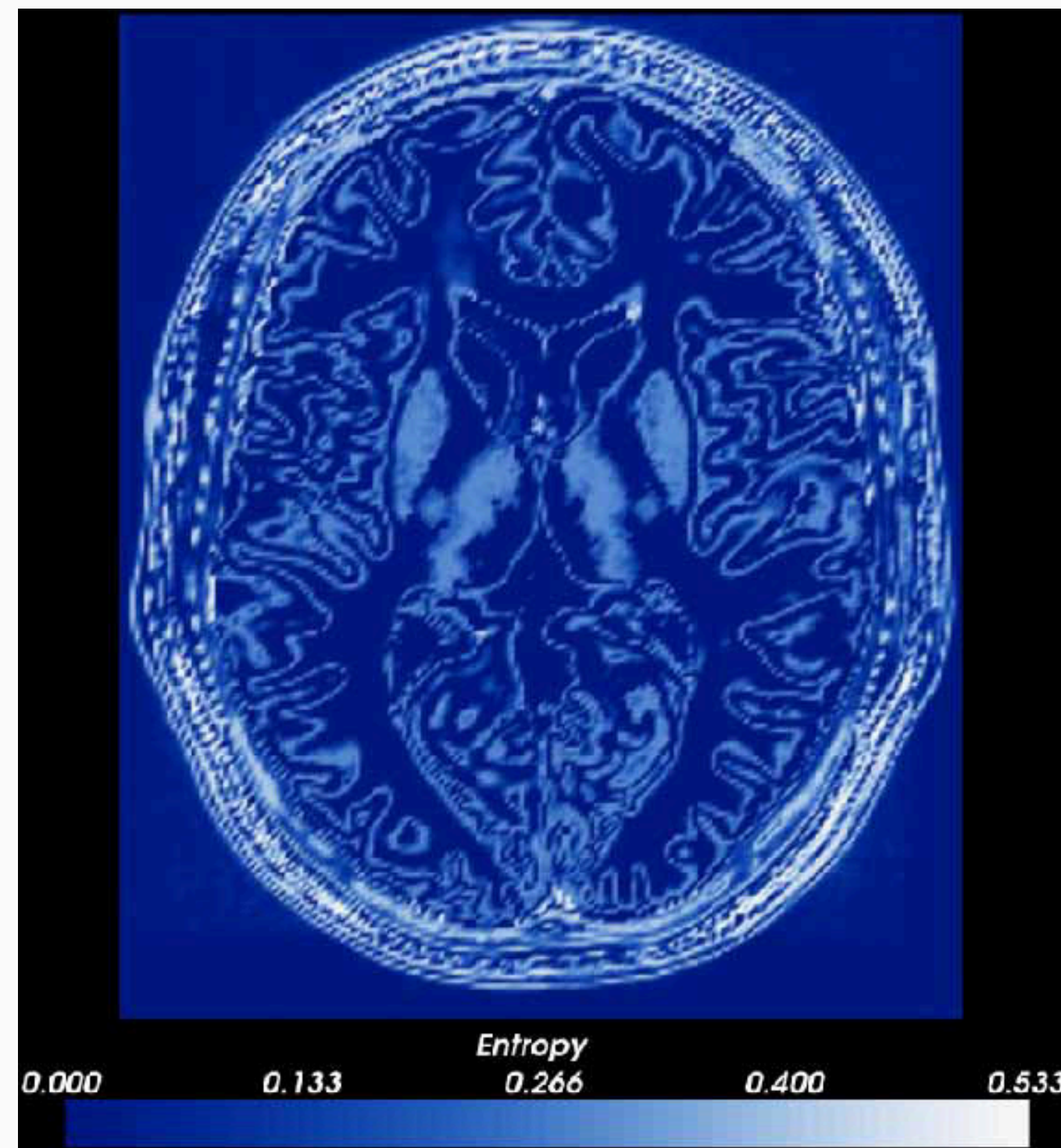
Visualization Challenges

- Well studied in Scientific Computing
 - Visualization?
- Uncertainty insights
 - Value uncertainty
 - Positional uncertainty
- Data insights
 - Assessments despite the uncertainty
 - Infinitely-many...
 - Uncertain data-set **as a whole**

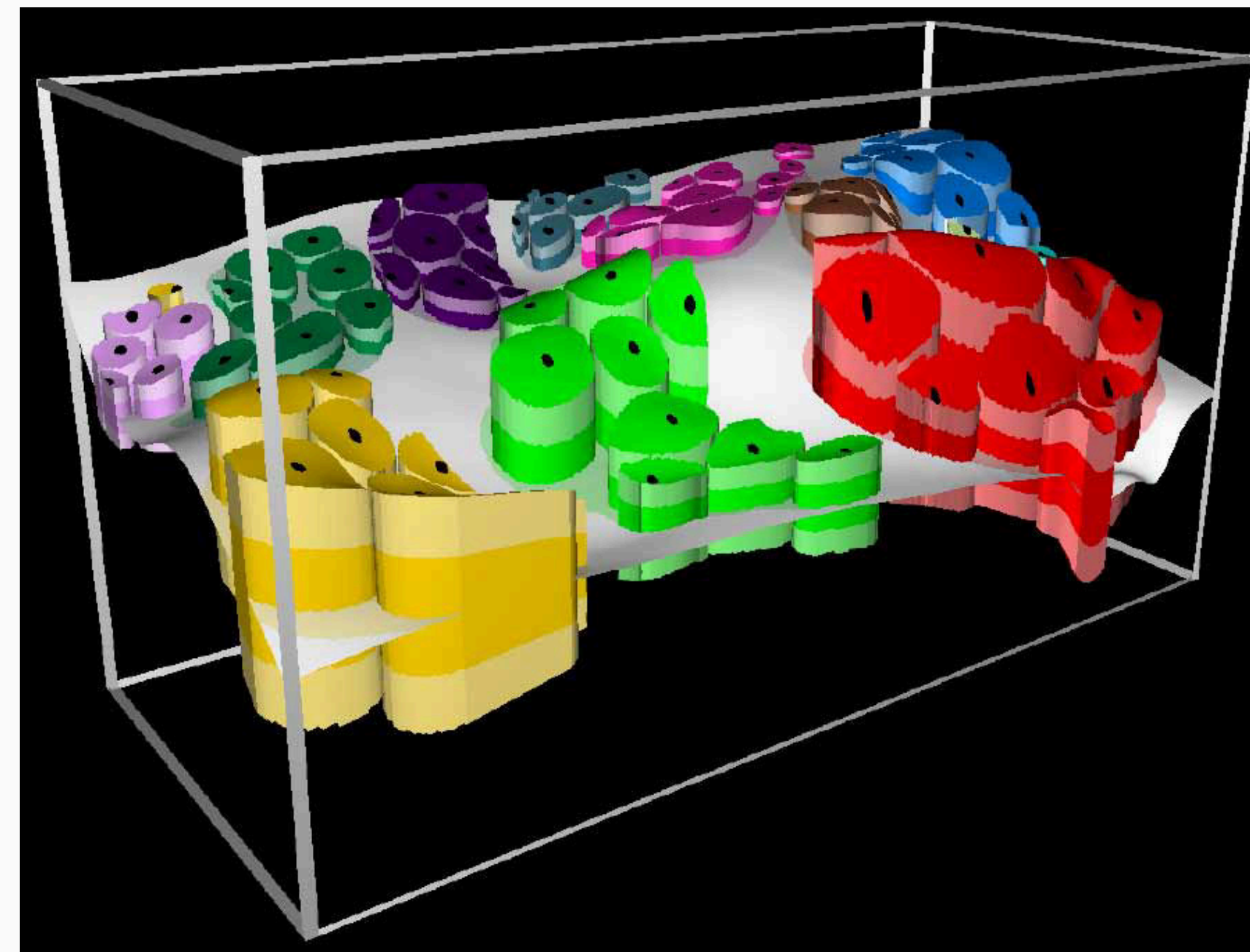


Uncertainty in Visualization

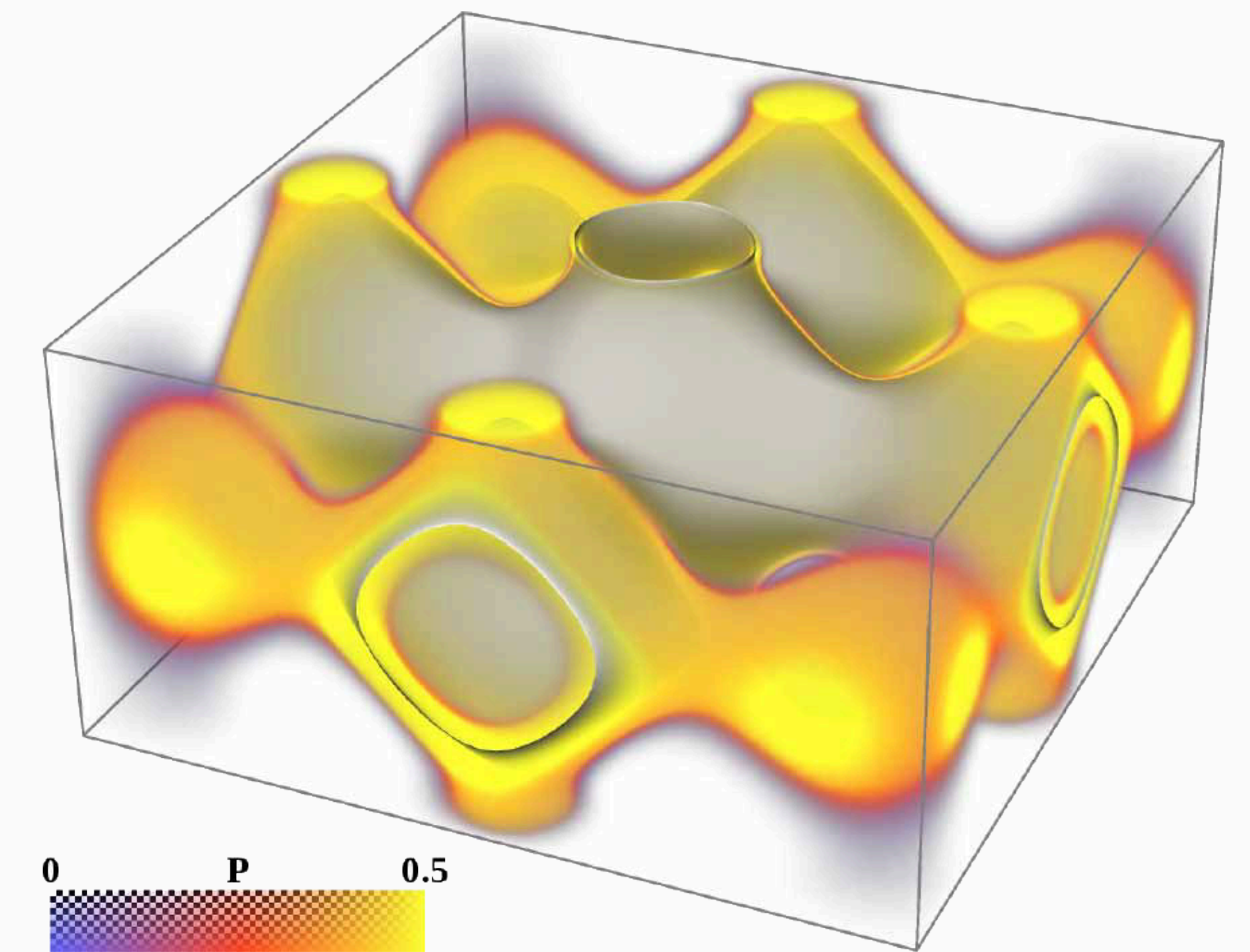
- Qualitative evaluation of positional uncertainty



[Potter et al., IEEE Comp. Graph. App. 2013]



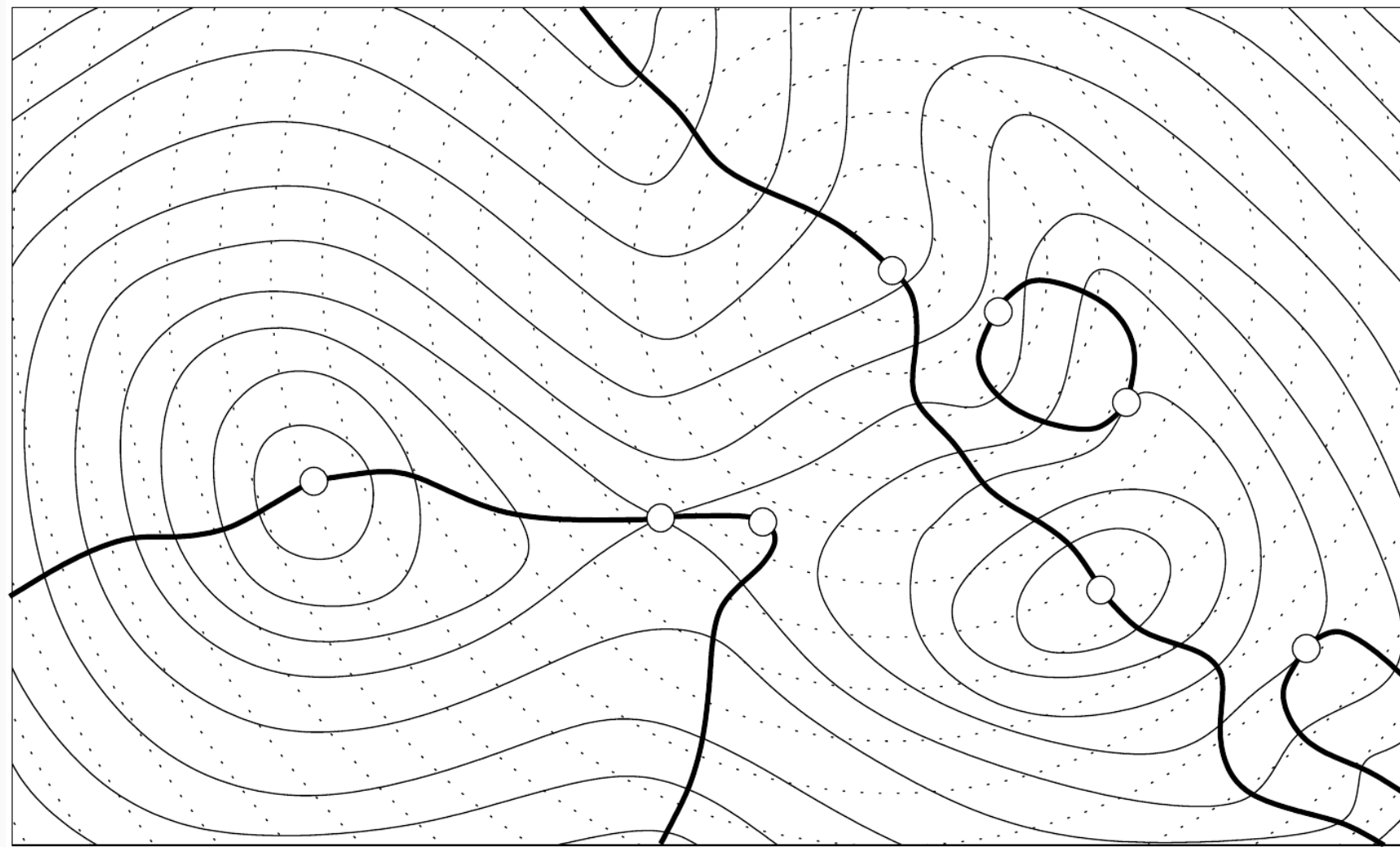
[Pfaffmoser et al., Comp. Graph. For. 2012]



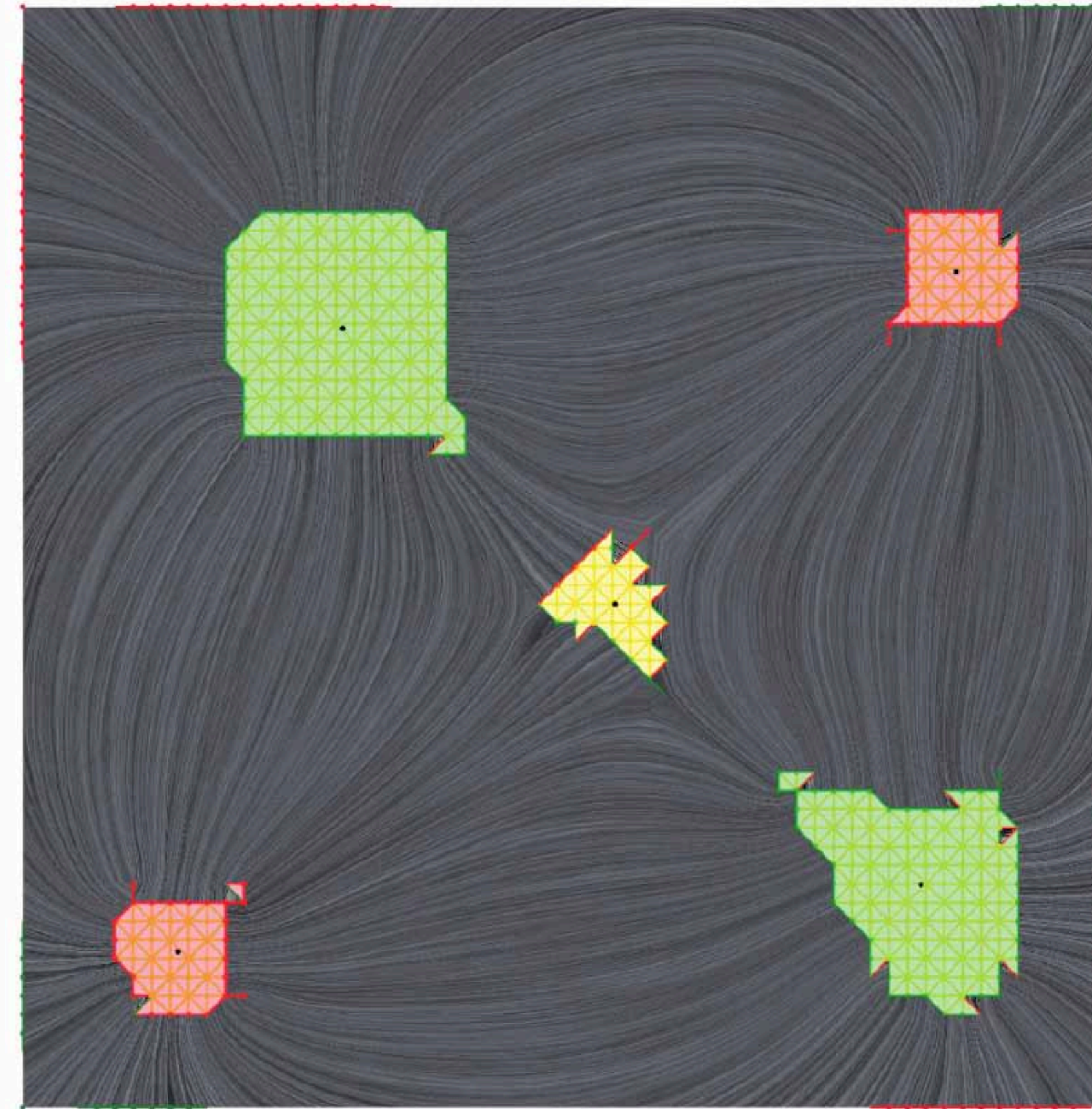
[Poethkow et al., Comp. Graph. For. 2011]

Topological Techniques for Uncertainty

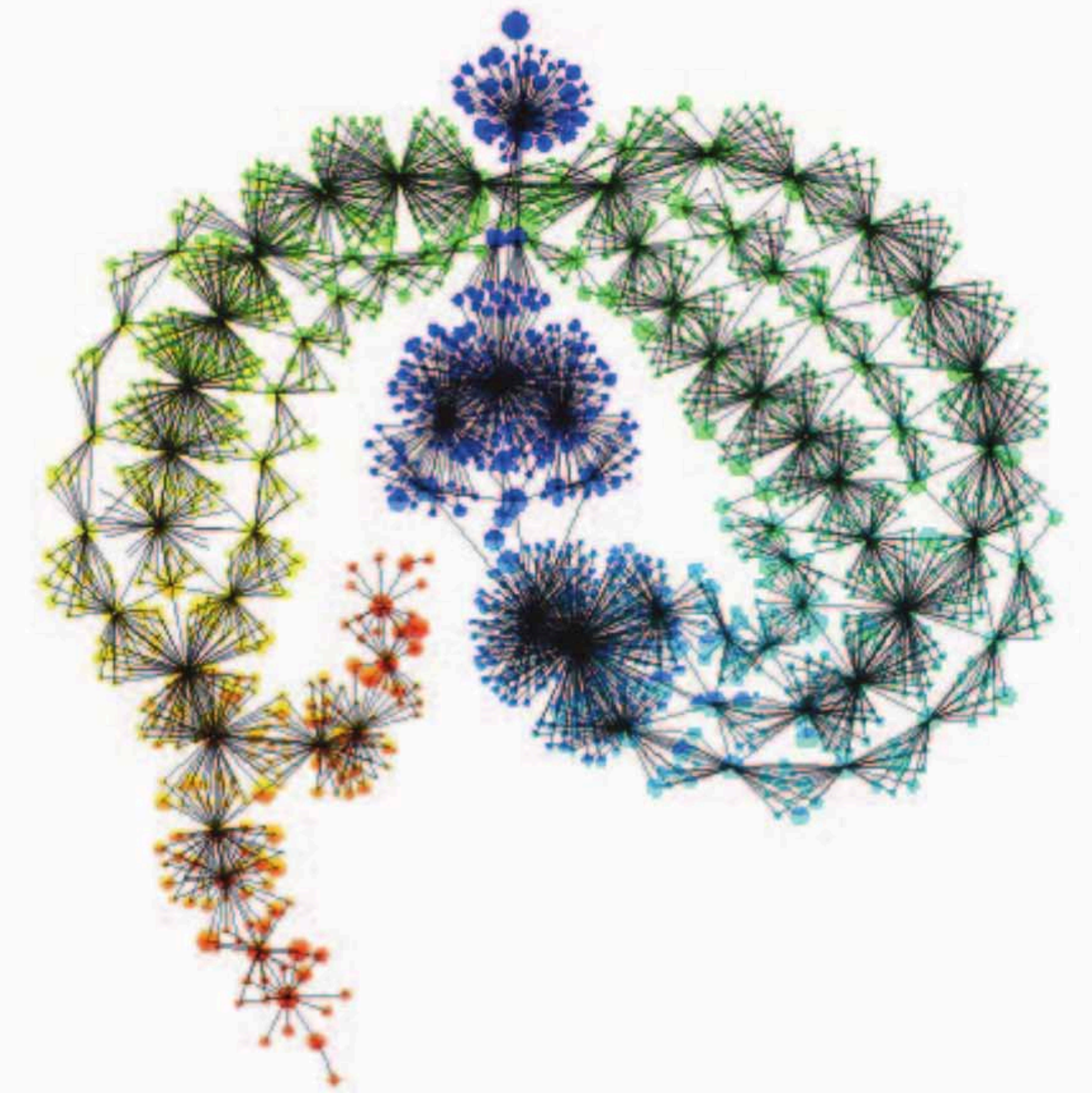
- Multiple scalar fields



[Hedelsbrunner04]



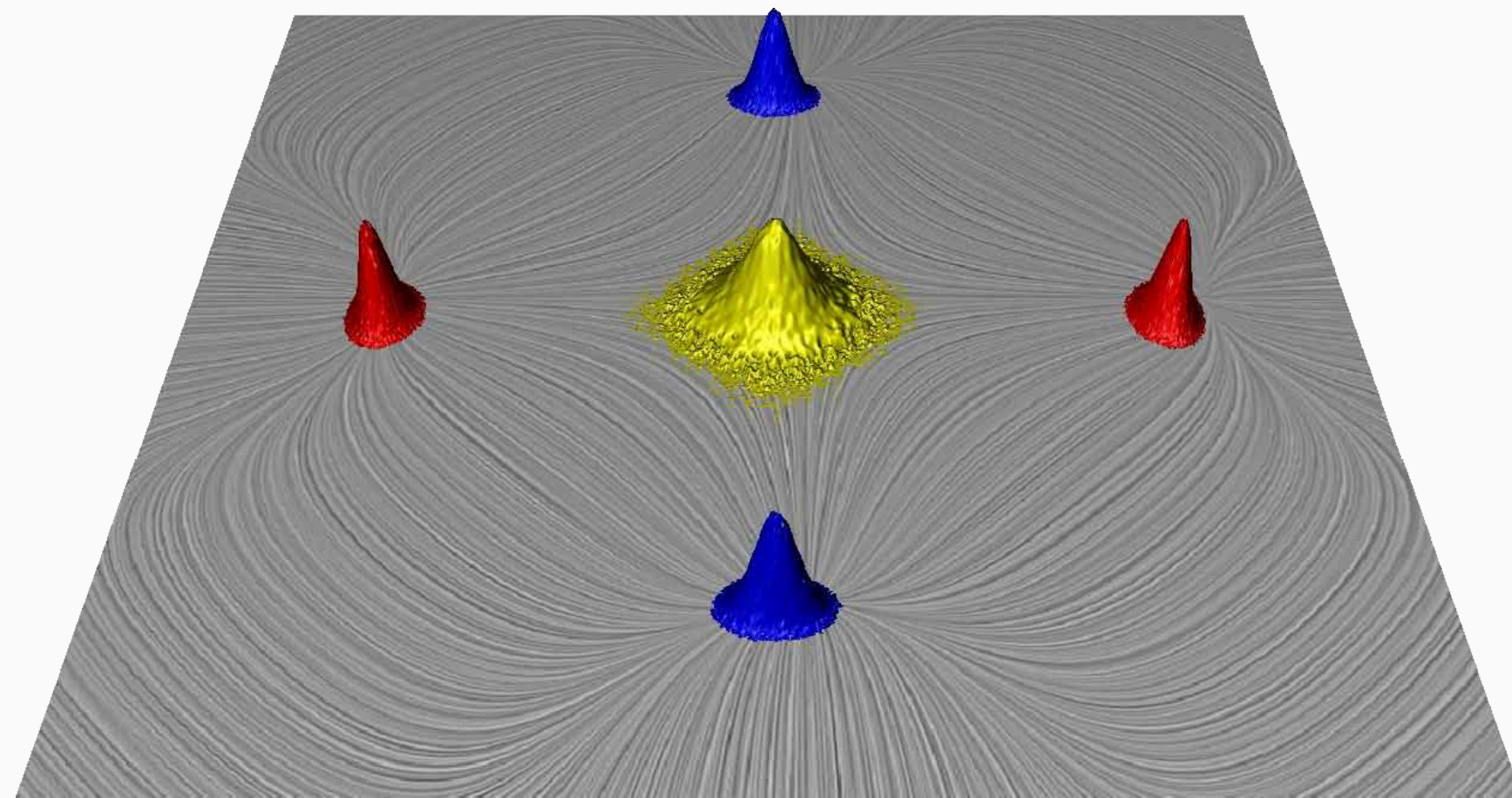
[Huttenberger13]



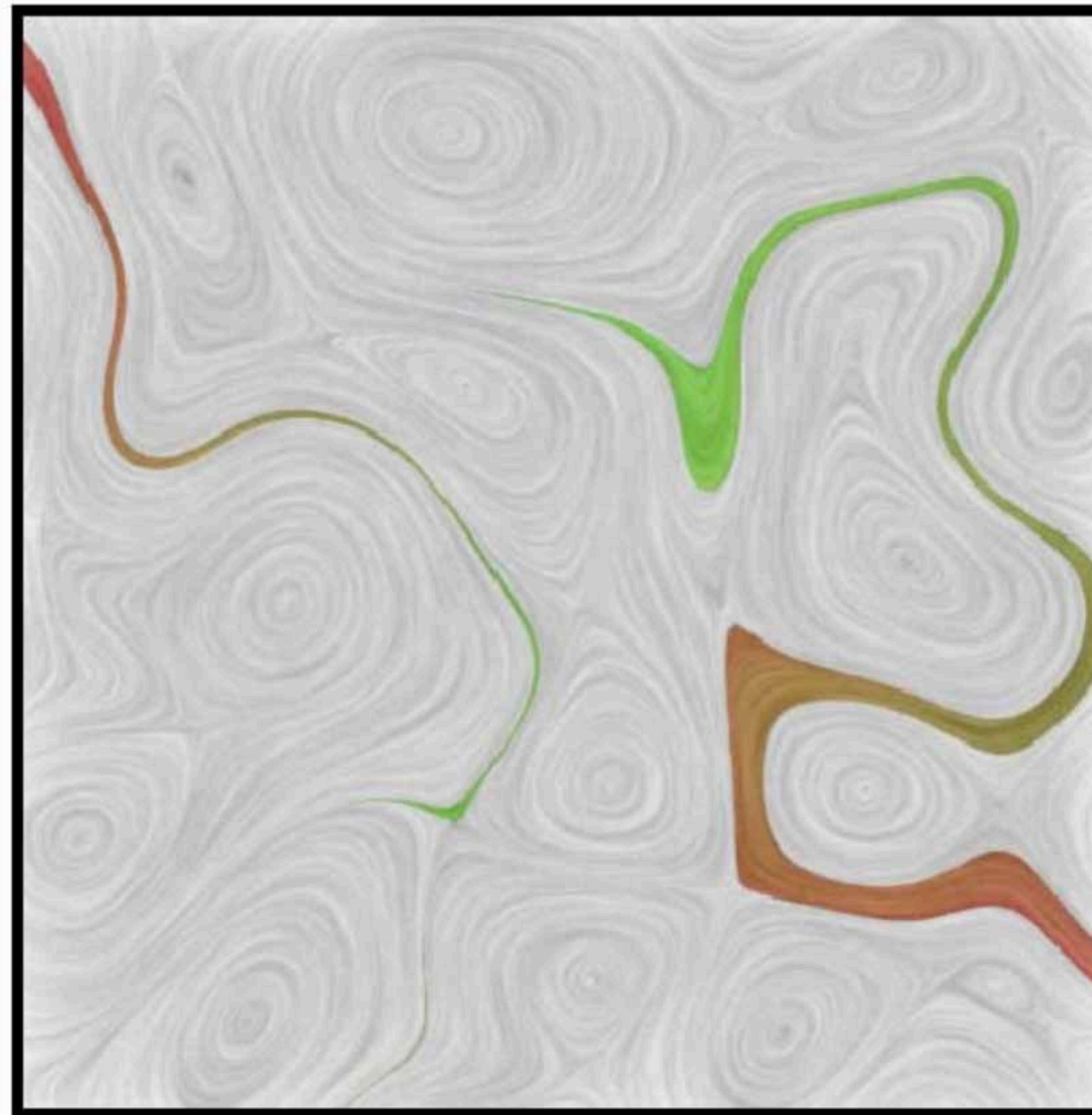
[Carr13]

Topological Techniques for Uncertainty

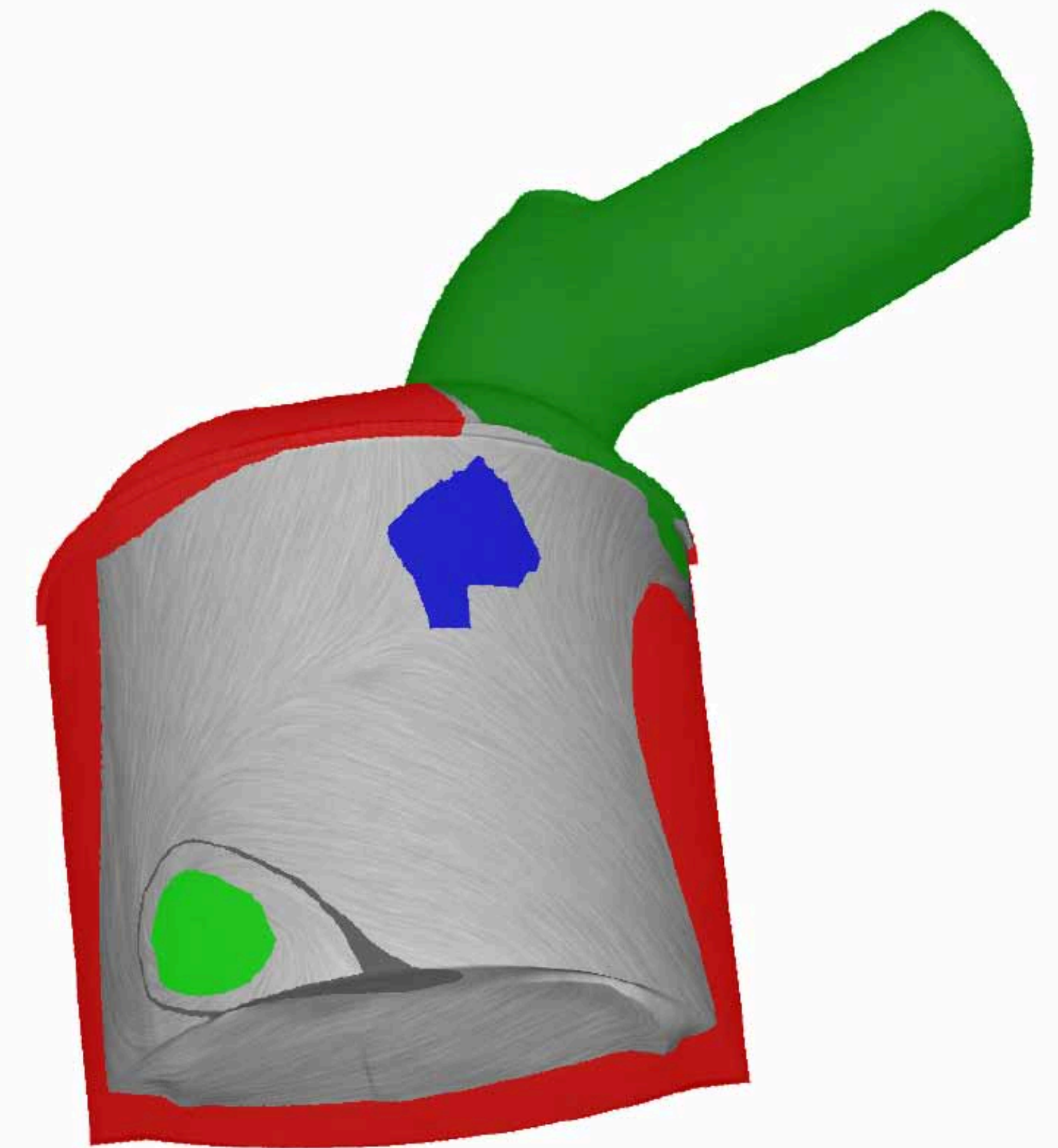
- Vector fields with errors



[Otto10]



[Bhatia12]



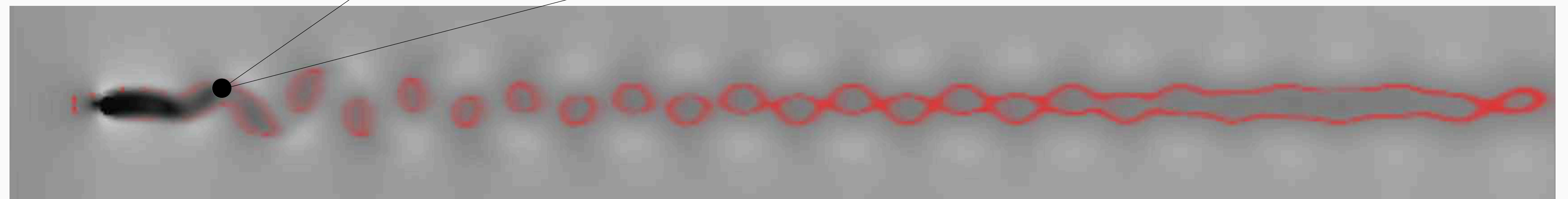
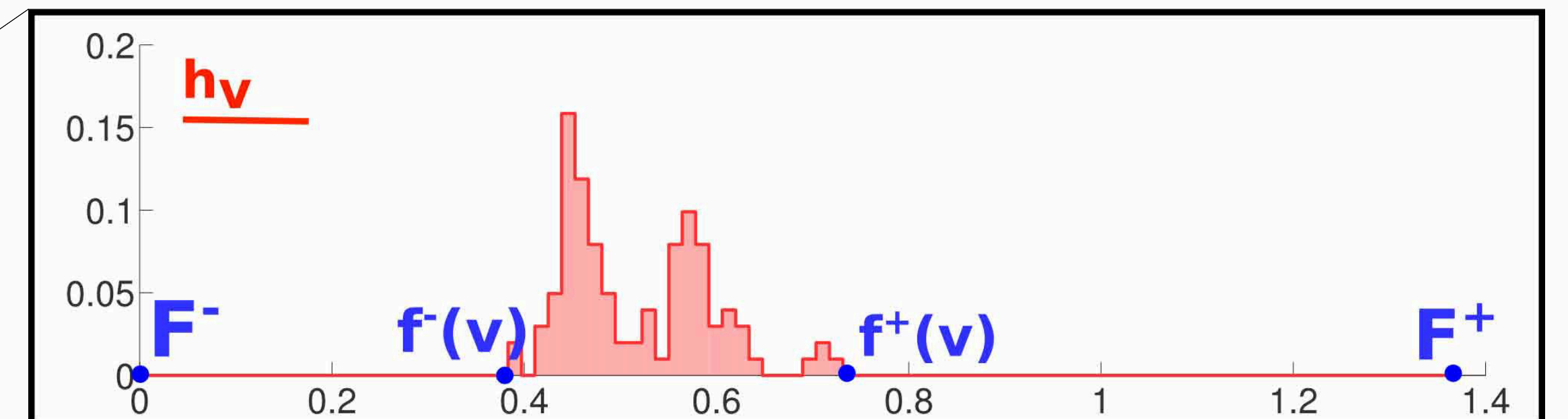
[Szymczak13]

Contributions

- A non-local characterization of *mandatory* critical points;
- A combinatorial algorithm for their extraction;
- A tree representation capturing their global relation;
- A mandatory critical point pair simplification scheme.

Data Representation

- Triangulated domain: $\mathcal{S} \subset \mathbb{R}^2$
- Uncertain Scalar Field F
 - Random Variable $F(v) \quad \forall v \in \mathcal{S}$
 - Compact support $f^- : \mathcal{S} \rightarrow \mathbb{R}, f^+ : \mathcal{S} \rightarrow \mathbb{R}$
- Realization Field $g : \mathcal{S} \rightarrow \mathbb{R}$

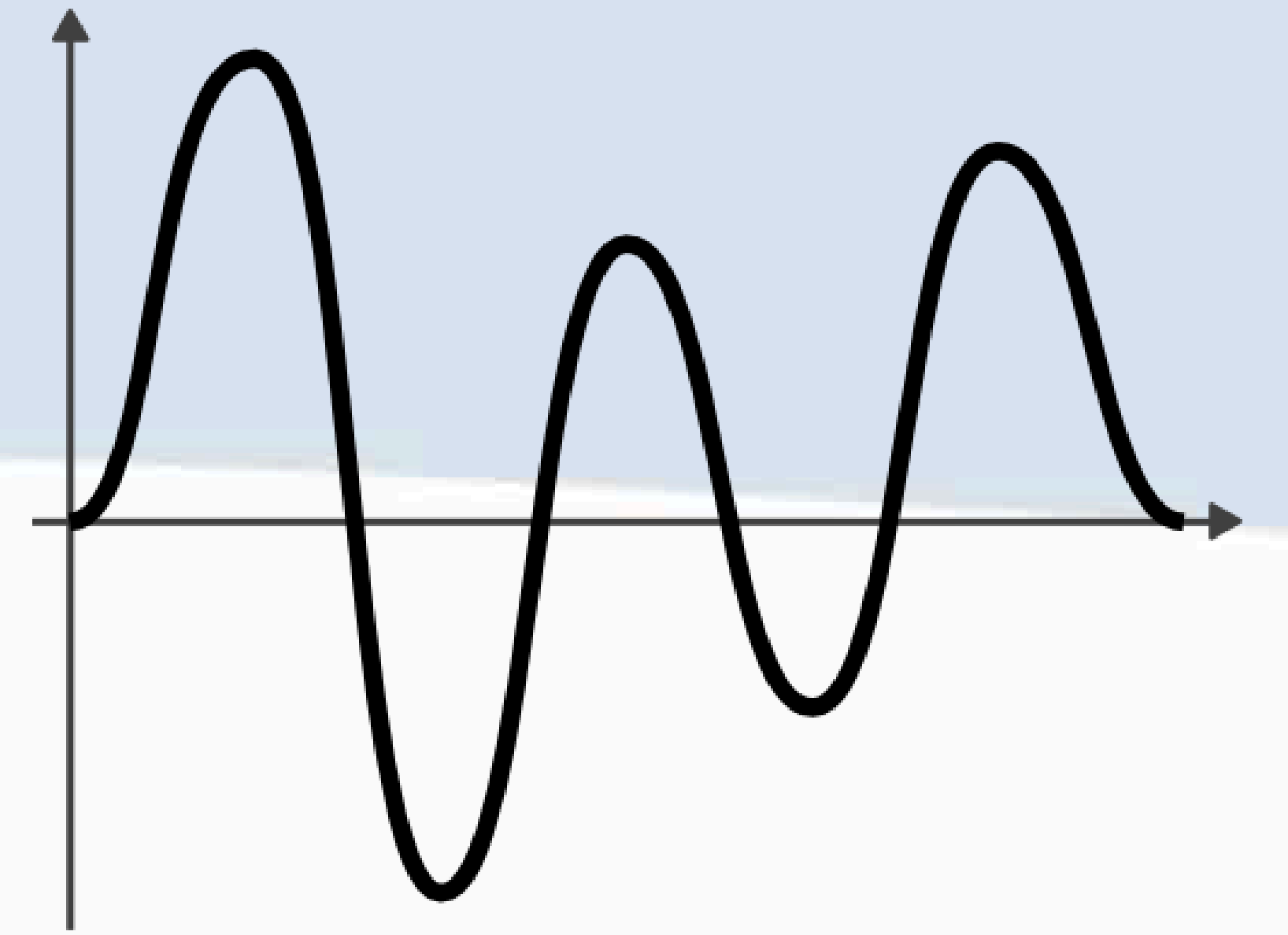


Critical Points of a Realization

- Sub-level set: $L_g^-(i) = \{p \in \mathcal{S} \mid g(p) \leq i\}$

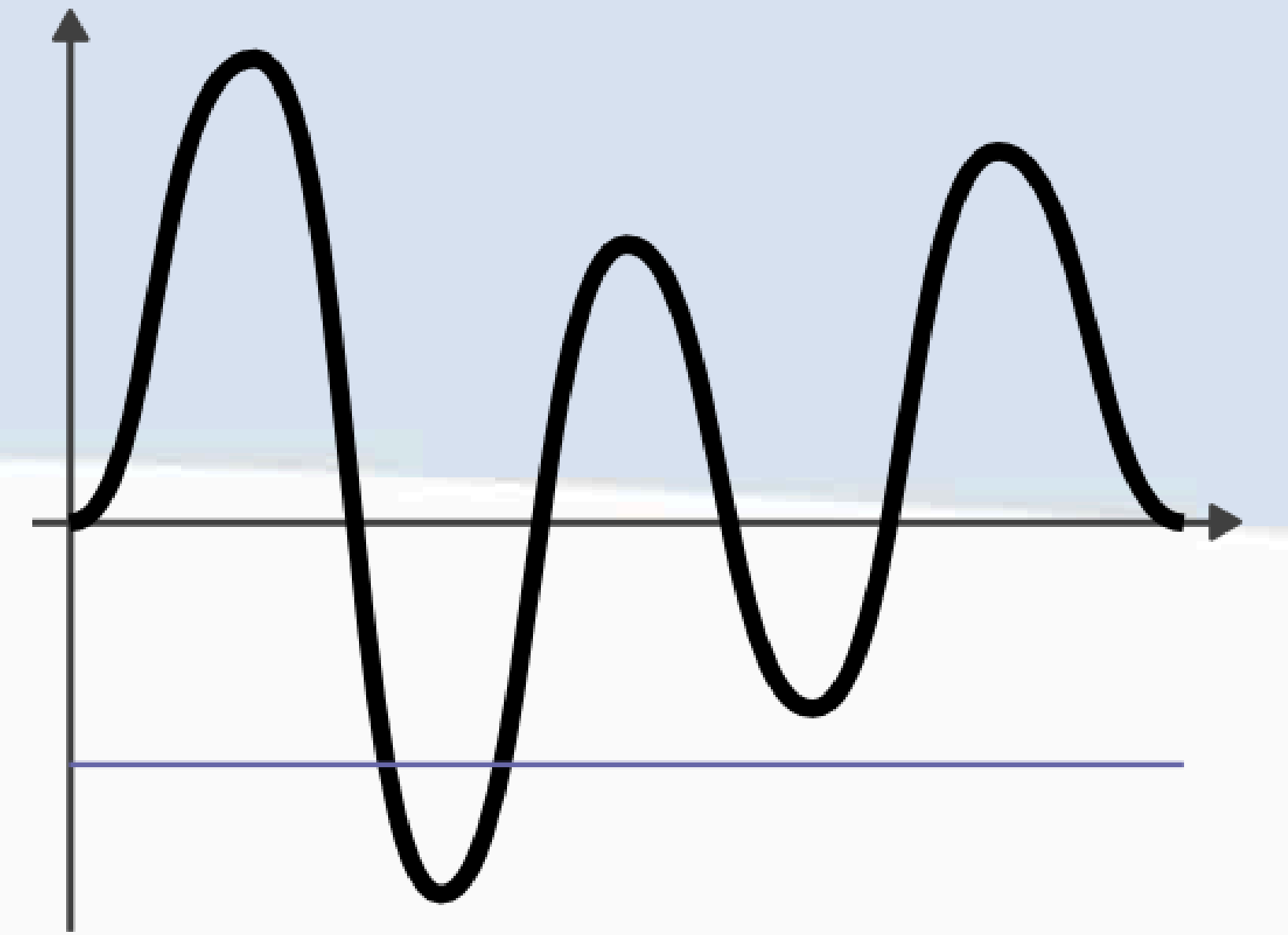
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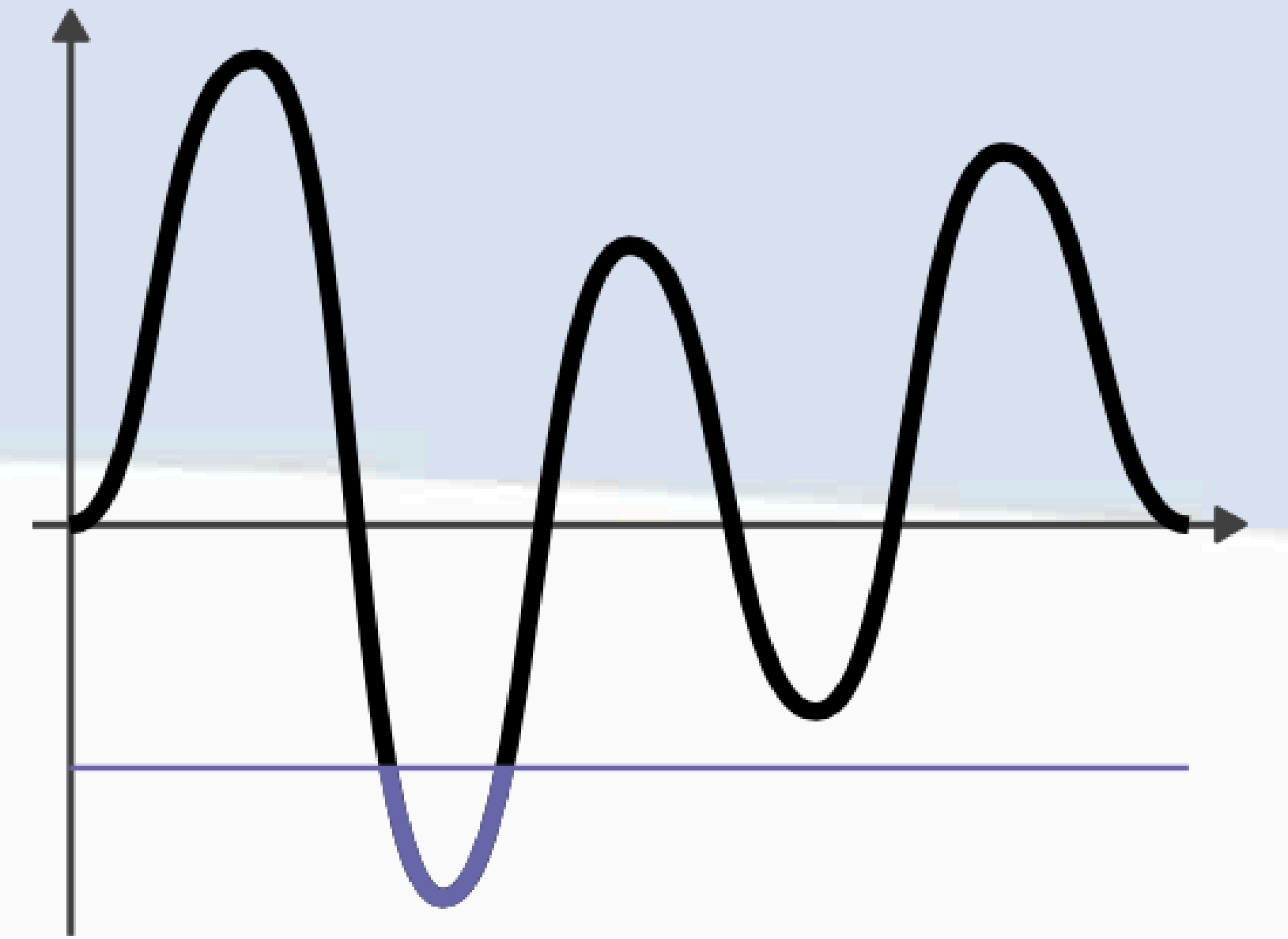
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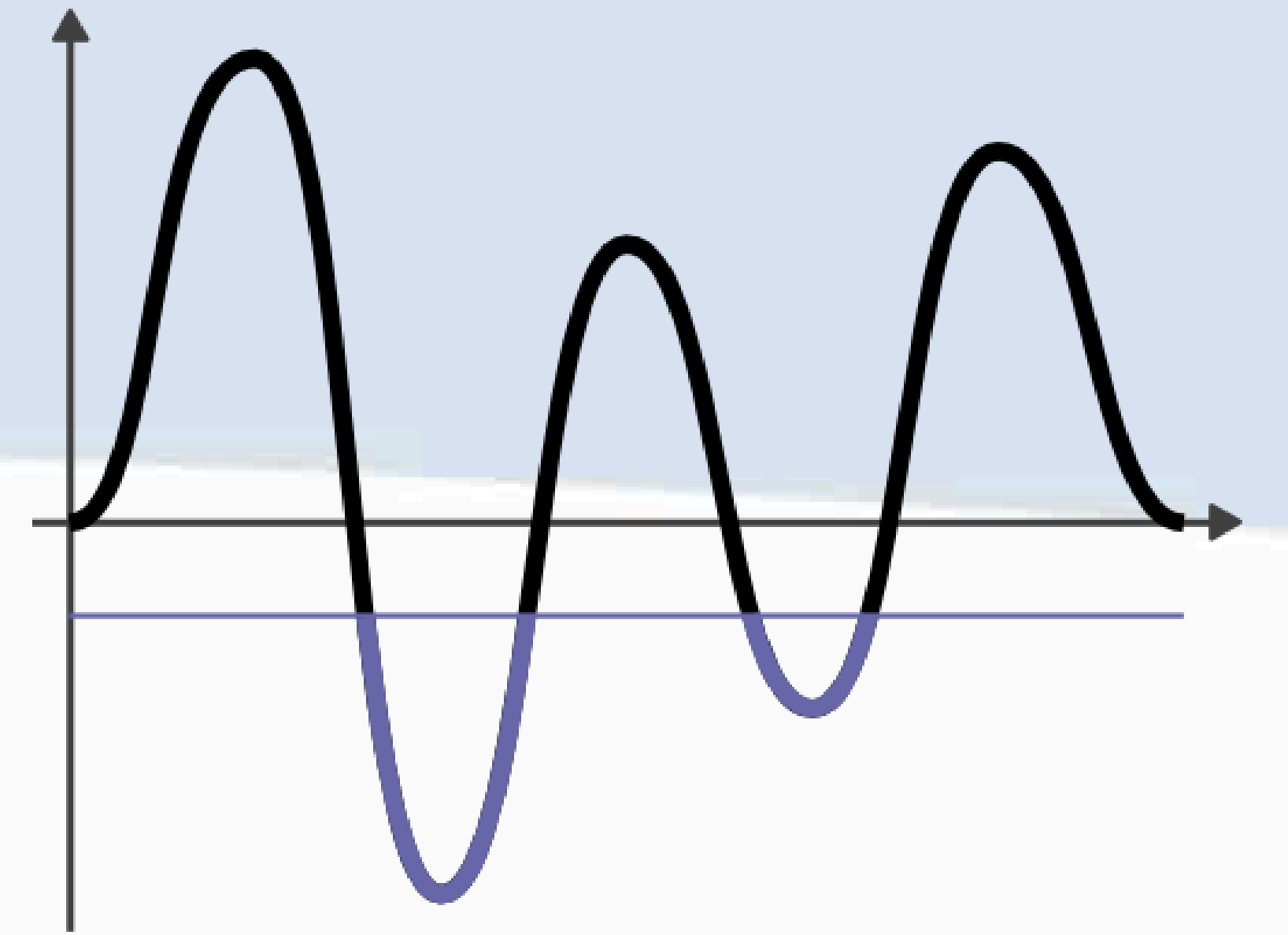
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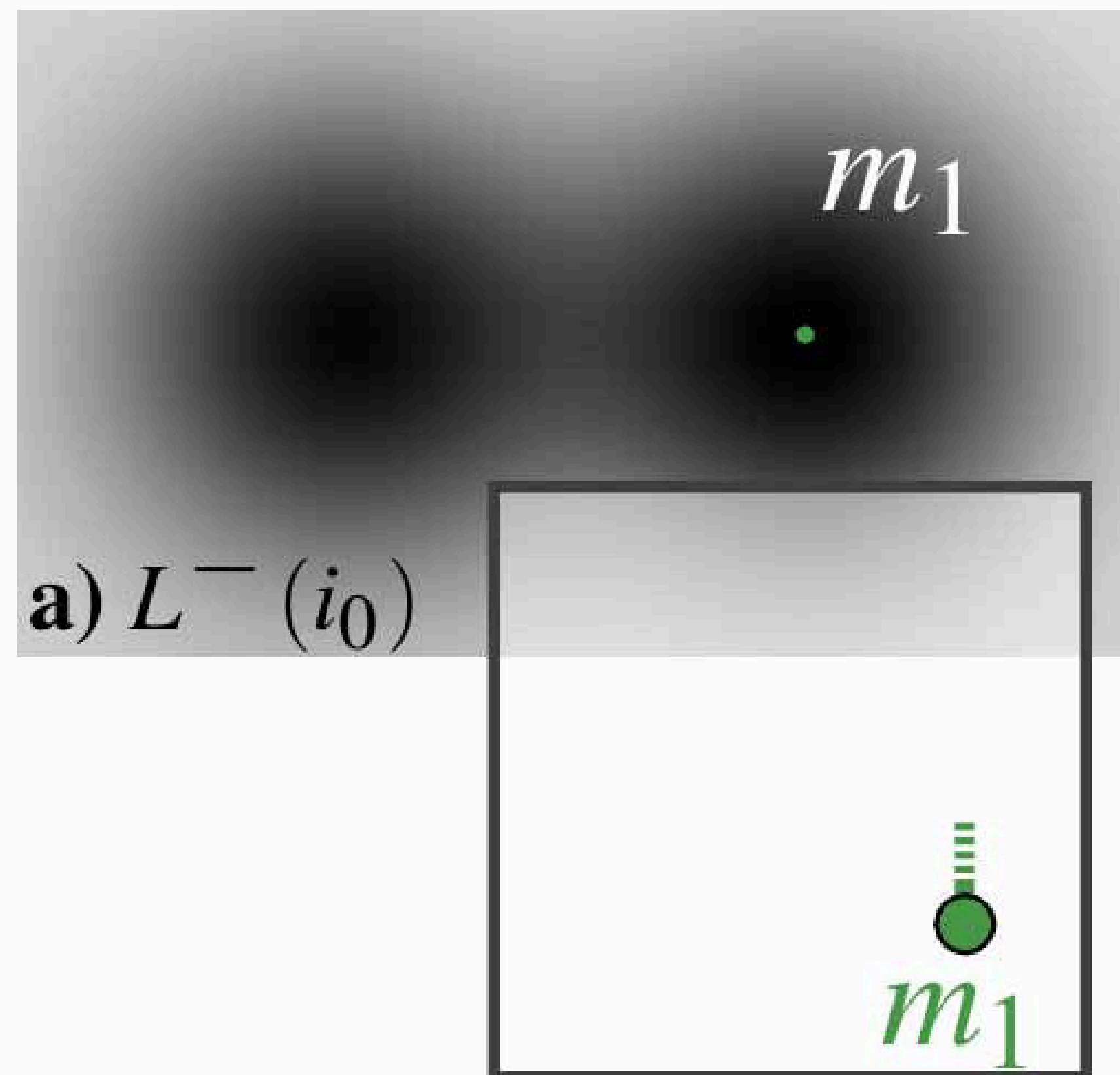
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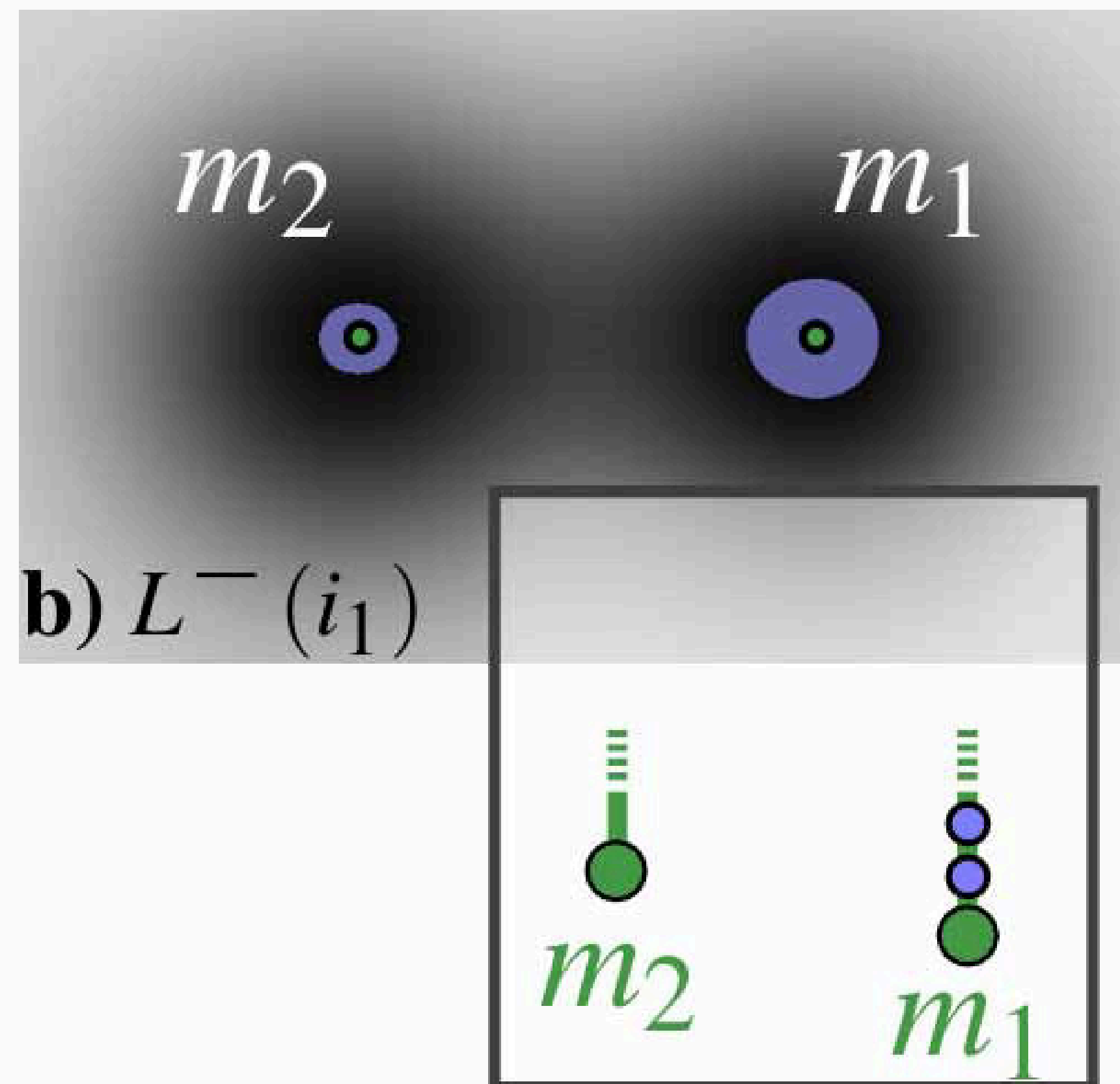
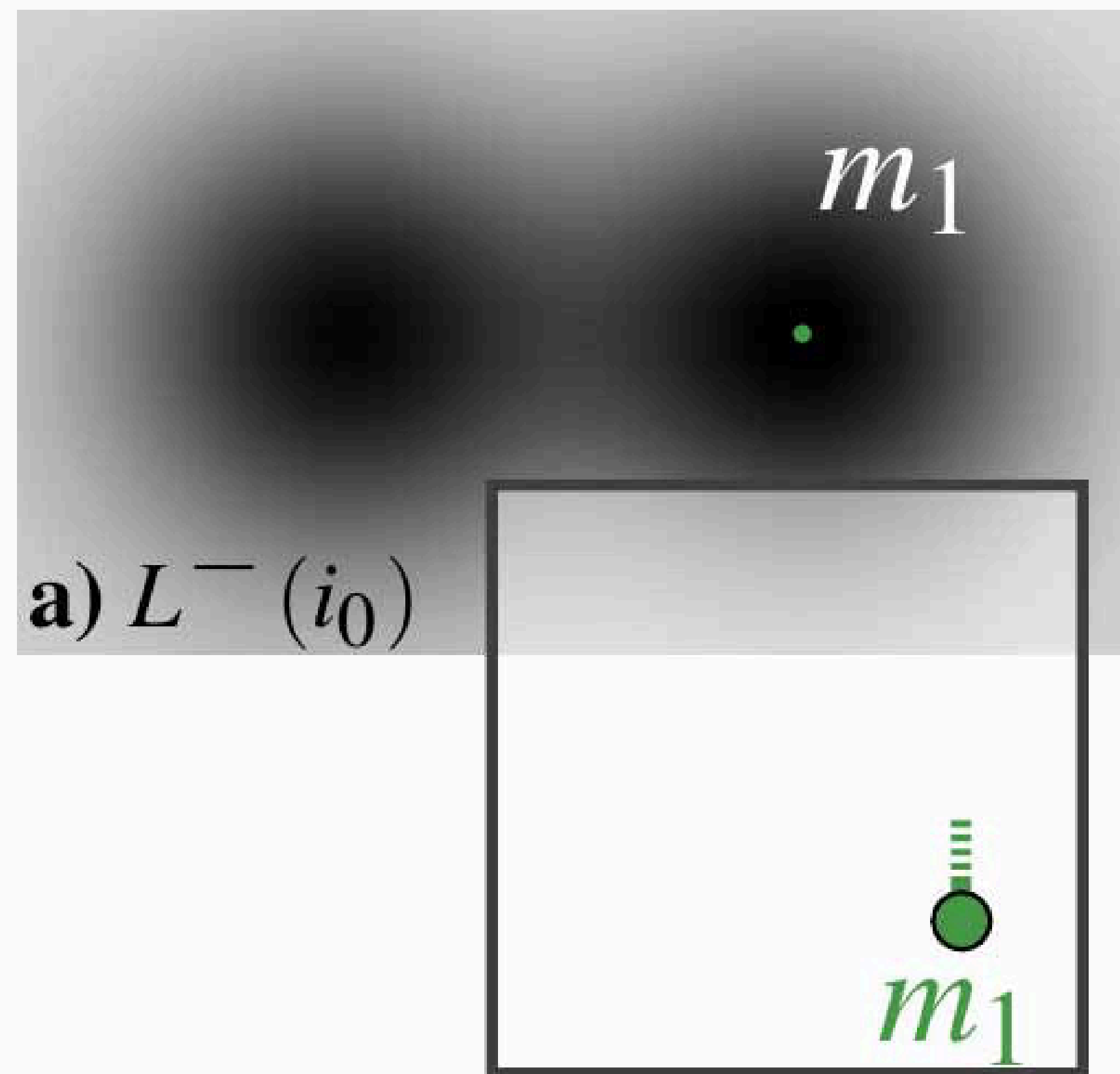
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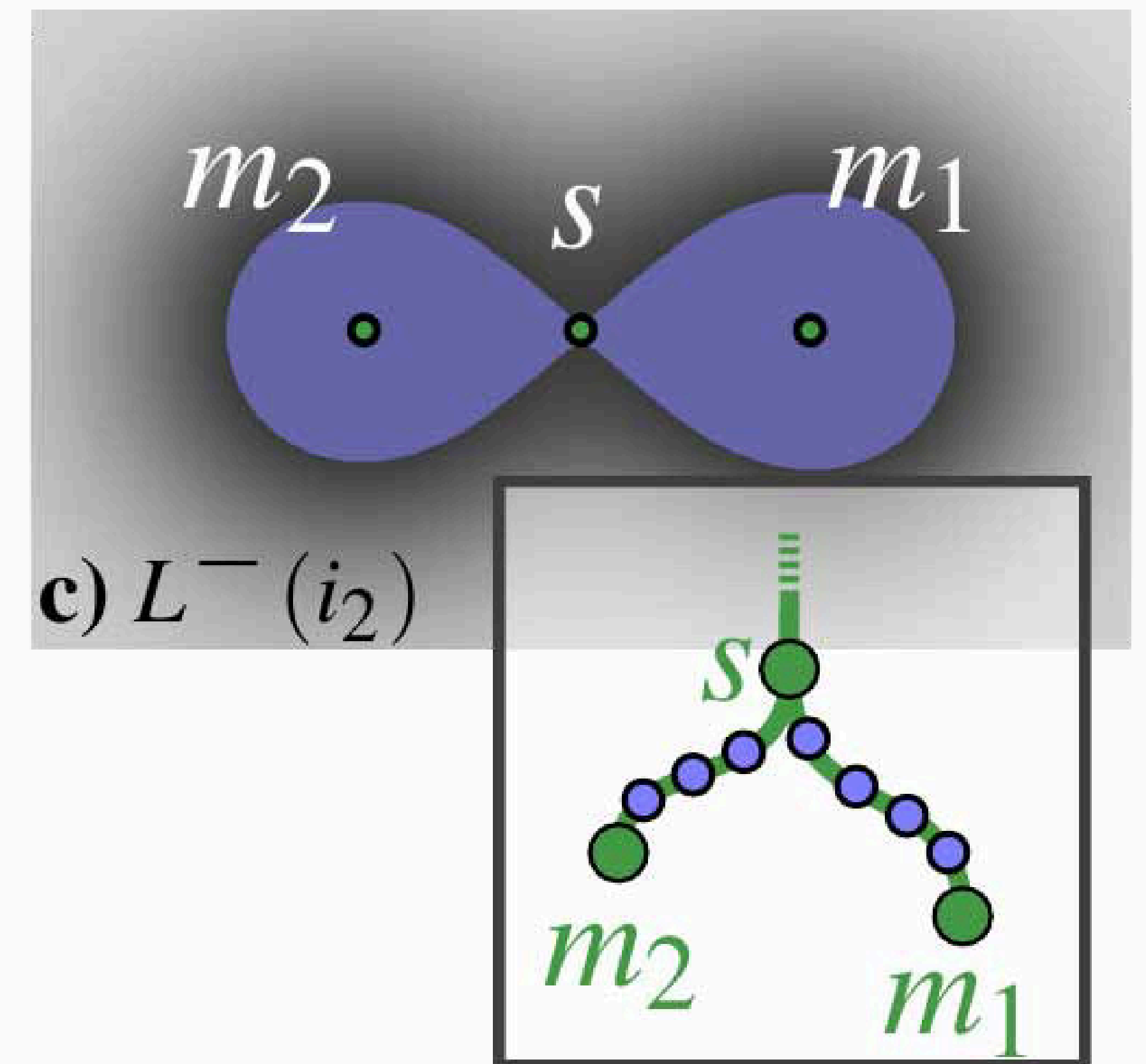
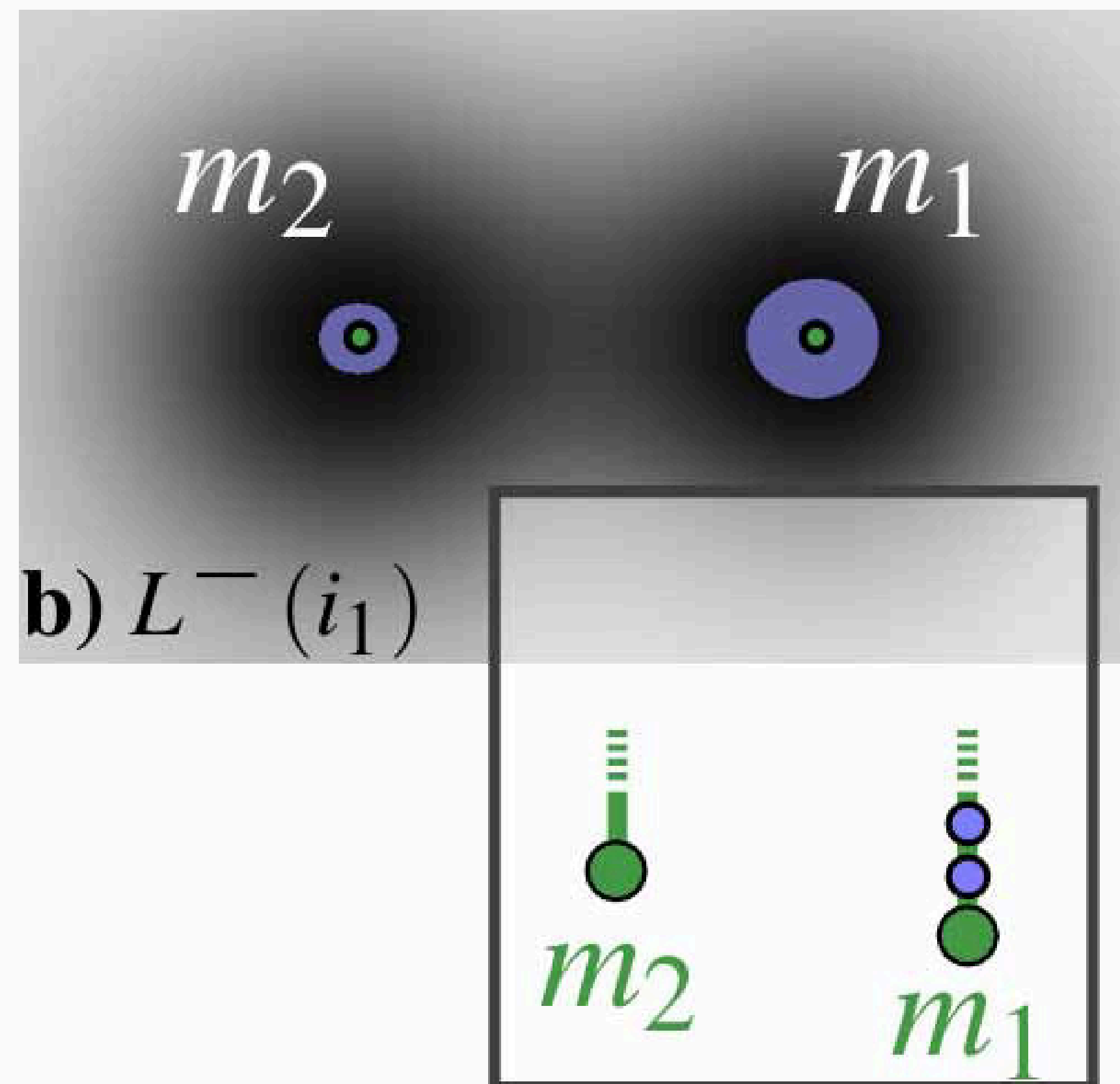
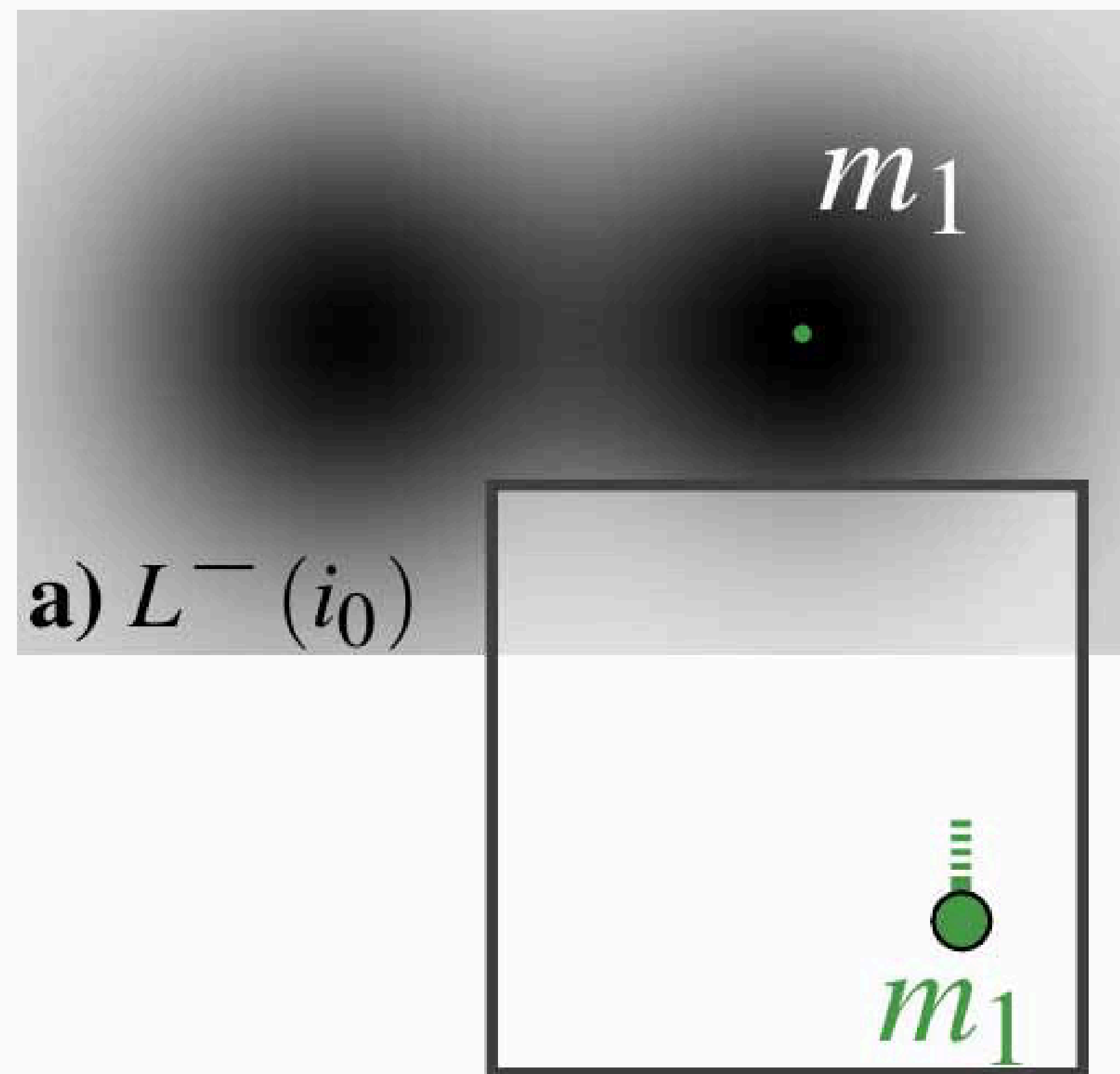
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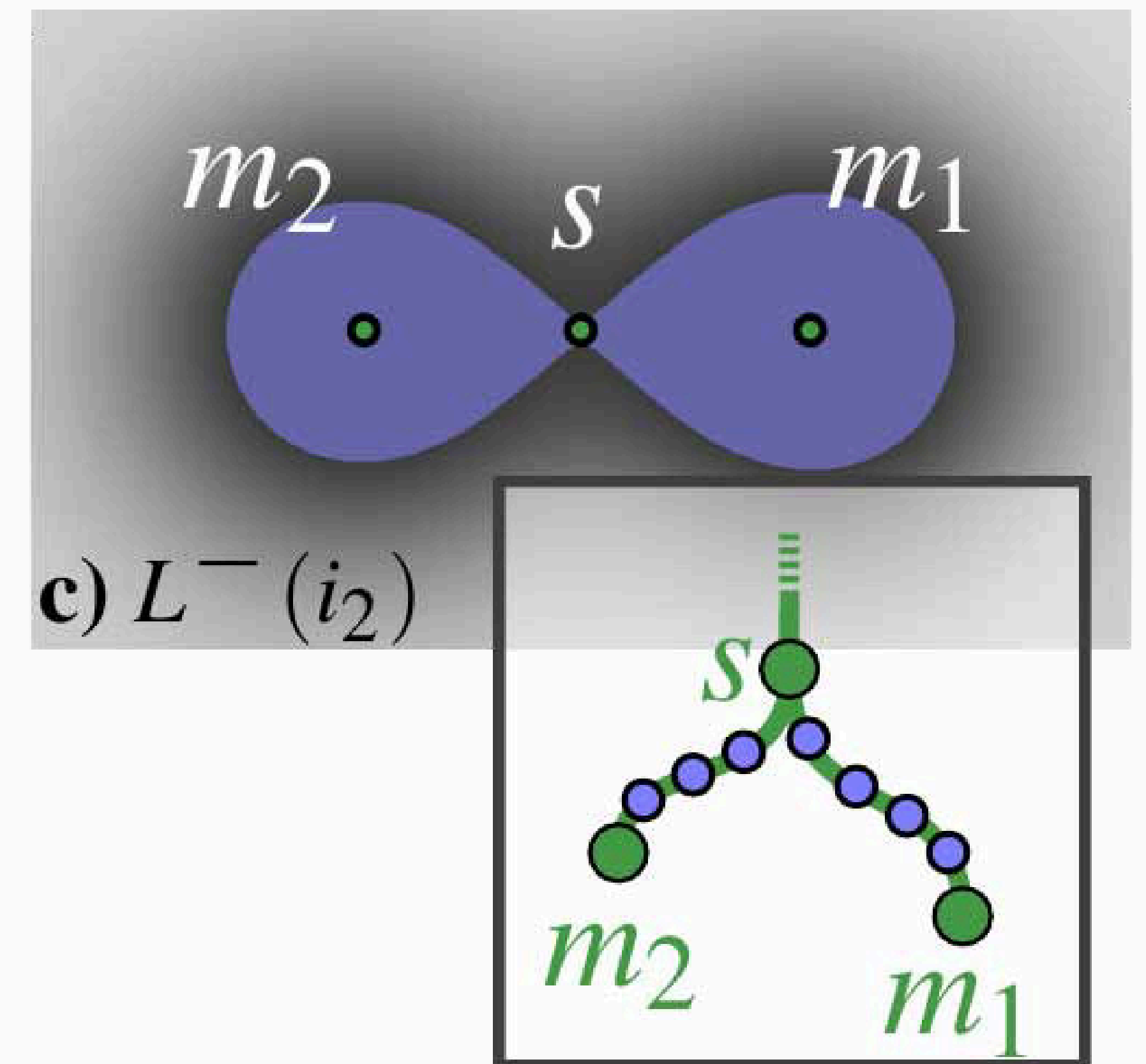
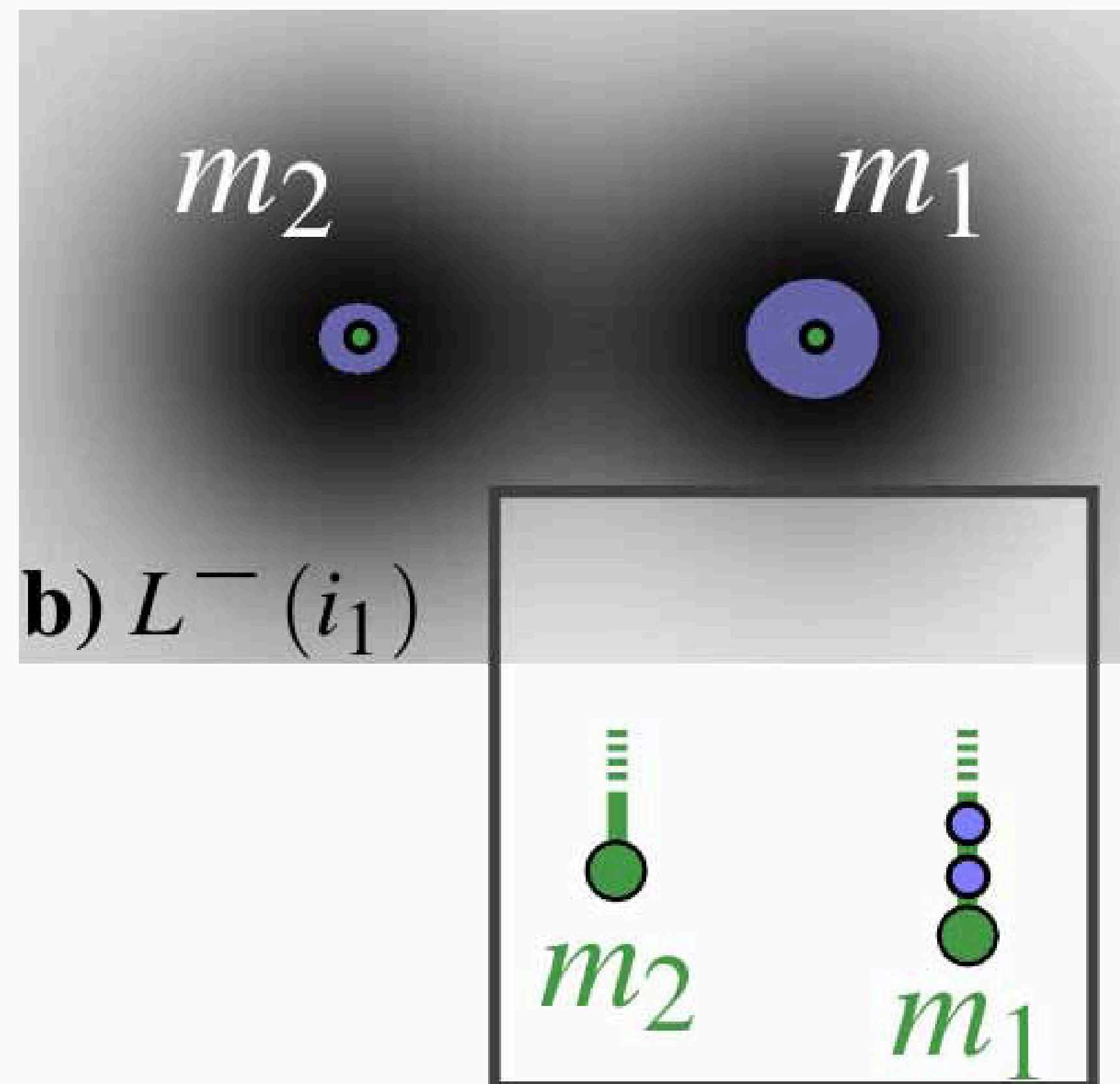
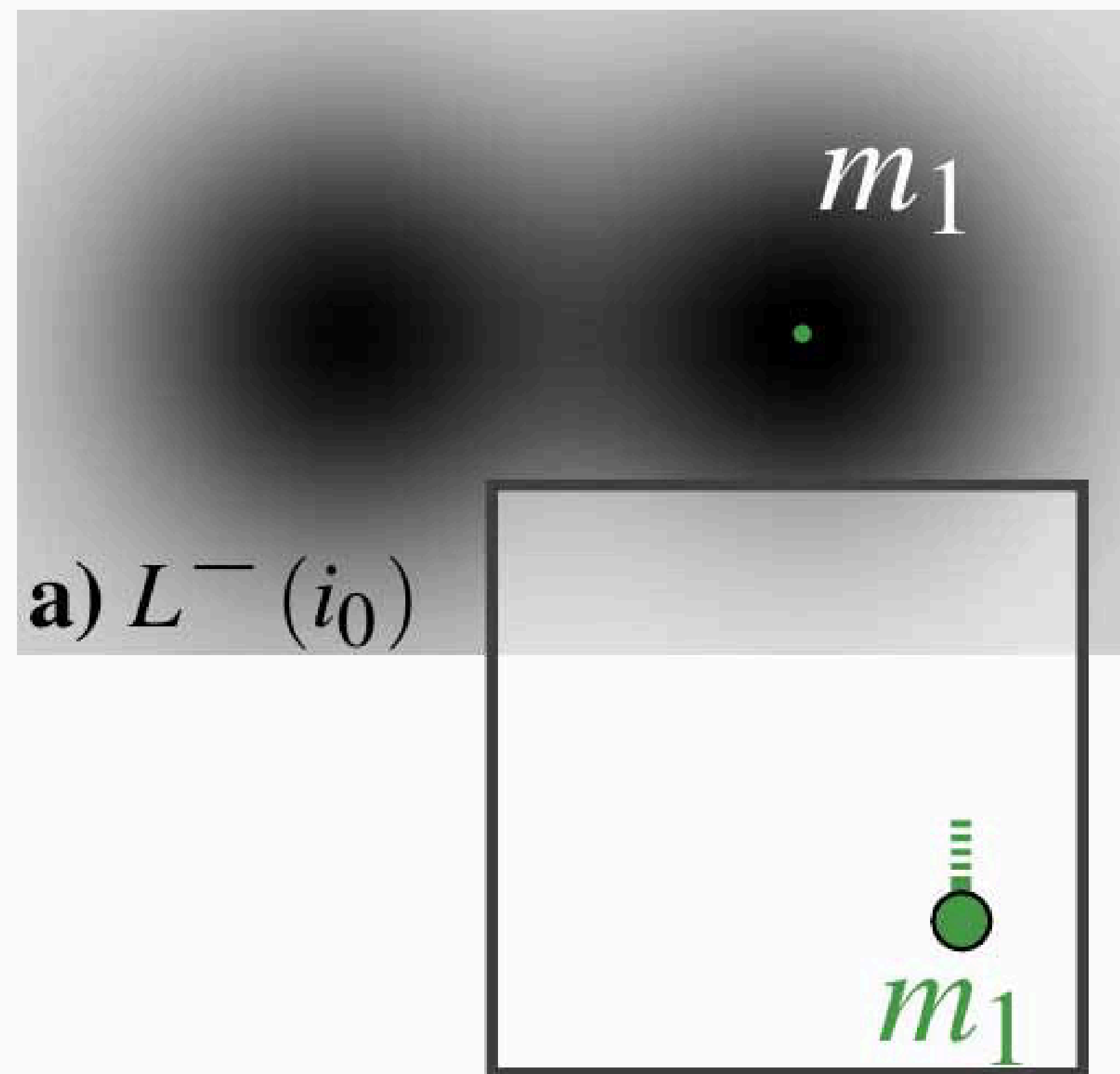
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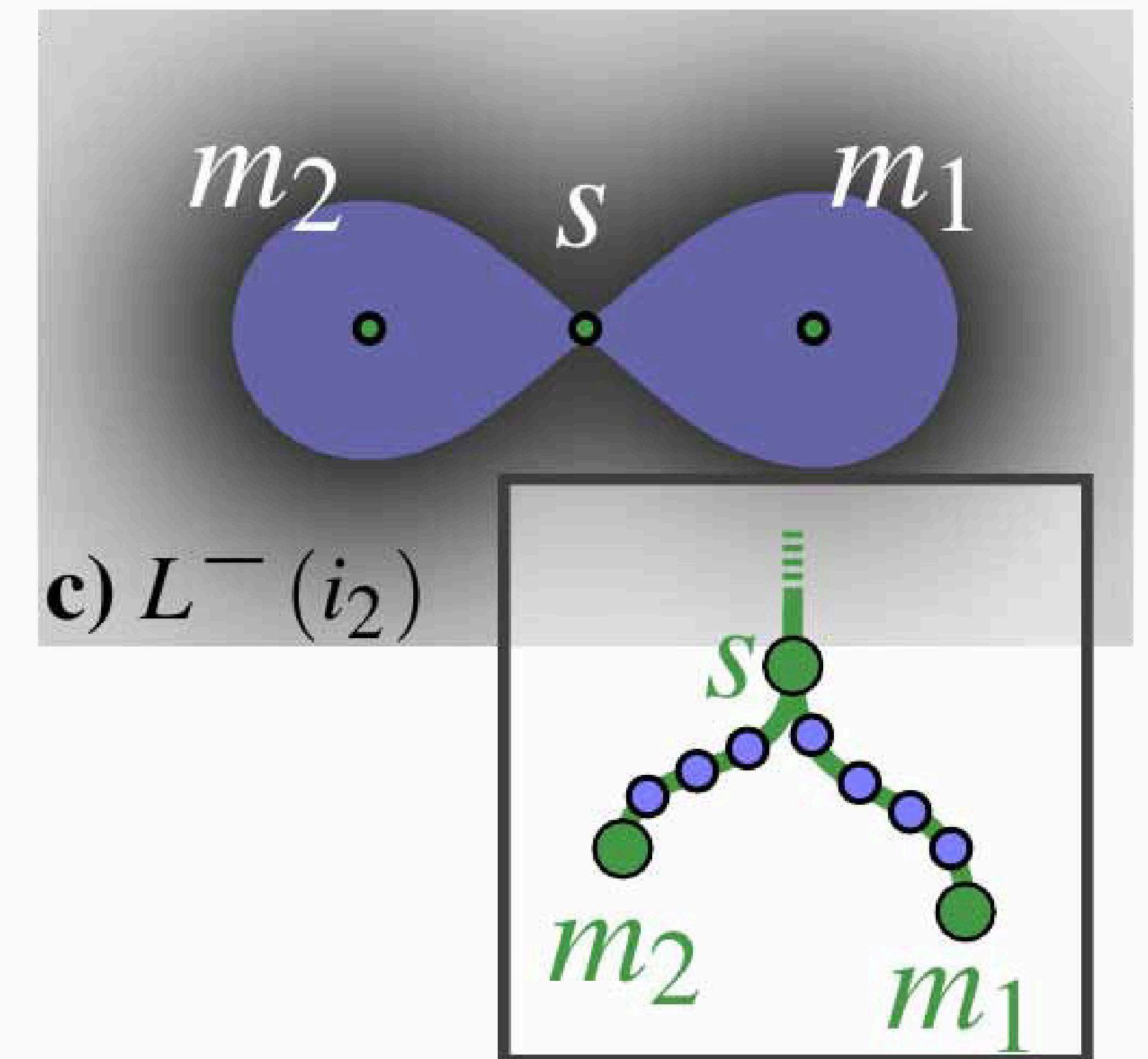
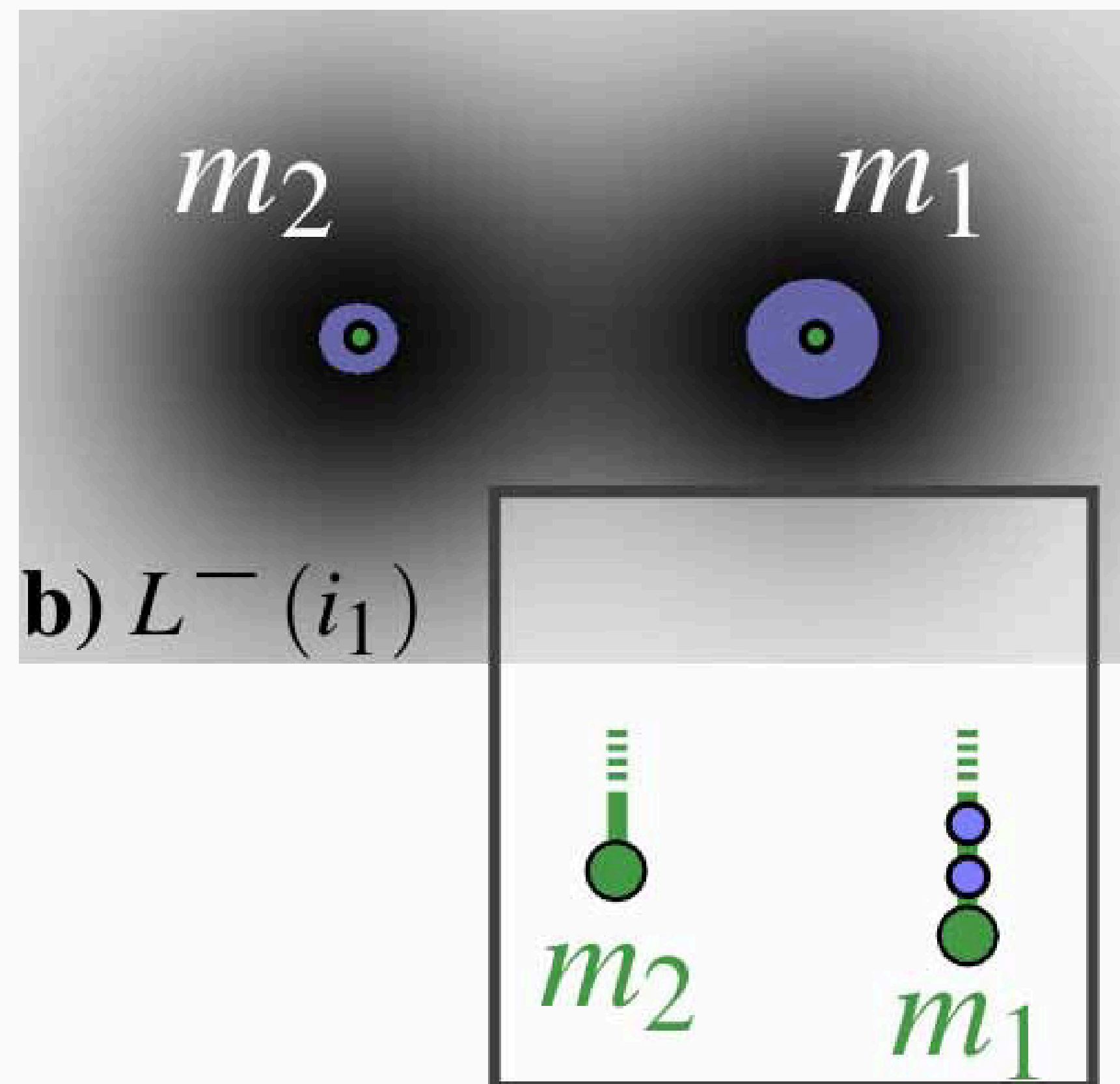
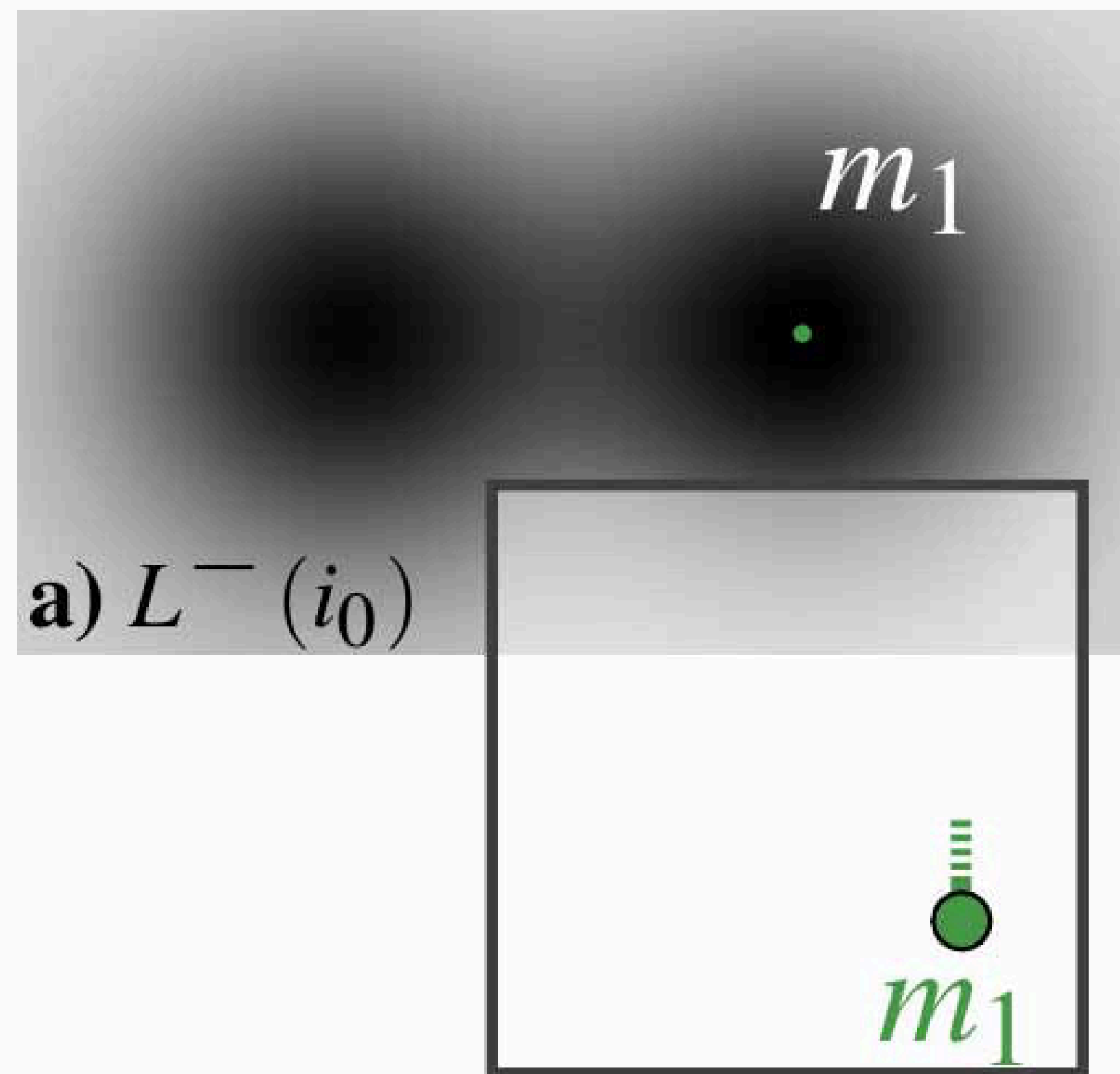
Critical Points of a Realization

- Sub-level set: $L_g^-(i) = \{p \in \mathcal{S} \mid g(p) \leq i\}$
 $L_g^-(i) \subseteq L_g^-(j) \quad \text{if } i \leq j$



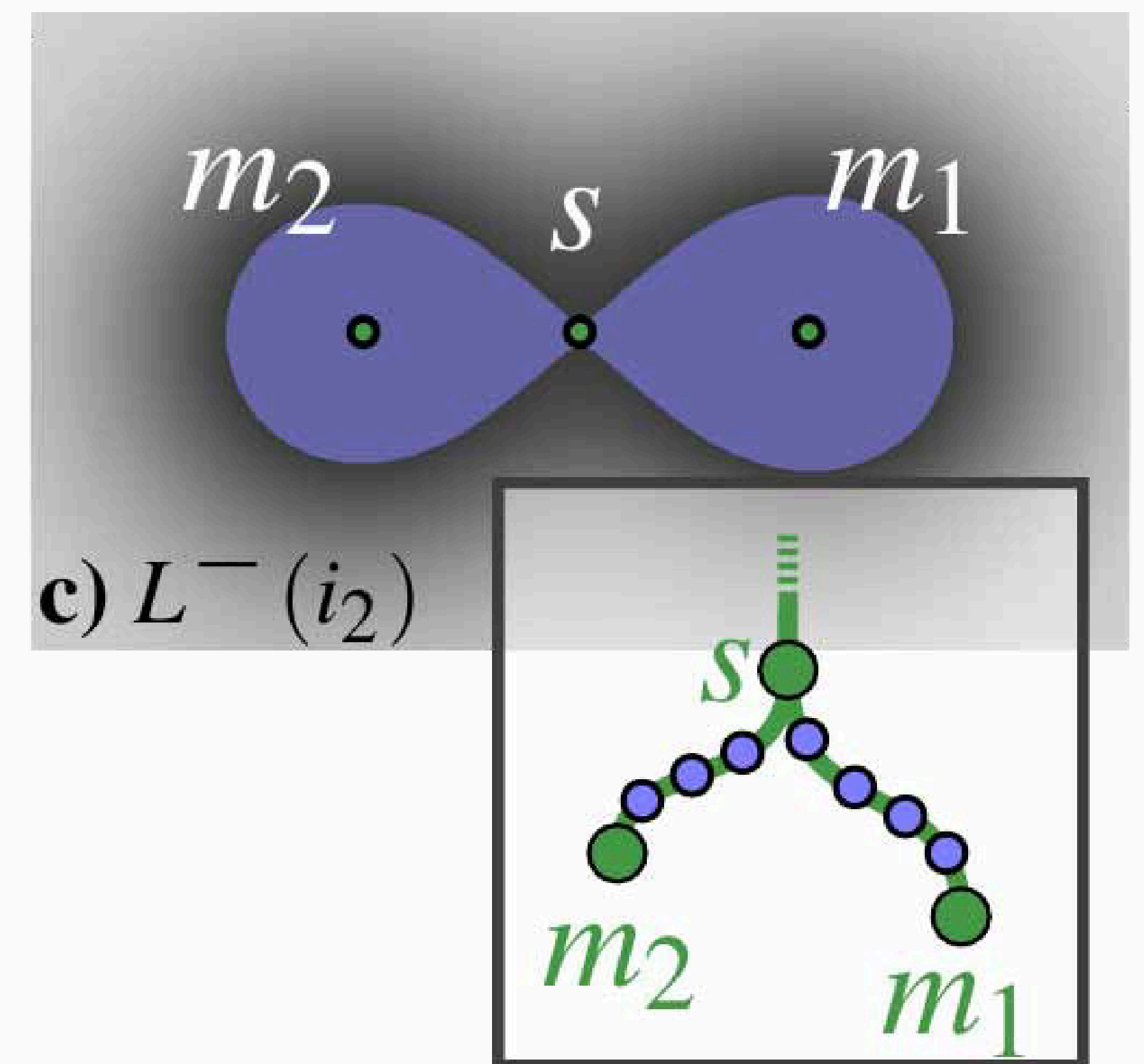
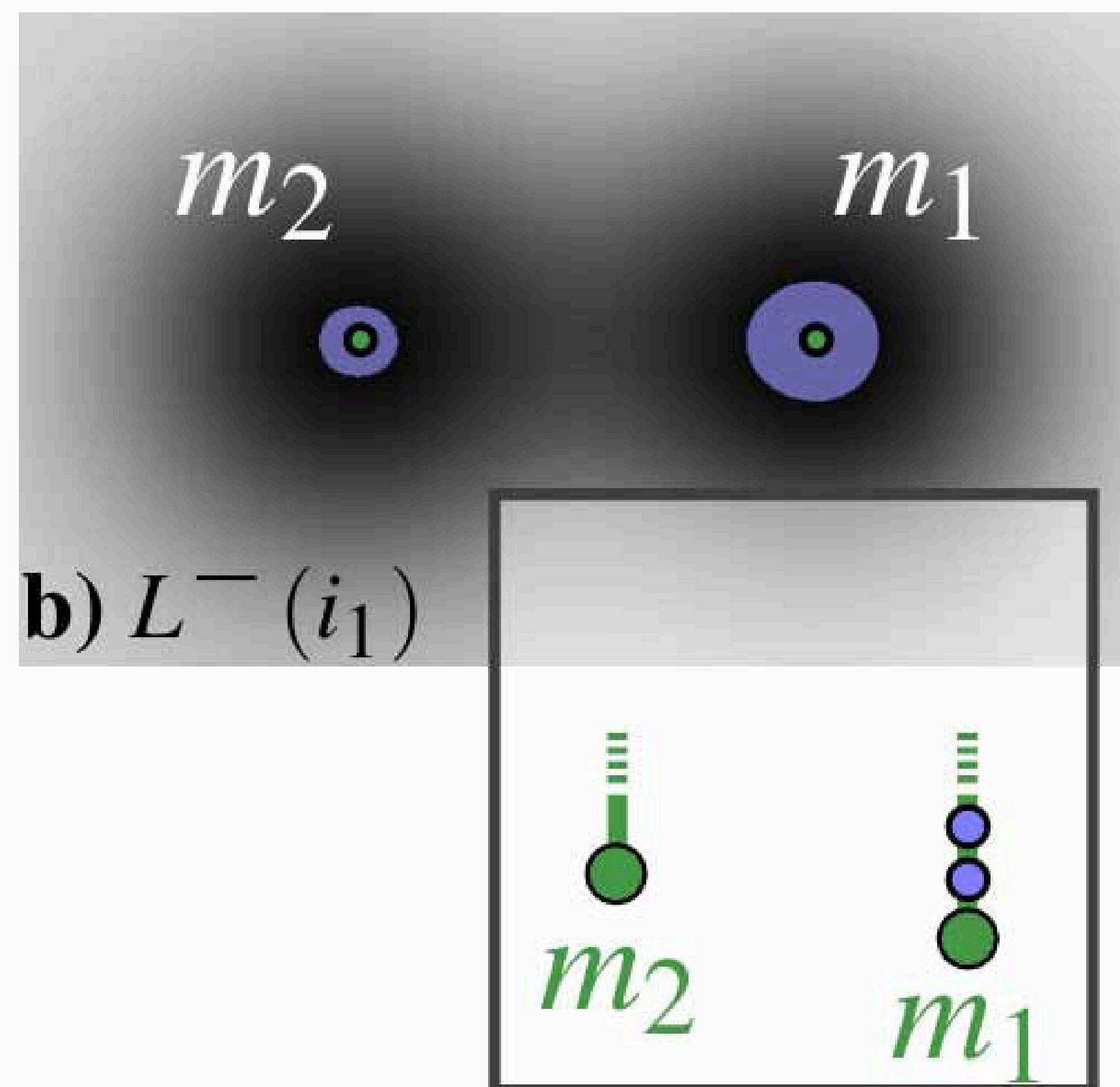
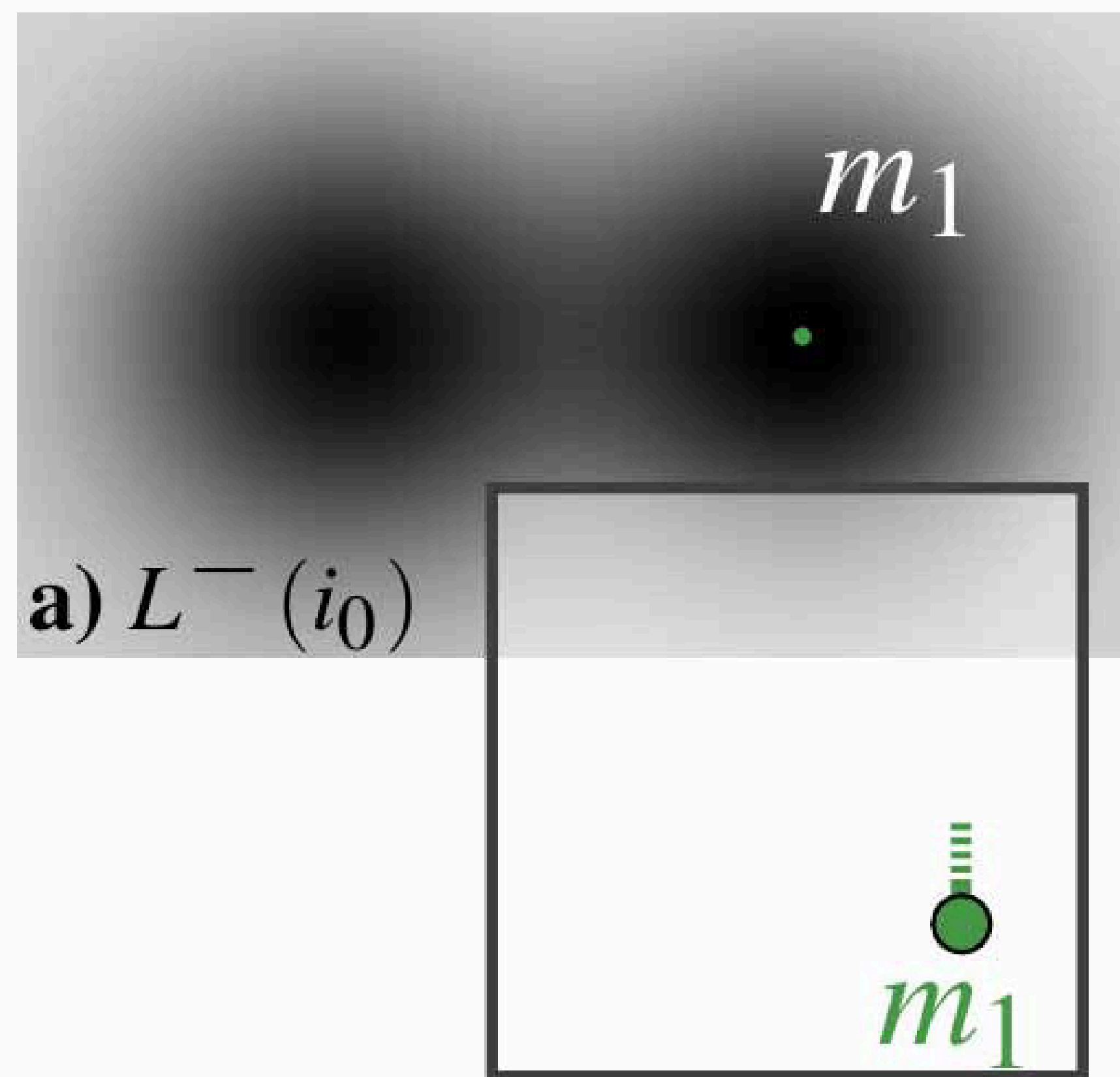
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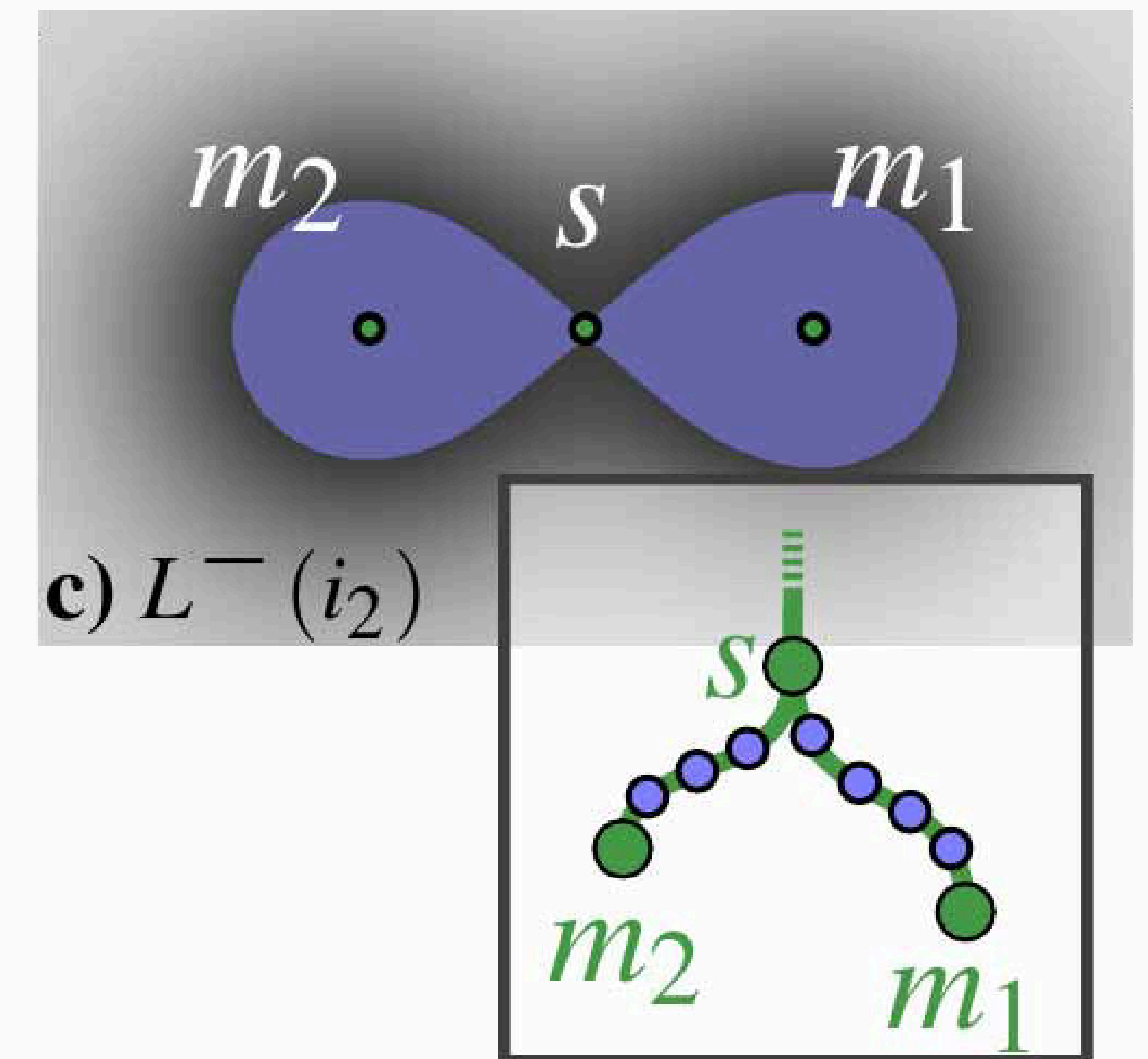
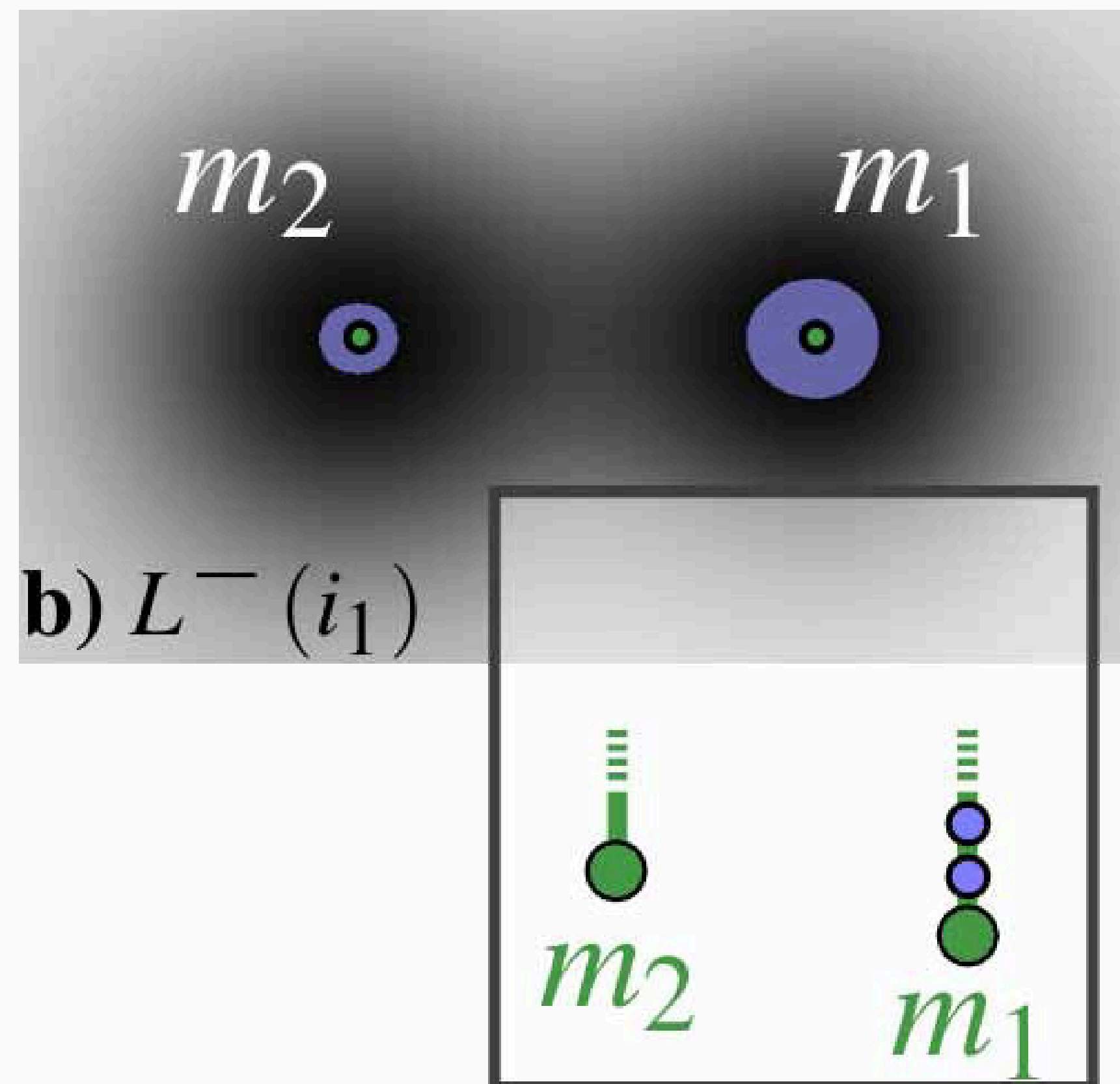
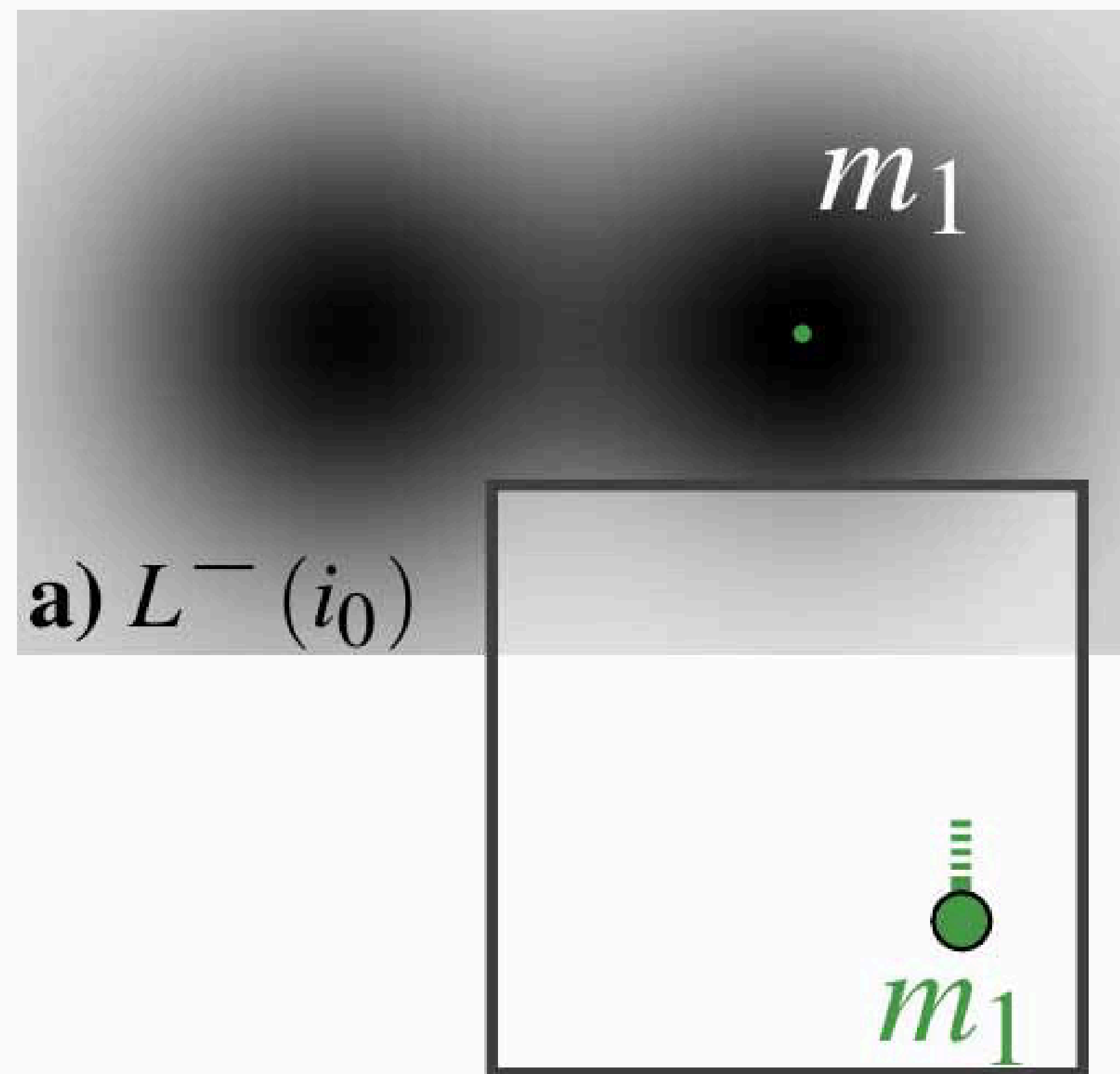
Critical Points of a Realization

- Join tree $\mathcal{T}(g)$



Critical Points of a Realization

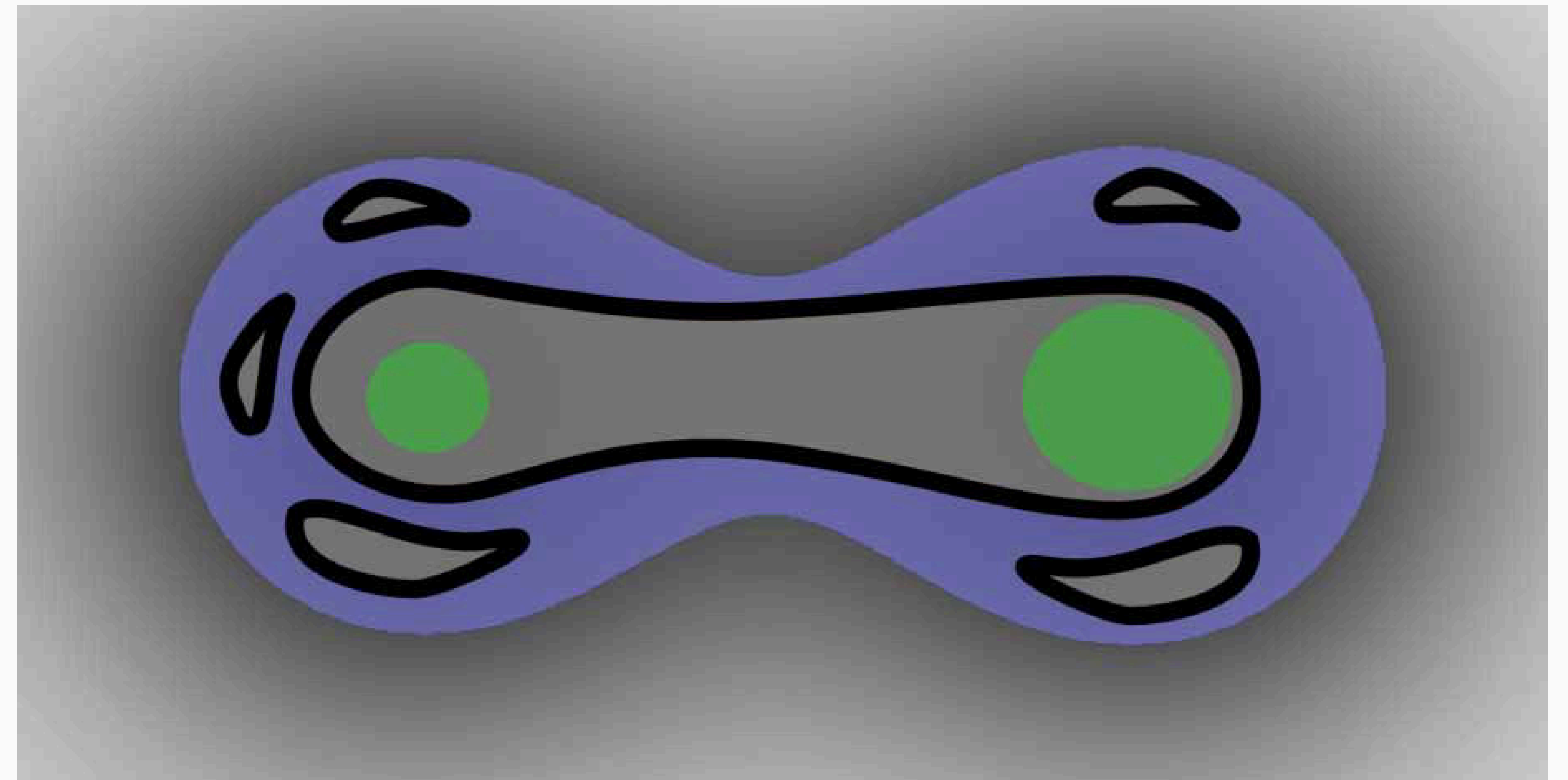
- Join tree $\mathcal{T}(g)$
 - 1-dimensional simplicial complex
 - Retraction of sub-level set components



Mandatory Critical Points: Intuition

- Key intuition:

$$f^- \leq g \leq f^+$$

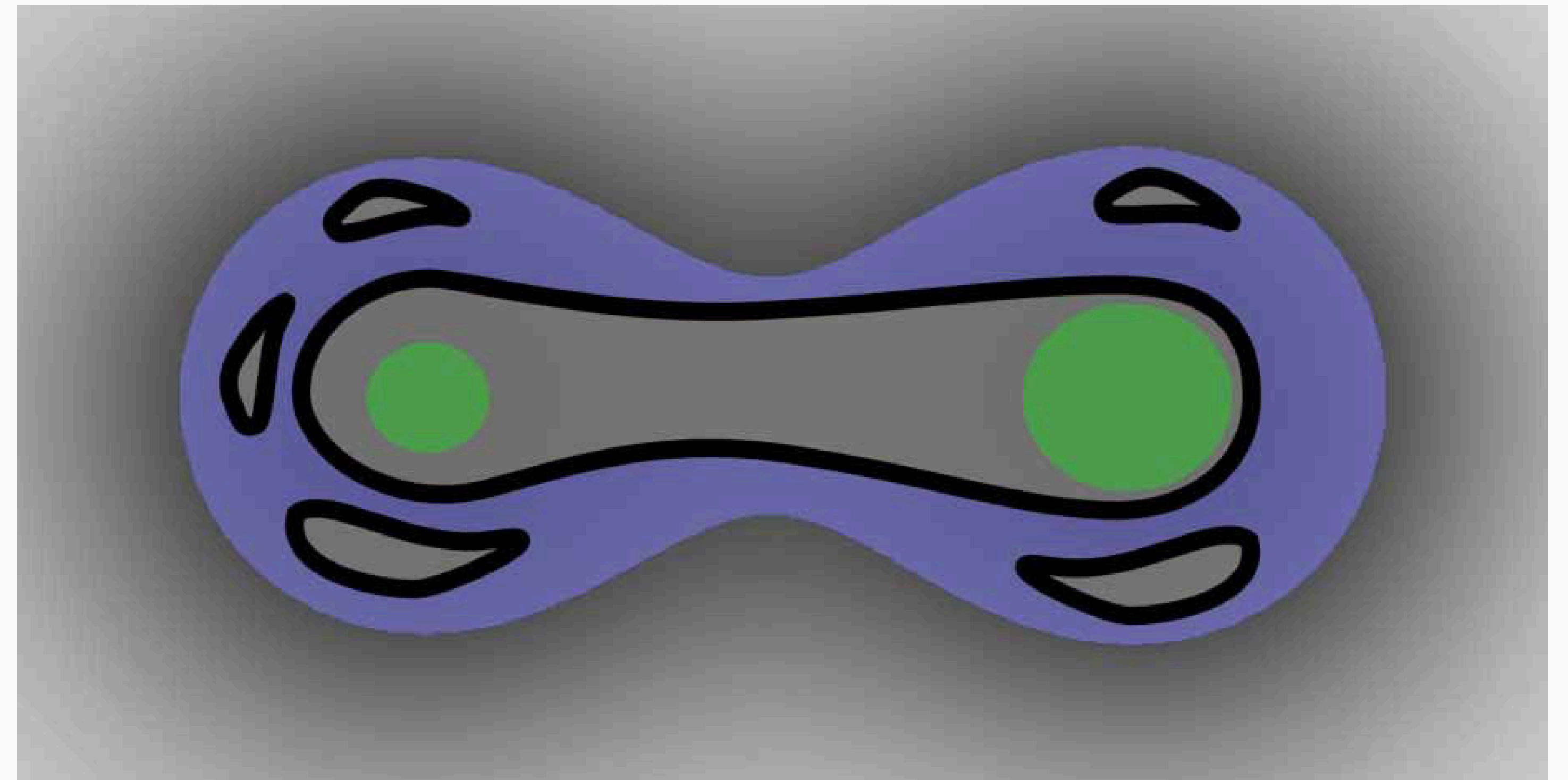


Mandatory Critical Points: Intuition

- Key intuition:

$$f^- \leq g \leq f^+$$

$$L_{f^+}^-(i) \subset L_g^-(i) \subset L_{f^-}^-(i)$$



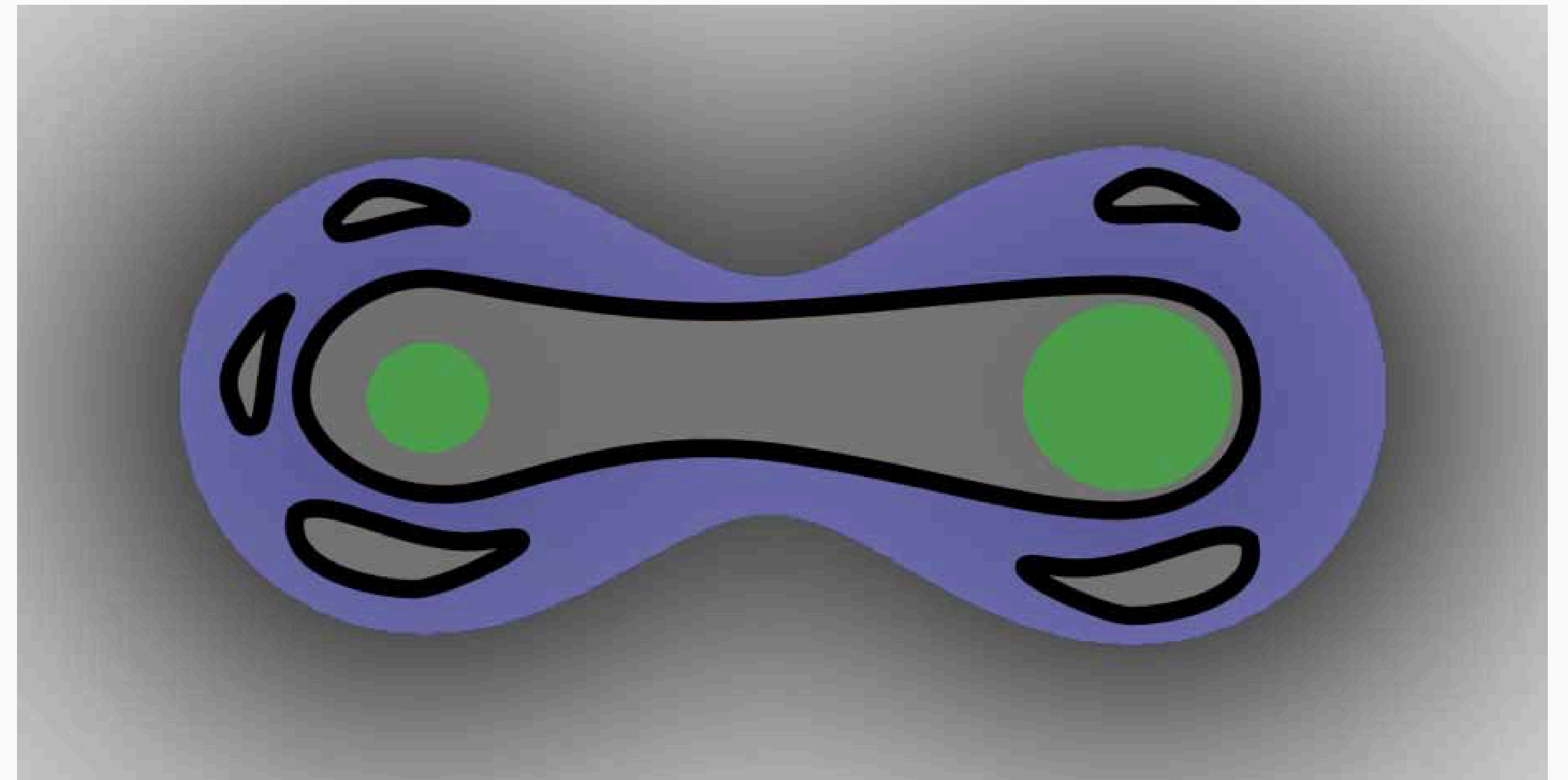
Mandatory Critical Points: Intuition

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Nested!



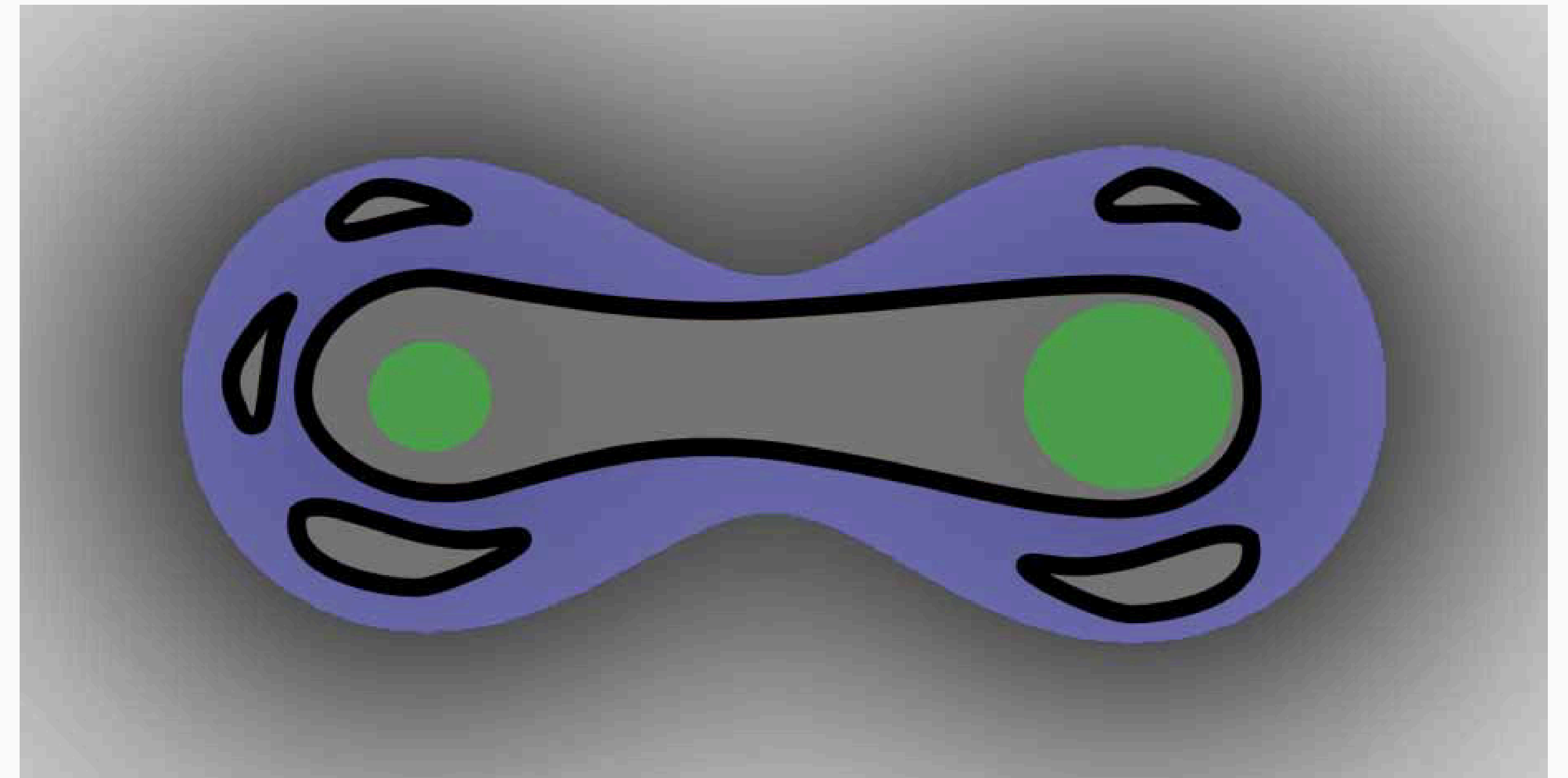
Mandatory Critical Points: Intuition

- Key intuition:

$$f^- \leq g \leq f^+$$

$$L_{f^+}^-(i) \subset L_g^-(i) \subset L_{f^-}^-(i)$$

Nested!

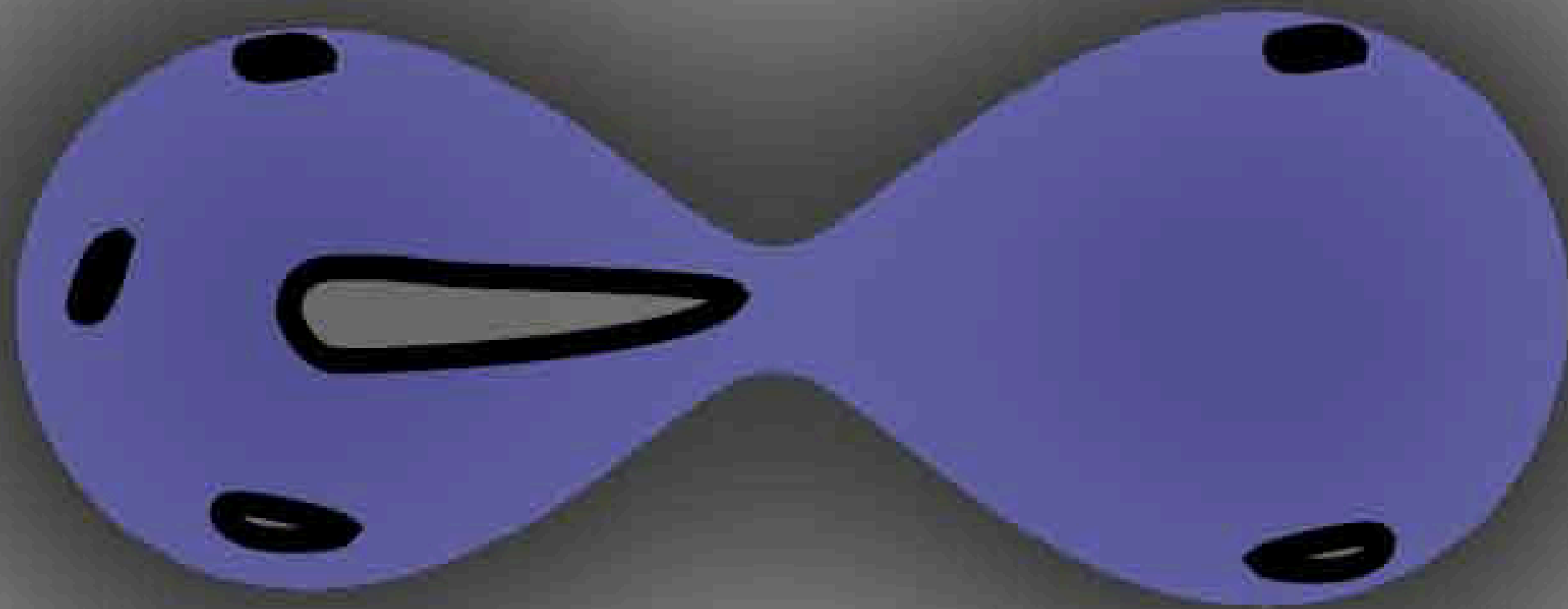


- Demo!

Mandatory Minimum: Intuition

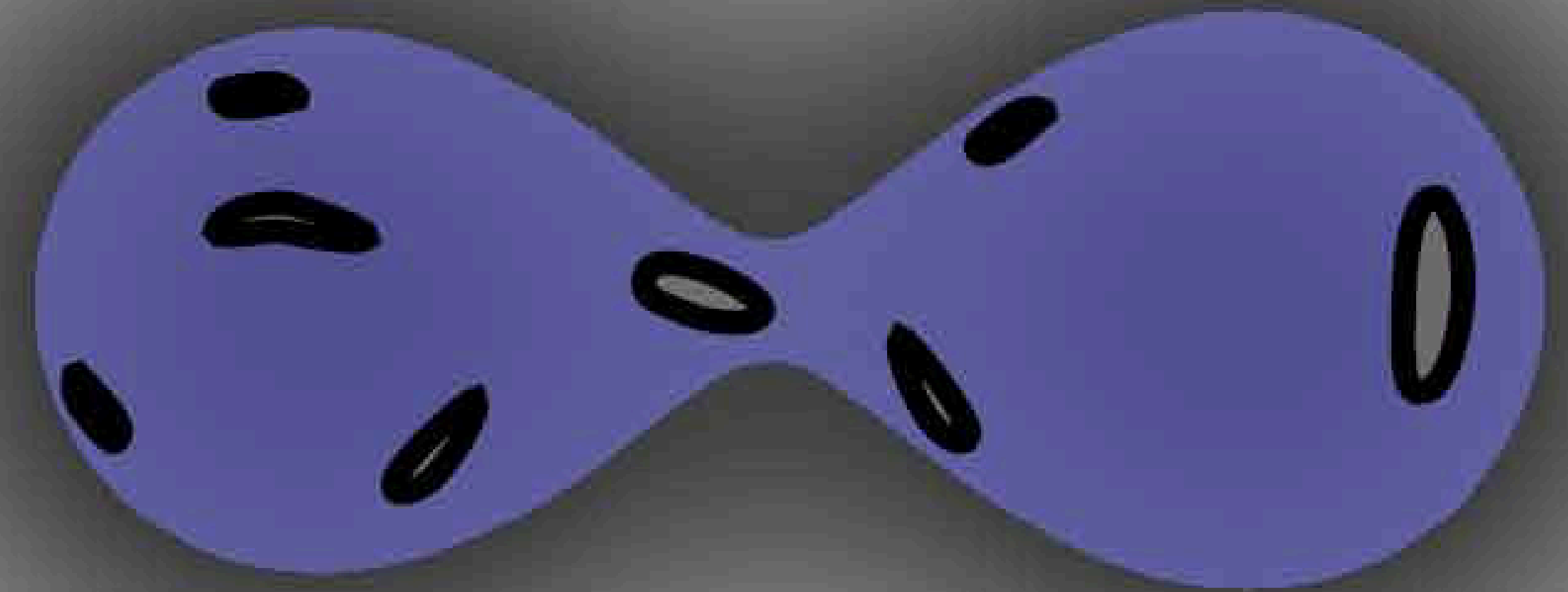
g_a

a) Isovalue i



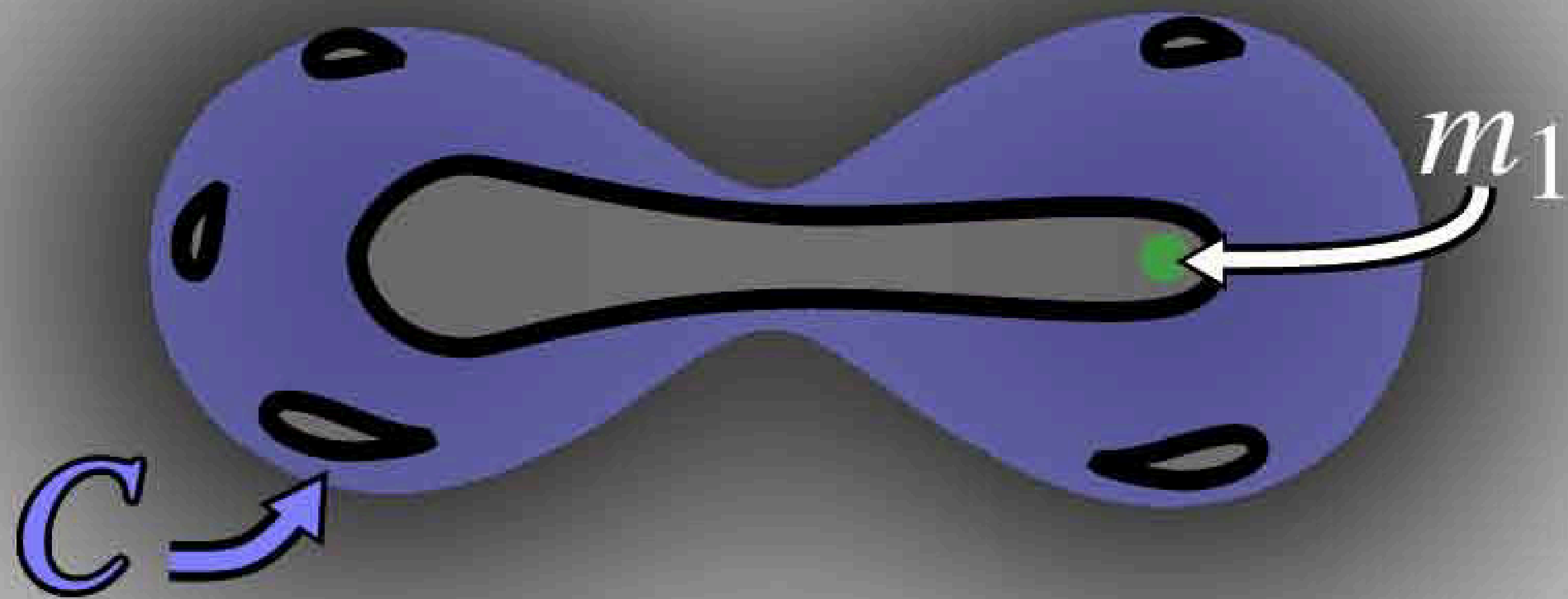
g_b

b) Isovalue i

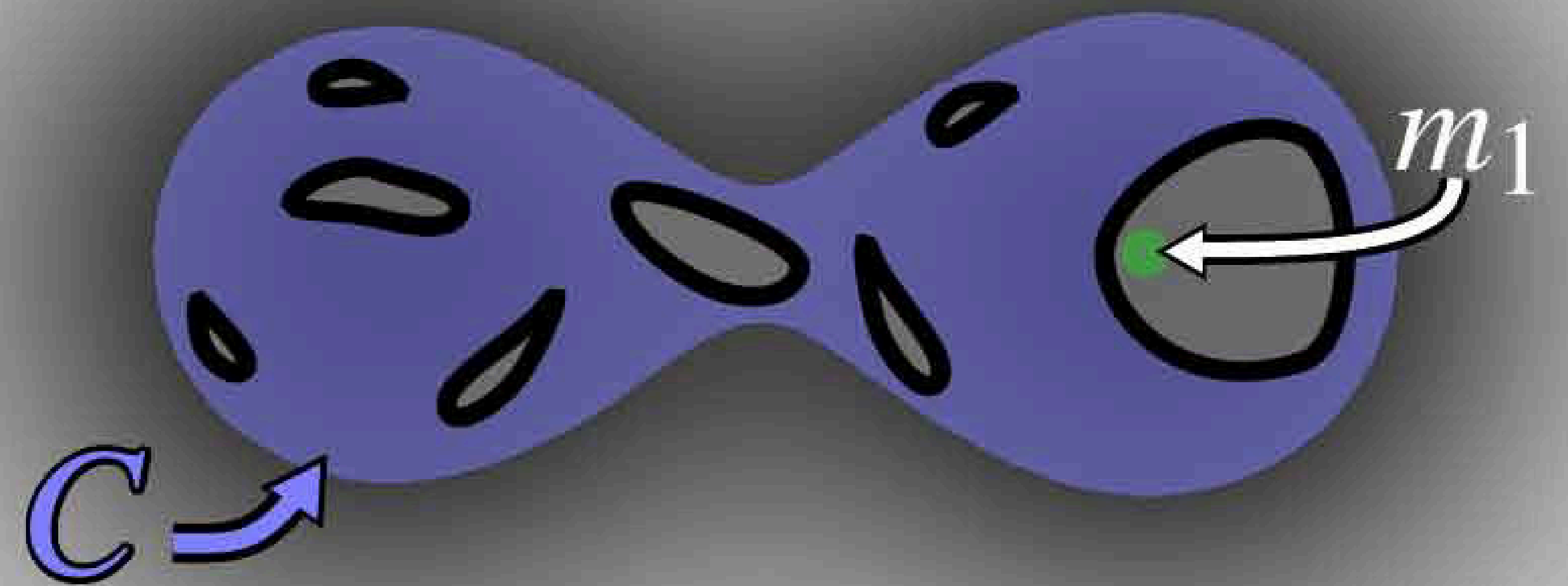


Mandatory Minimum: Intuition

c) Isovalue $f^+(m_1)$

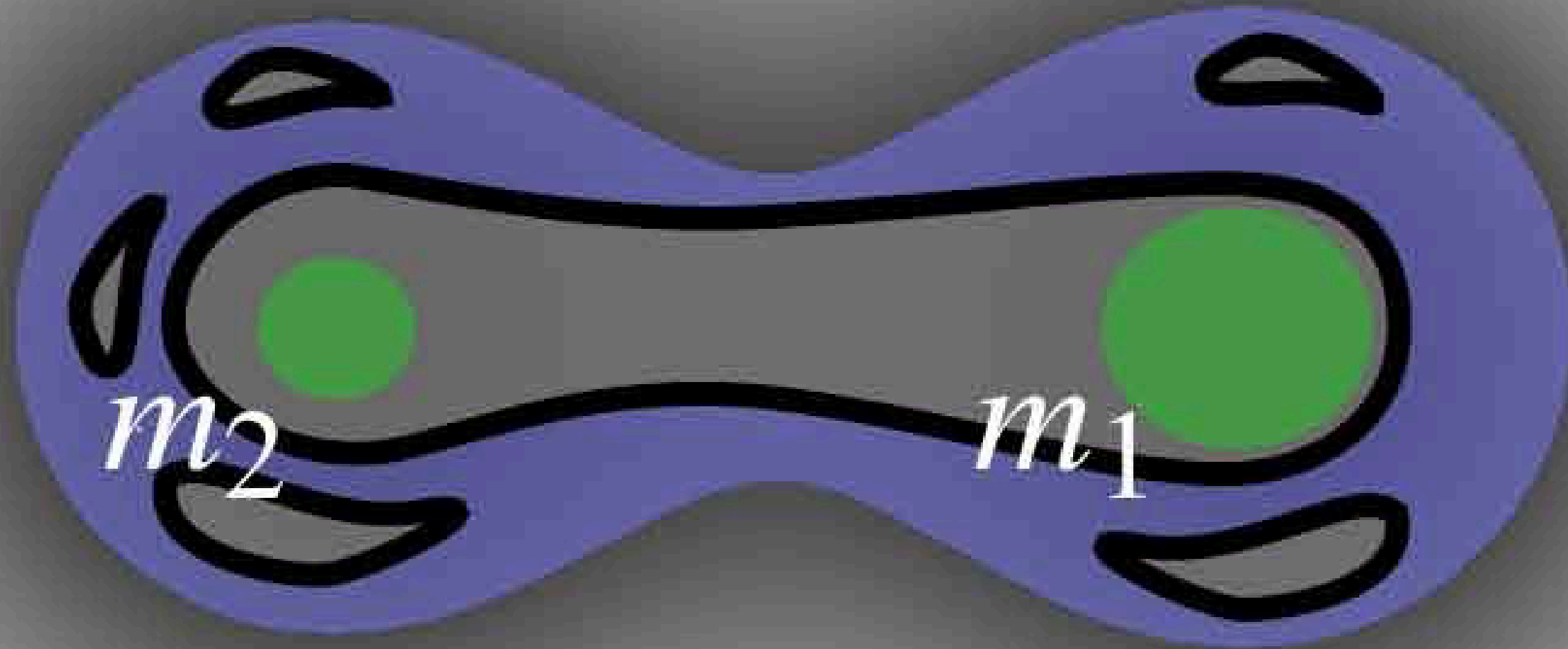


d) Isovalue $f^+(m_1)$



Mandatory Minimum: Intuition

e) Isovalue $f^+(m_2)$

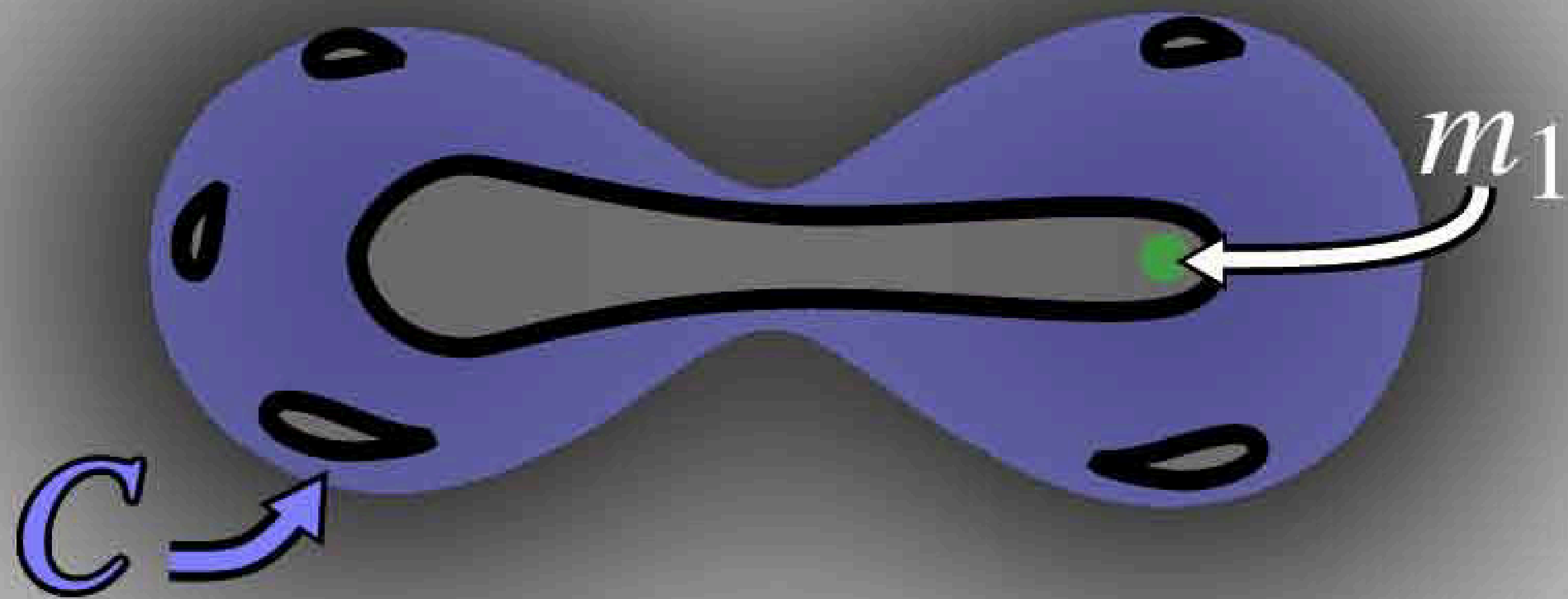


f) Isovalue $f^+(m_2)$

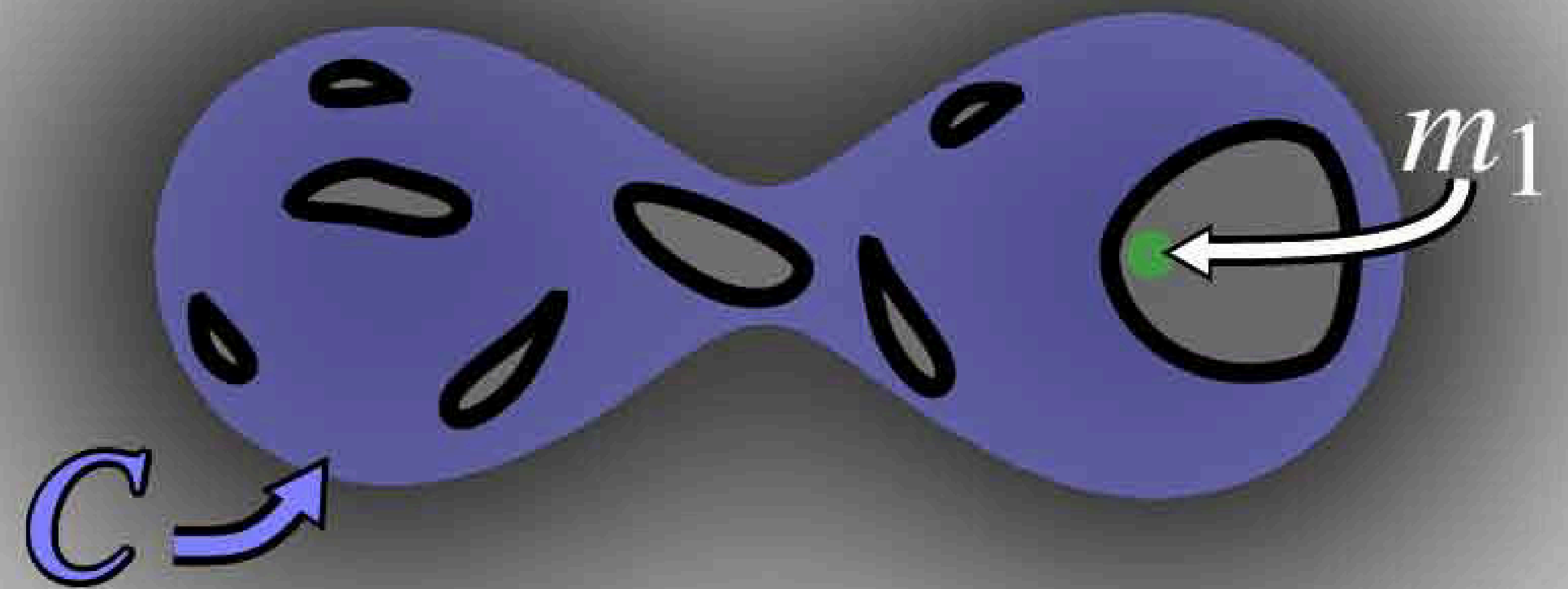


Mandatory Minimum: Intuition

c) Isovalue $f^+(m_1)$



d) Isovalue $f^+(m_1)$



Mandatory Minimum: Definition

- A Mandatory Minimum M

Mandatory Minimum: Definition

- A *Mandatory Minimum* M
 - Minimal connected component $C \subset \mathcal{S}$
 - Minimal interval $I \subset \mathbb{R}$

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- A *Mandatory Minimum* M
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 - Such that there exists for any g *at least* one minimum m

Mandatory Minimum: Definition

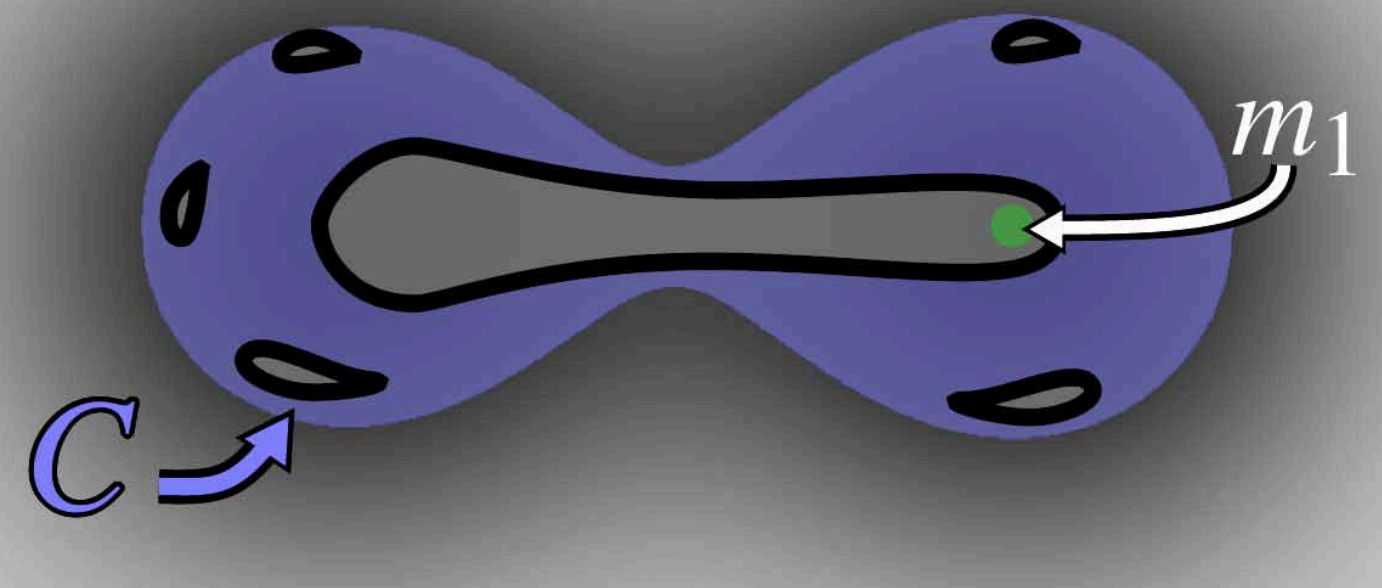
- A Mandatory Minimum M
 - Minimal connected component $C \subset \mathcal{S}$
 - Minimal interval $I \subset \mathbb{R}$
 - Such that there exists for any g at least one minimum m
 - $m \in C$
 - $g(m) \in I$

Mandatory Minimum: Definition

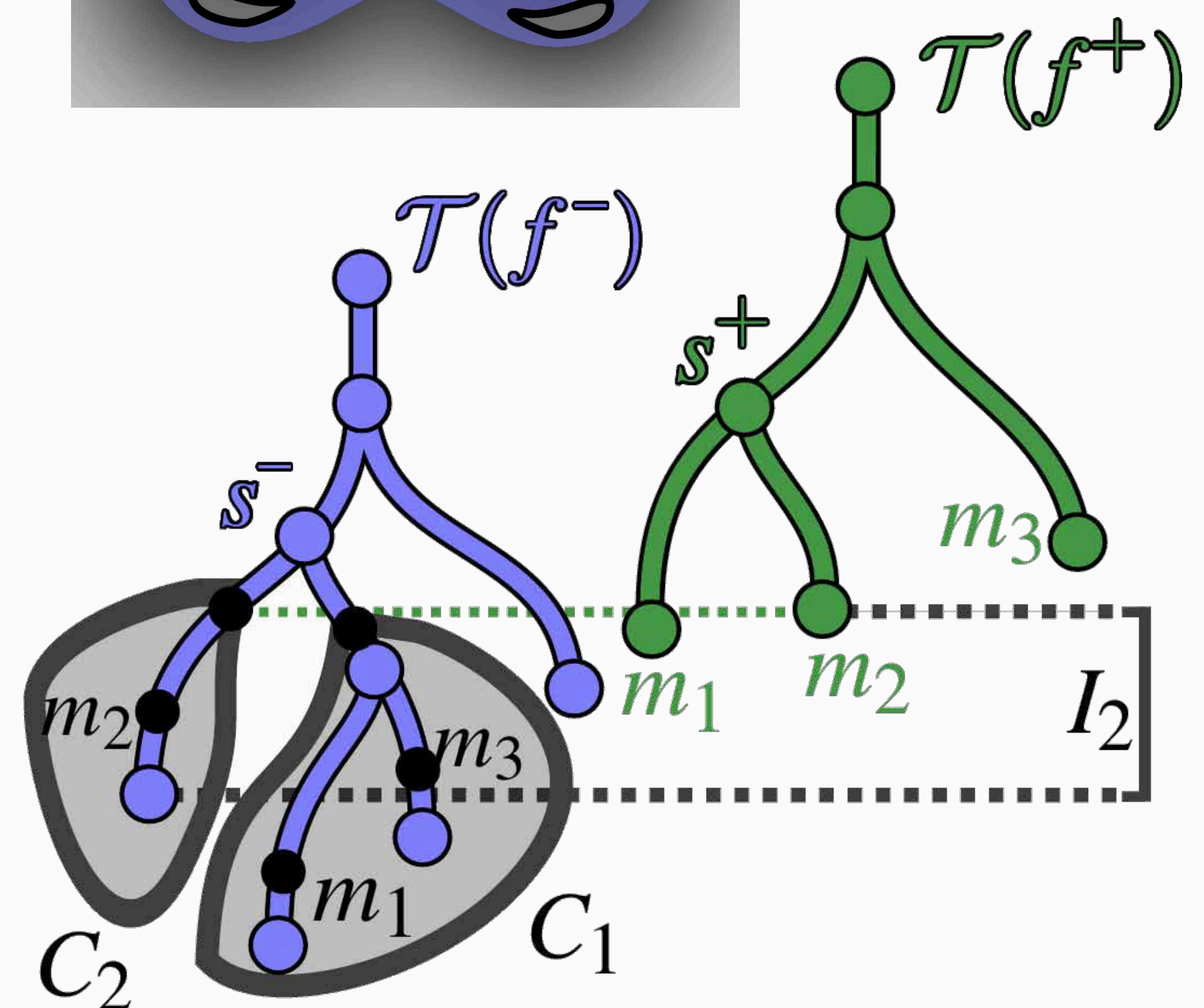
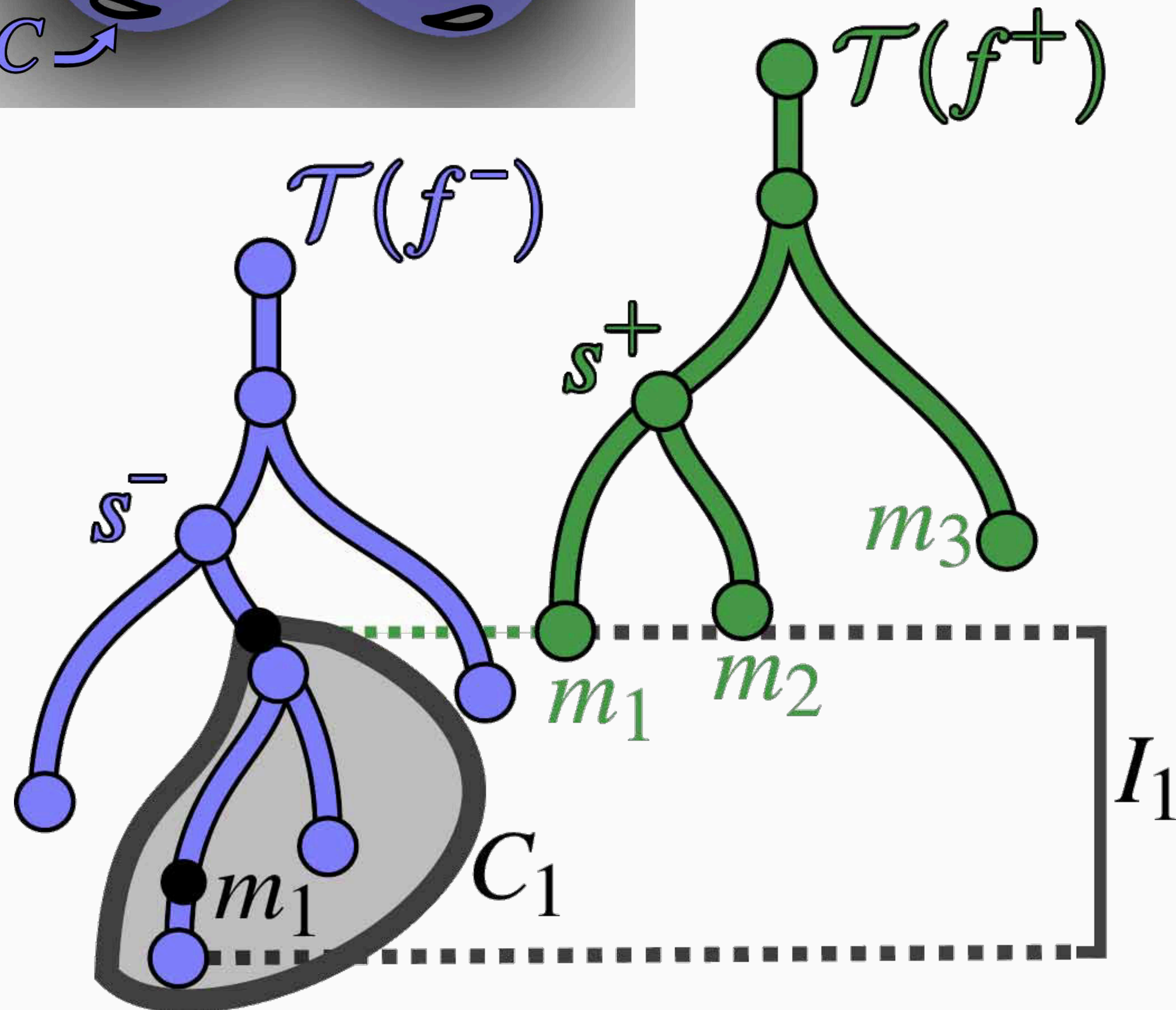
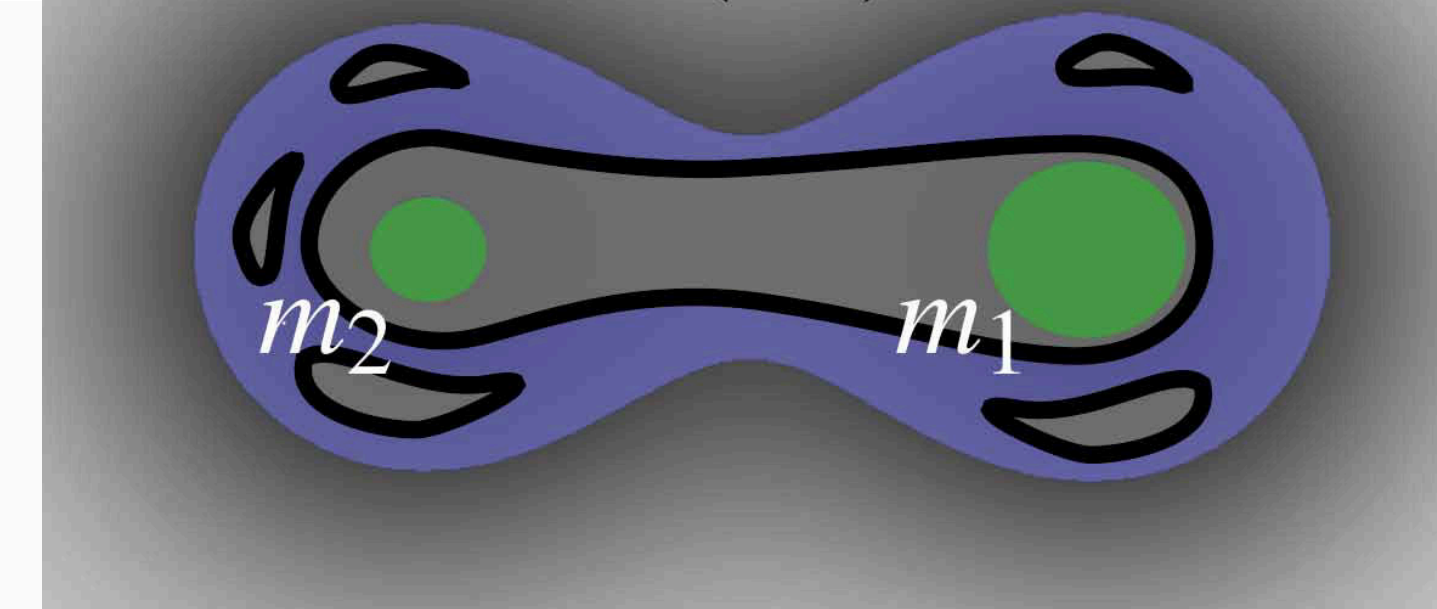
- A *Mandatory Minimum* M
 - Minimal connected **component** $C \subset \mathcal{S}$
 - Minimal **interval** $I \subset \mathbb{R}$
 - Such that there exists for any g *at least* one minimum m
 - $m \in C$
 - $g(m) \in I$

Mandatory Minimum Extraction Algorithm

c) Isovalue $f^+(m_1)$



e) Isovalue $f^+(m_2)$



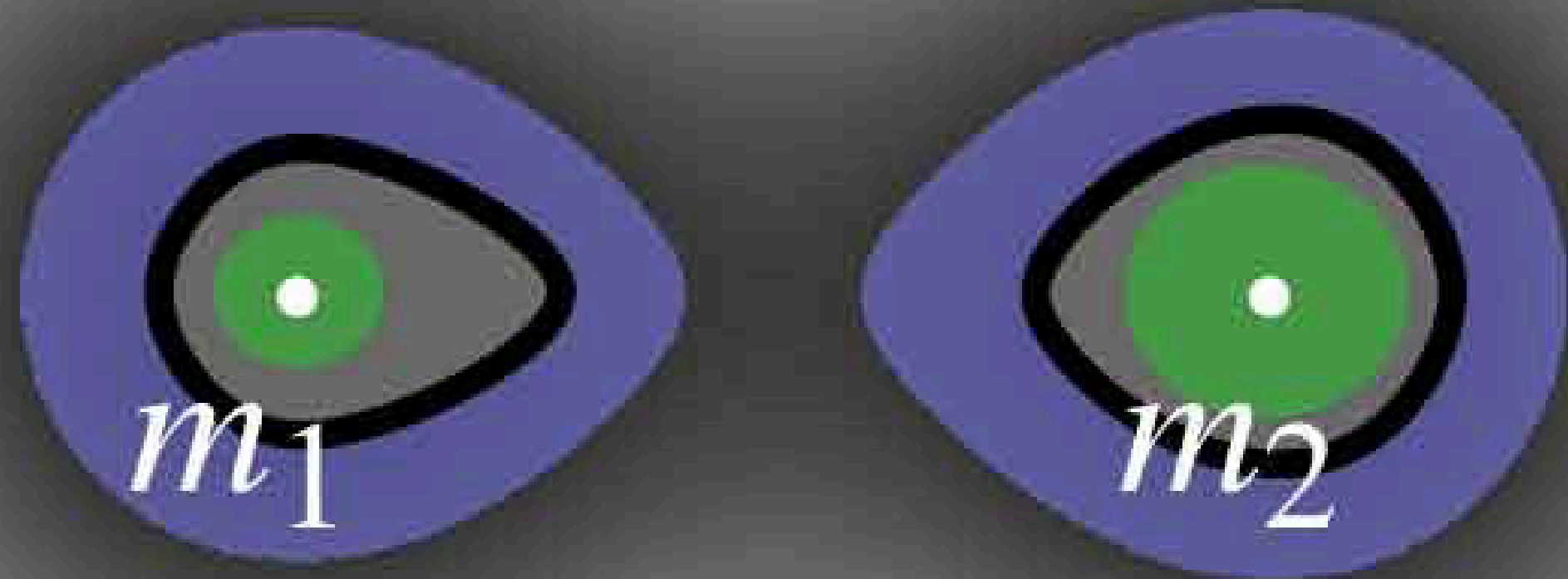
Mandatory Join Saddle: Definition

- A Mandatory Minimum \mathcal{S}
 - Minimal connected component $C \subset \mathcal{S}$
 - Minimal interval $I \subset \mathbb{R}$
 - Such that there exists for any g at least one join saddle s
 - $s \in C$
 - $g(s) \in I$

Mandatory Join Saddle: Intuition

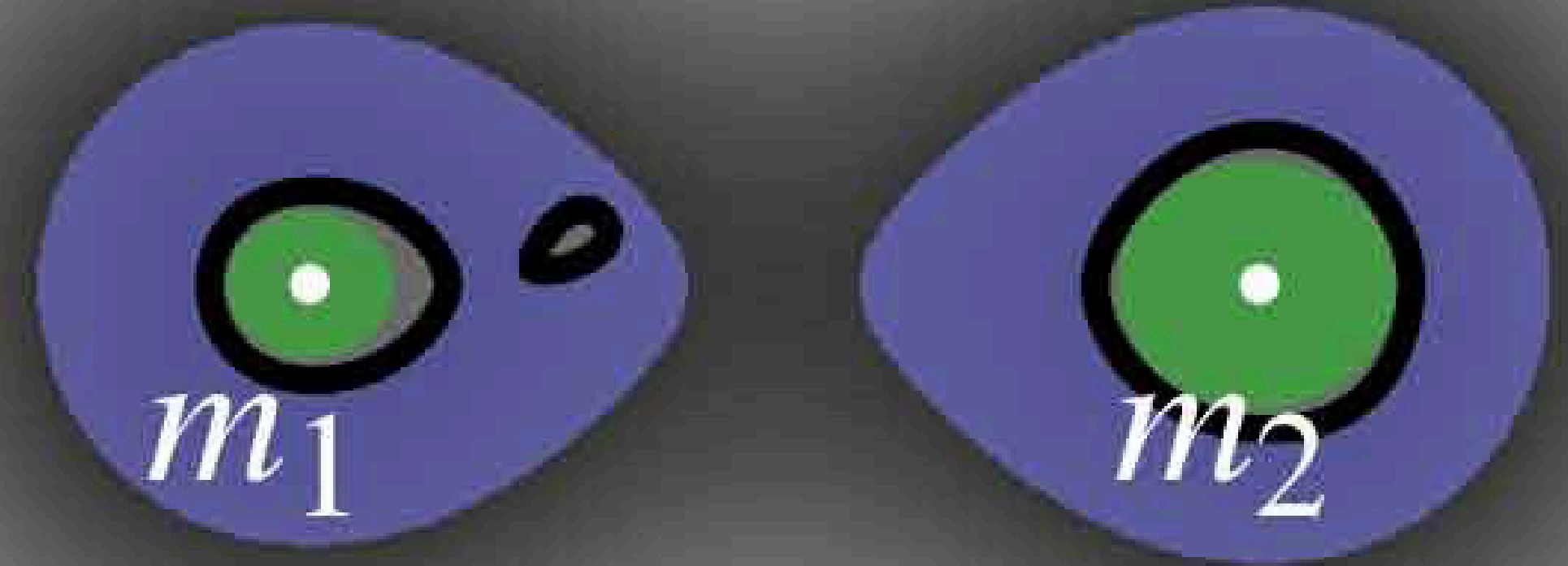
g_a

a) Isovalue ξ

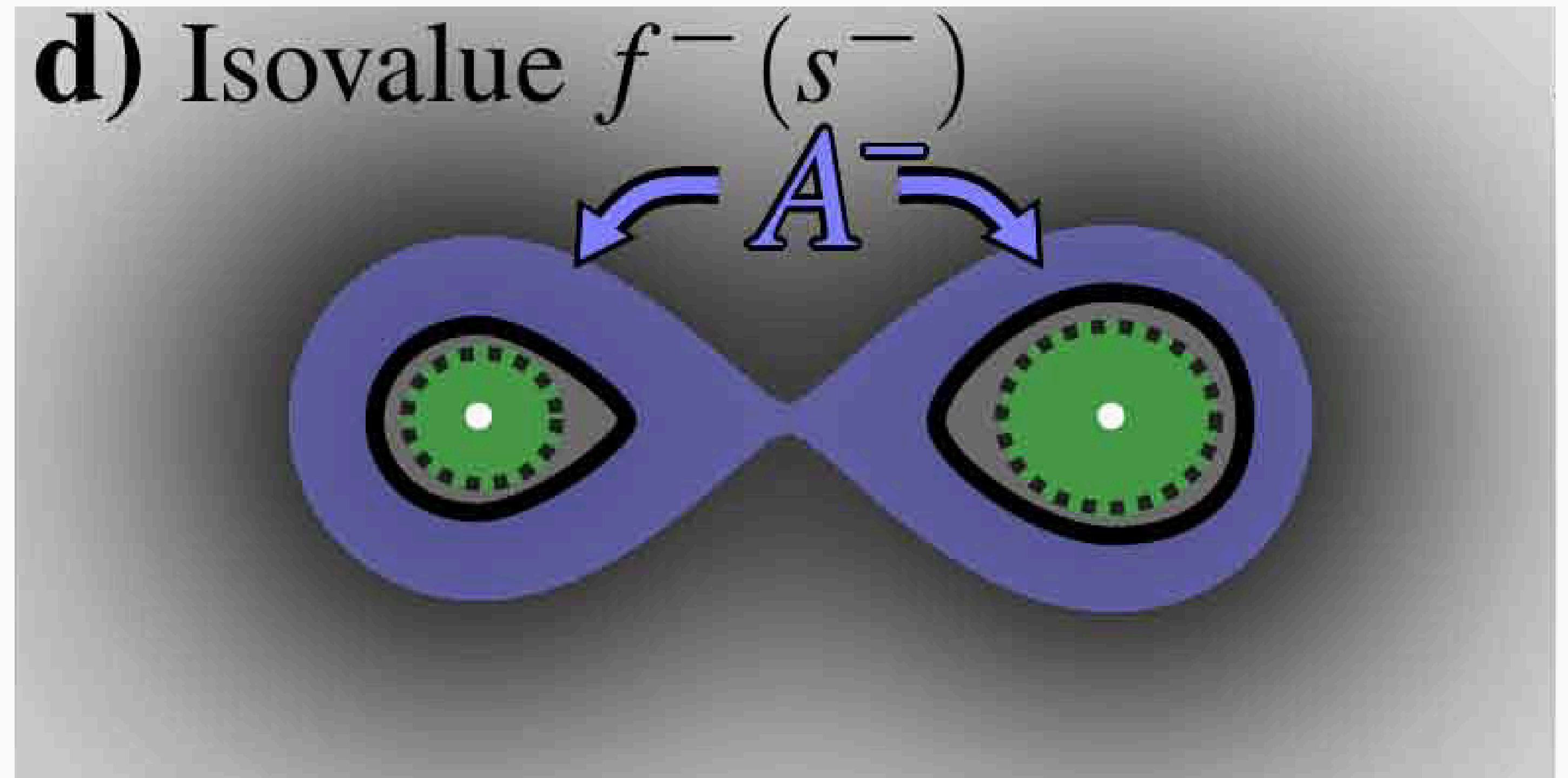
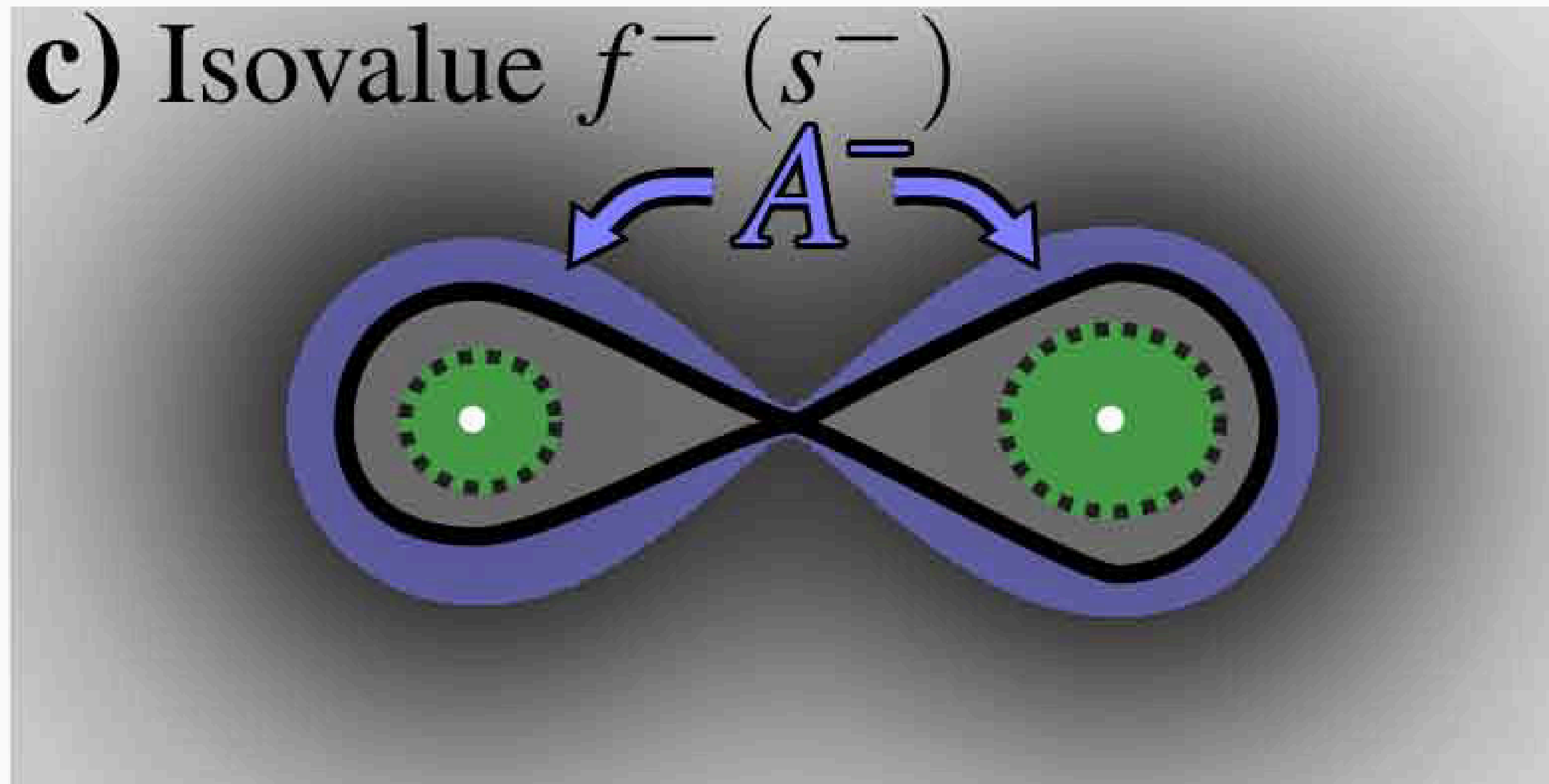


g_b

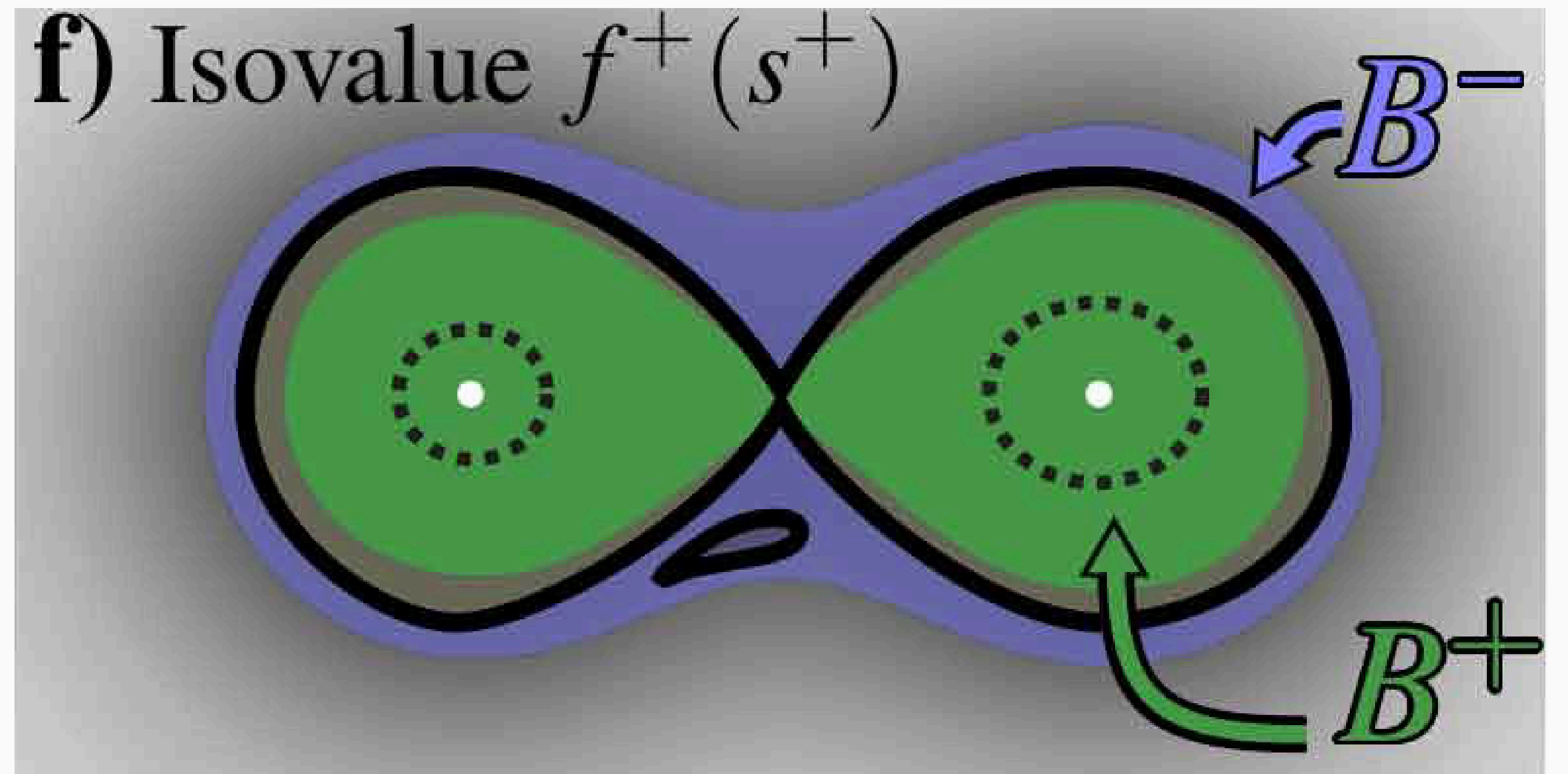
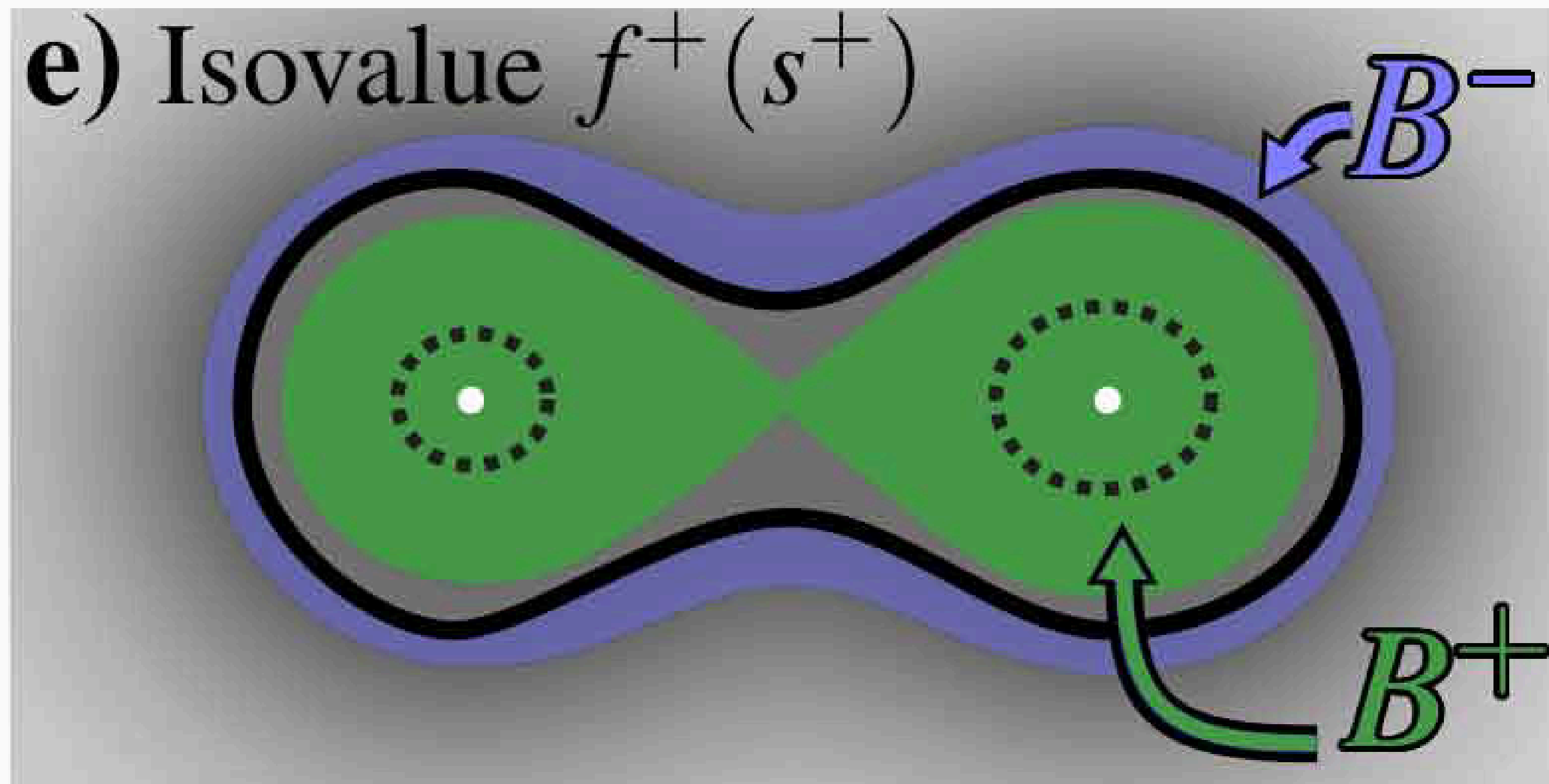
b) Isovalue ξ



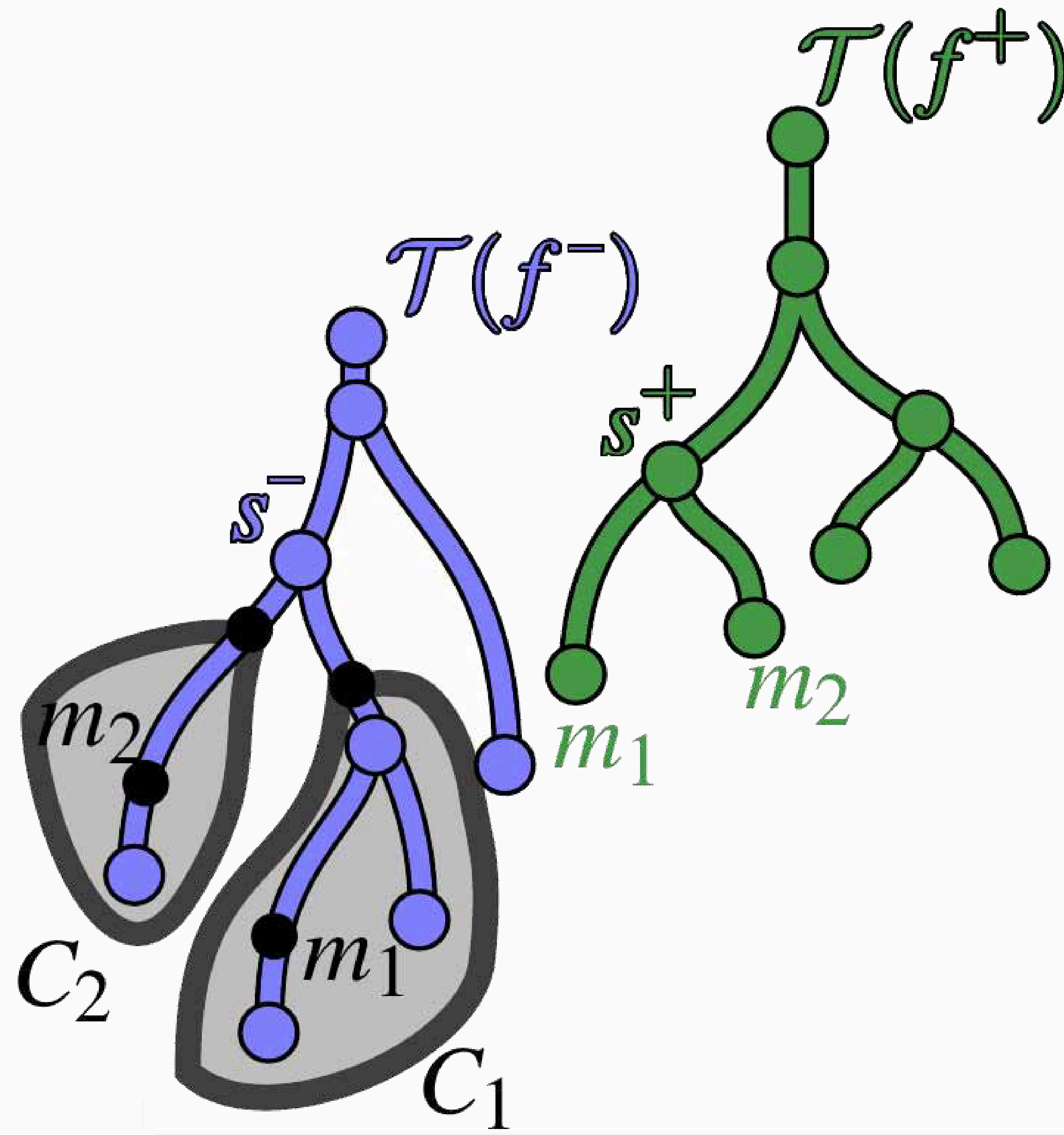
Mandatory Join Saddle: Intuition



Mandatory Join Saddle: Intuition

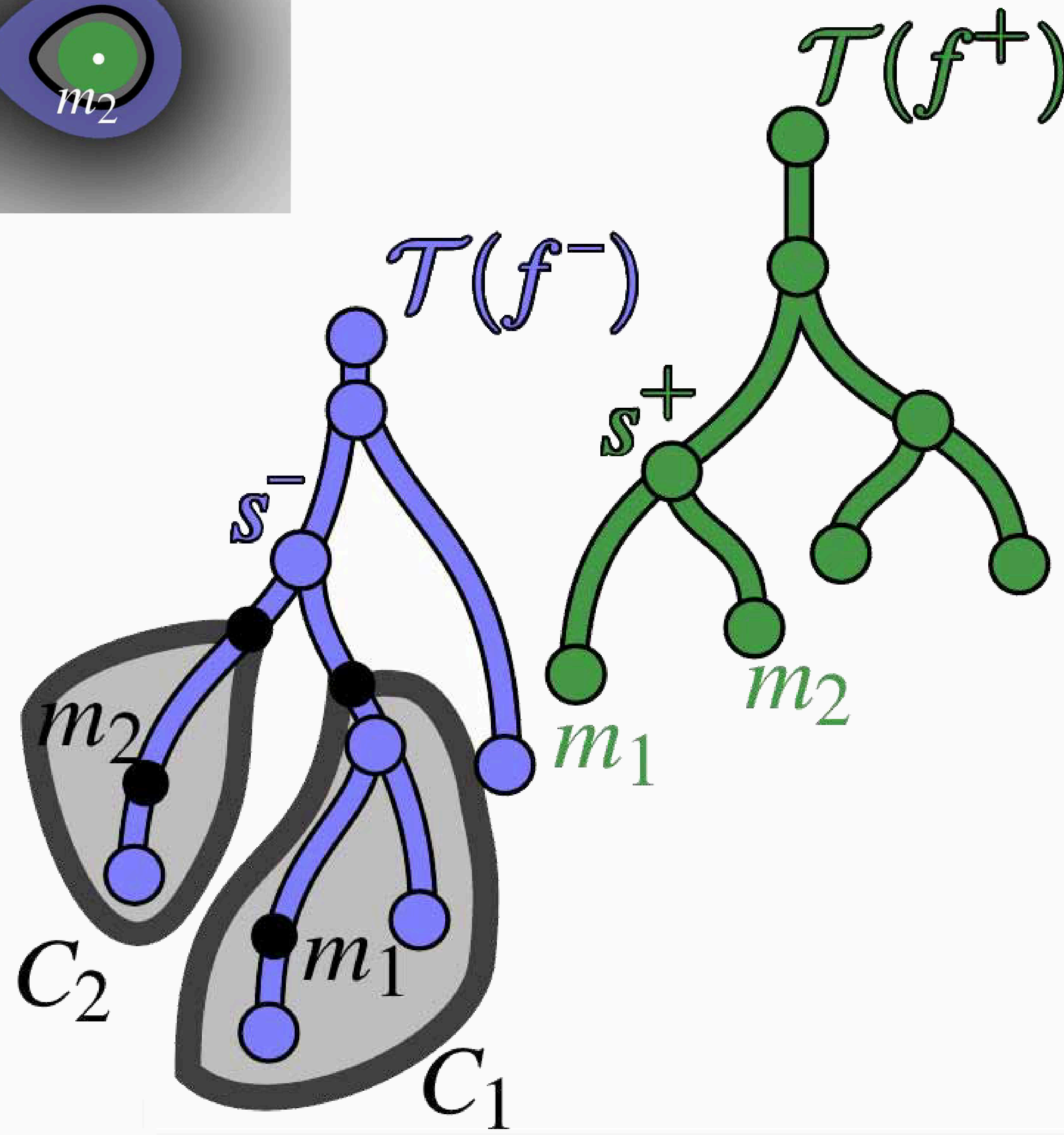
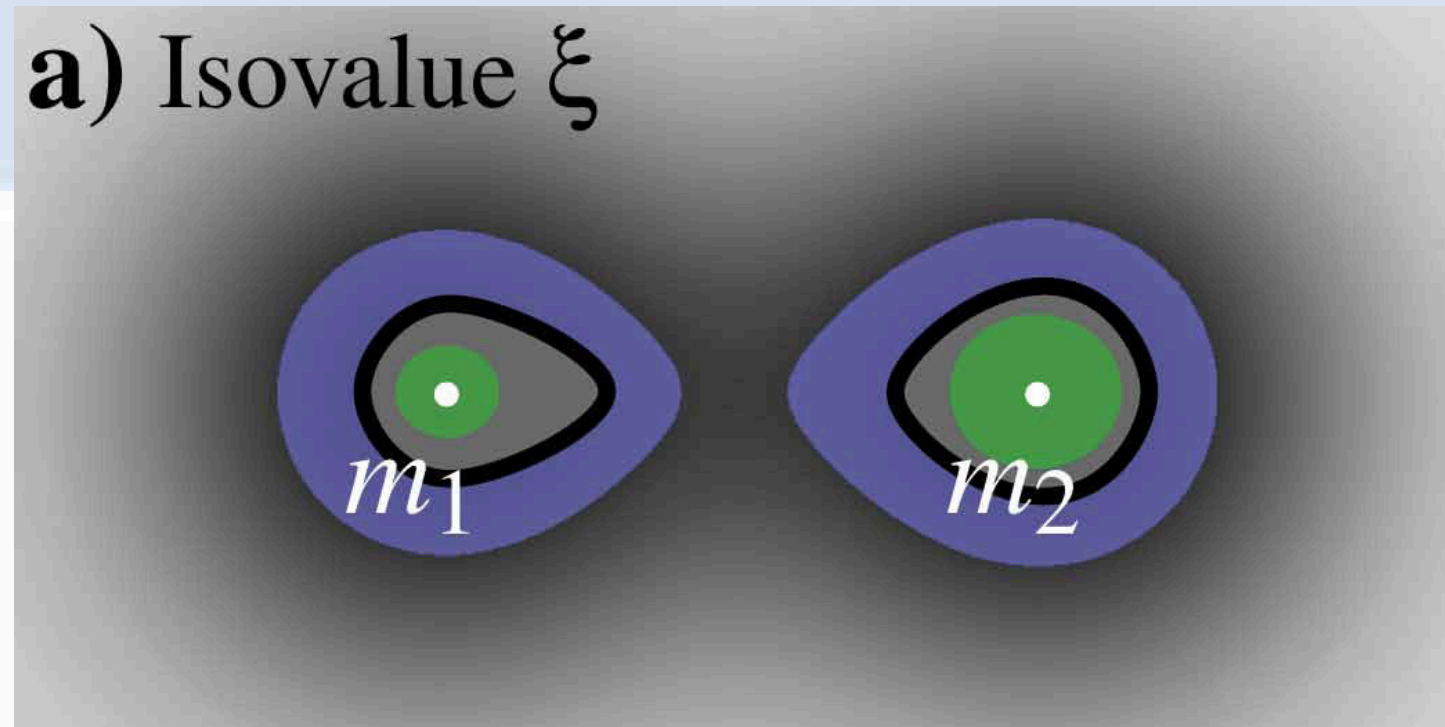


Mandatory Join Saddle: Critical Component

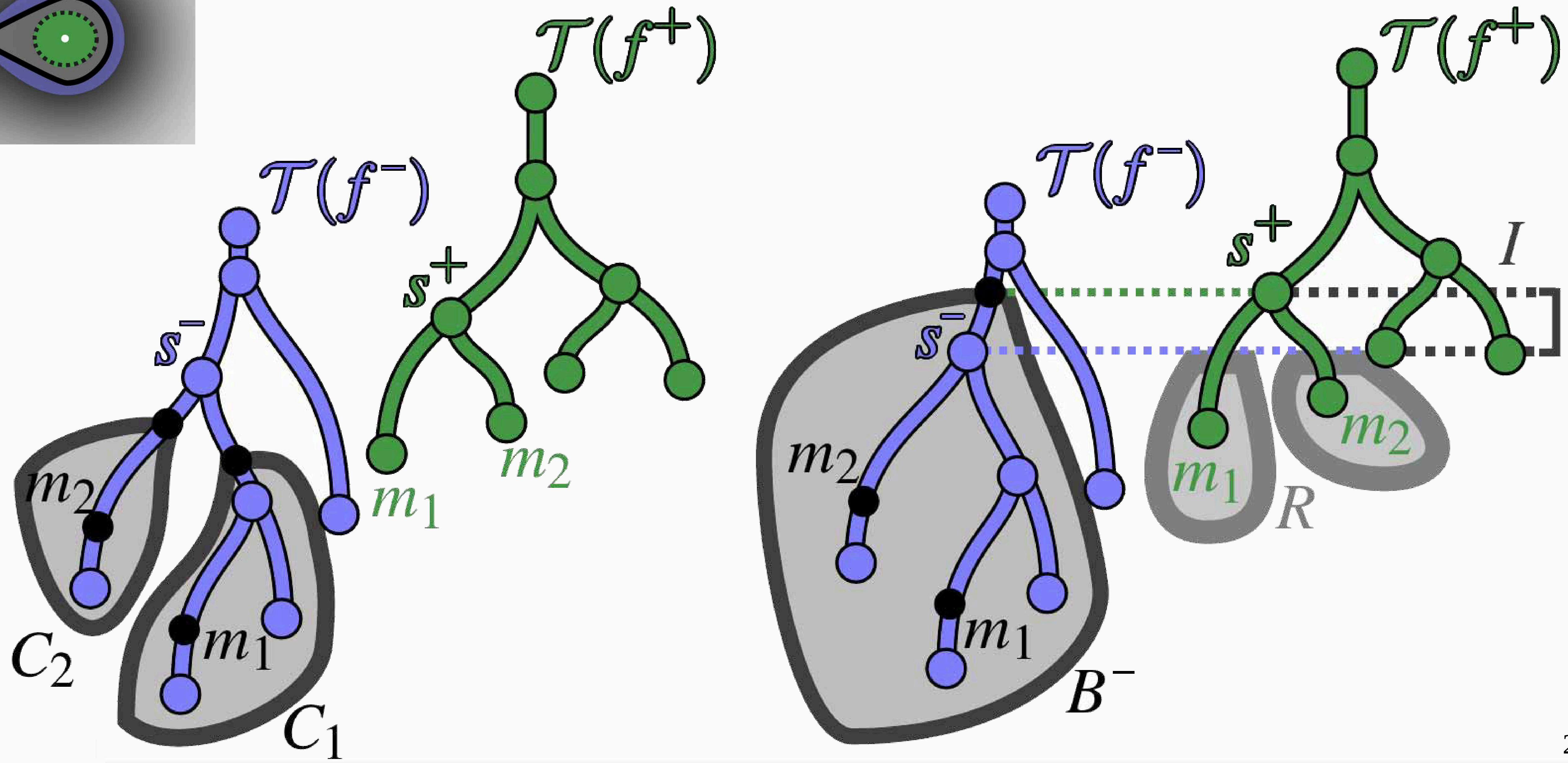
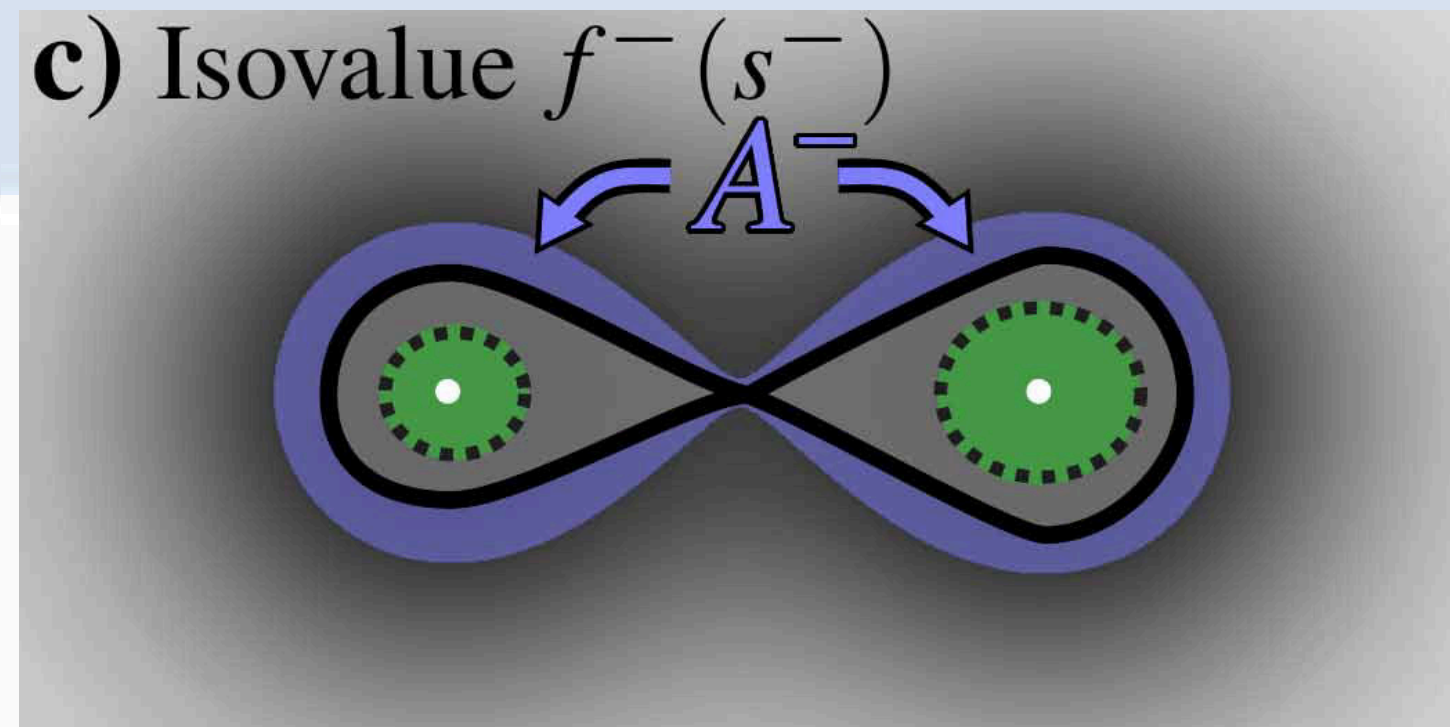


Mandatory Join Saddle: Critical Component

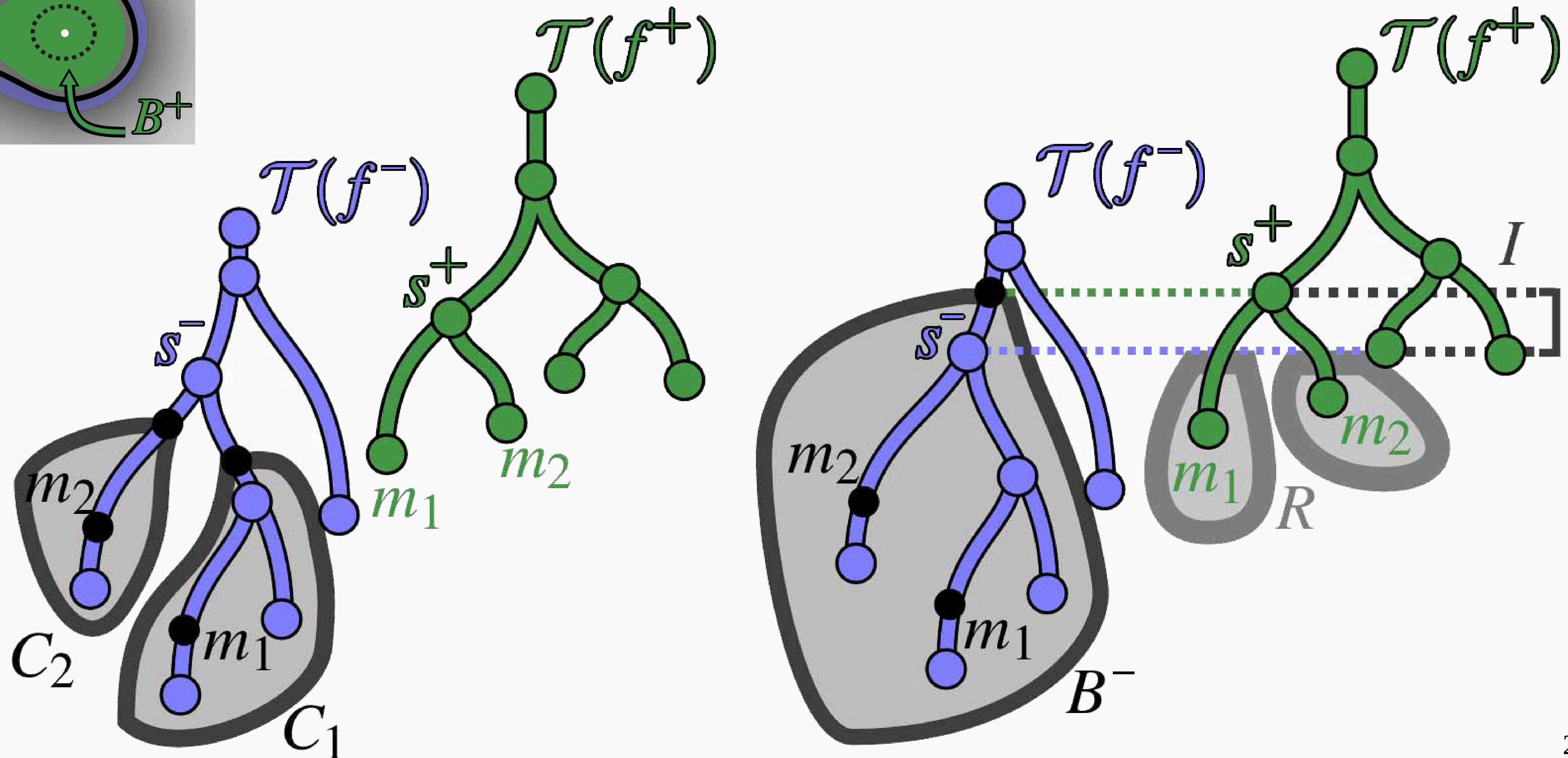
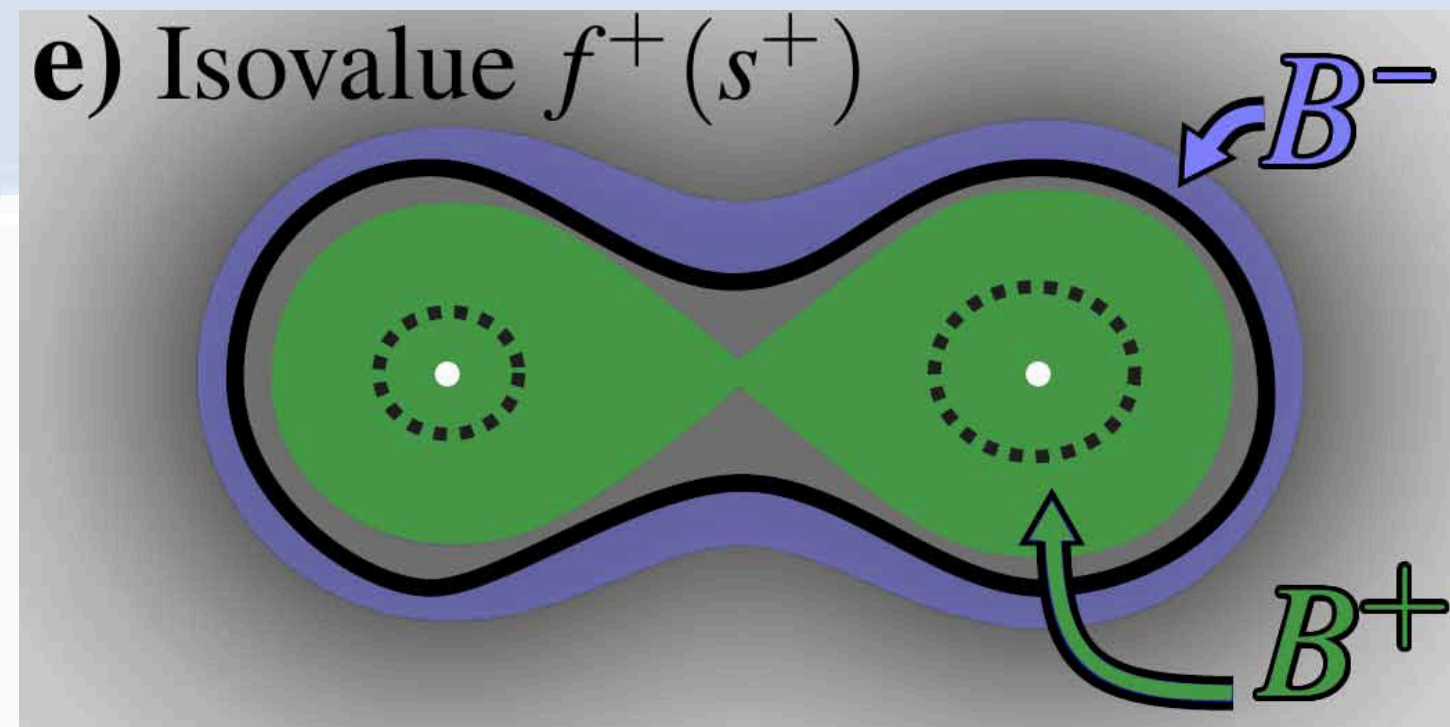
a) Isovalue ξ



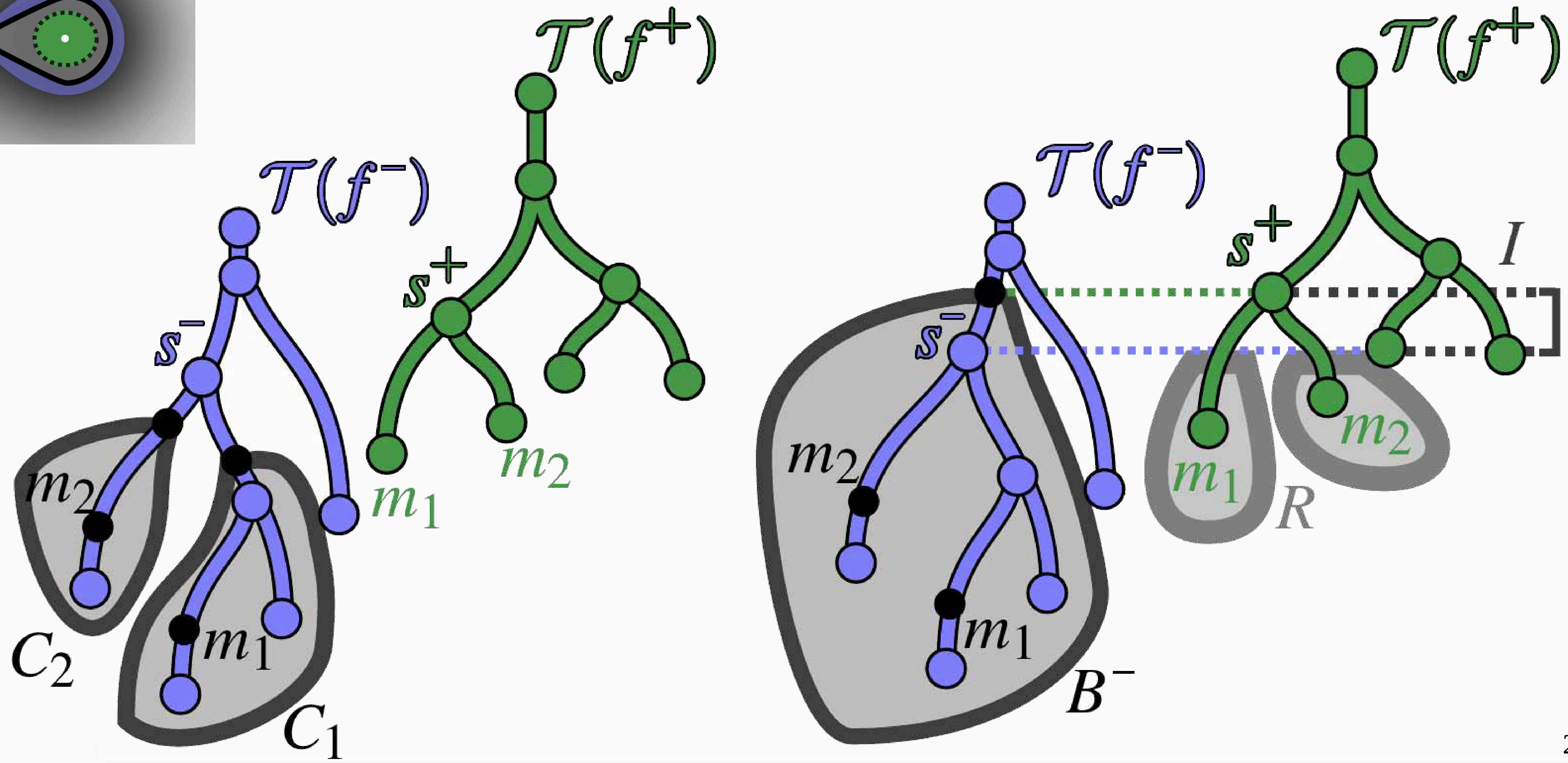
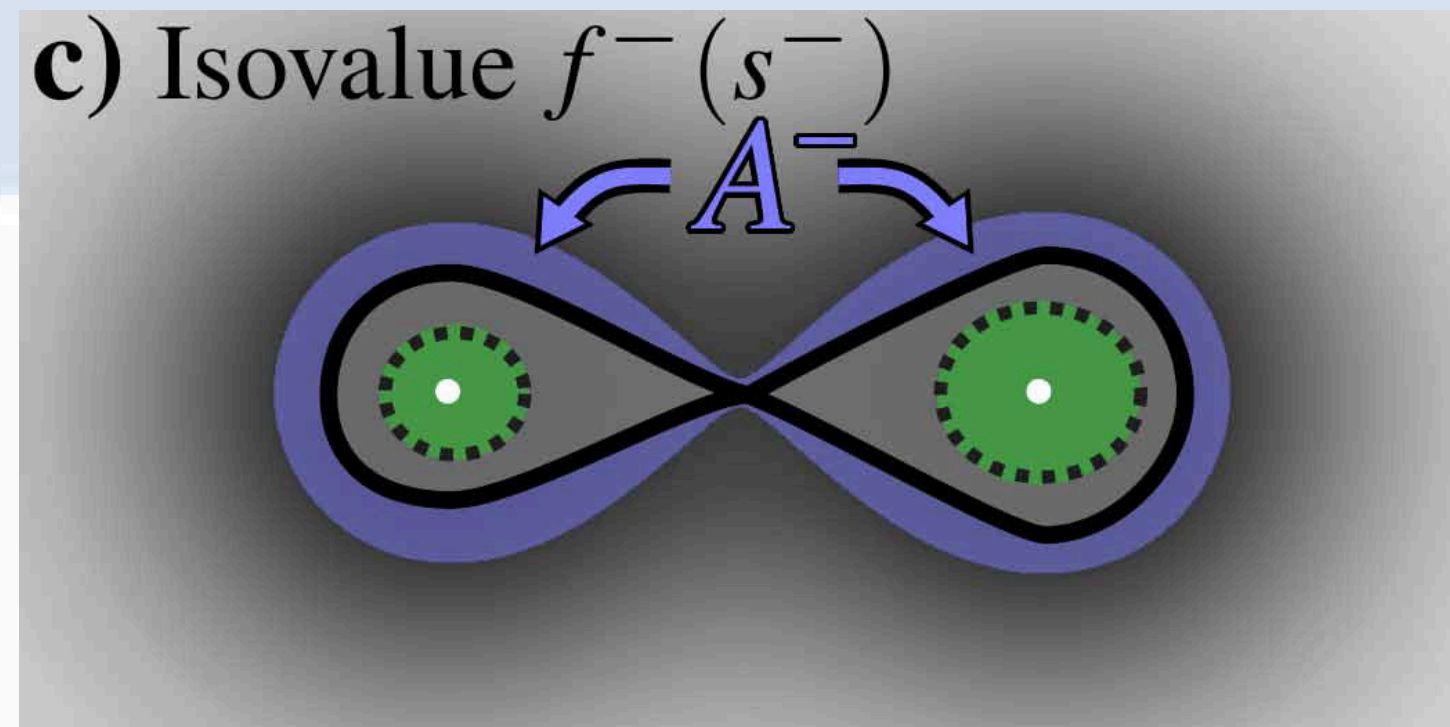
Mandatory Join Saddle: Critical Component



Mandatory Join Saddle: Critical Component



Mandatory Join Saddle: Critical Component



Uncertainty Visualization

- Leverage new visualization capabilities

Uncertainty Visualization

- Leverage new visualization capabilities
 - Positional extent of critical points
 - Value extent of critical points

Uncertainty Visualization

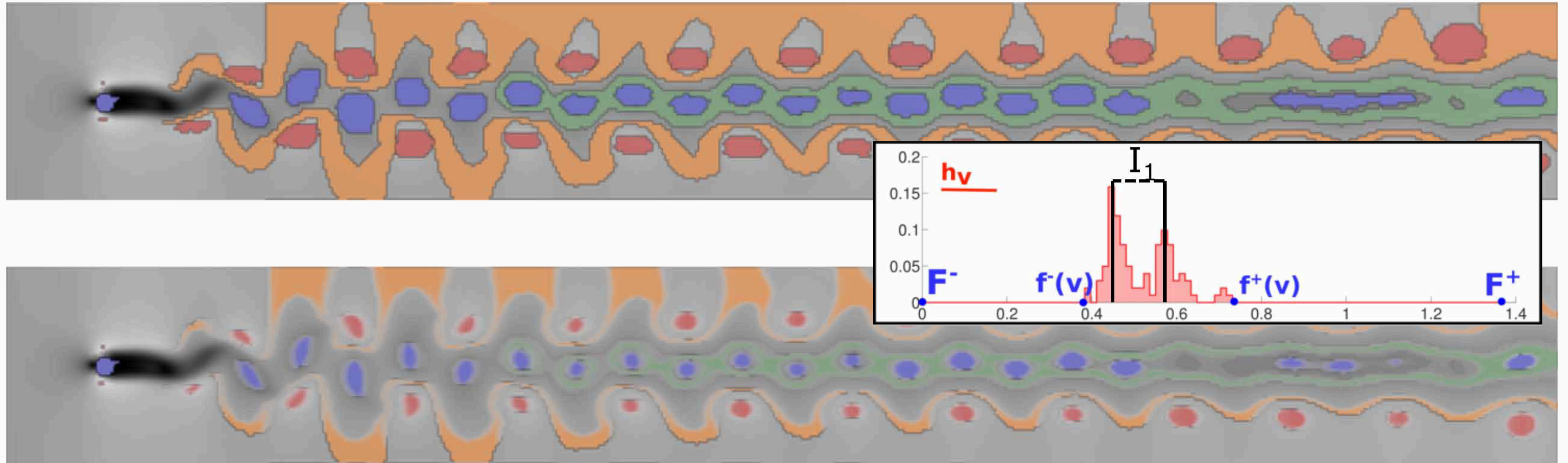
- Leverage new visualization capabilities
 - Positional extent of critical points
 - Value extent of critical points
 - Structural representation

Uncertainty Visualization

- Leverage new visualization capabilities
 - Positional extent of critical points
 - Value extent of critical points
 - Structural representation
 - Multi-scale exploration

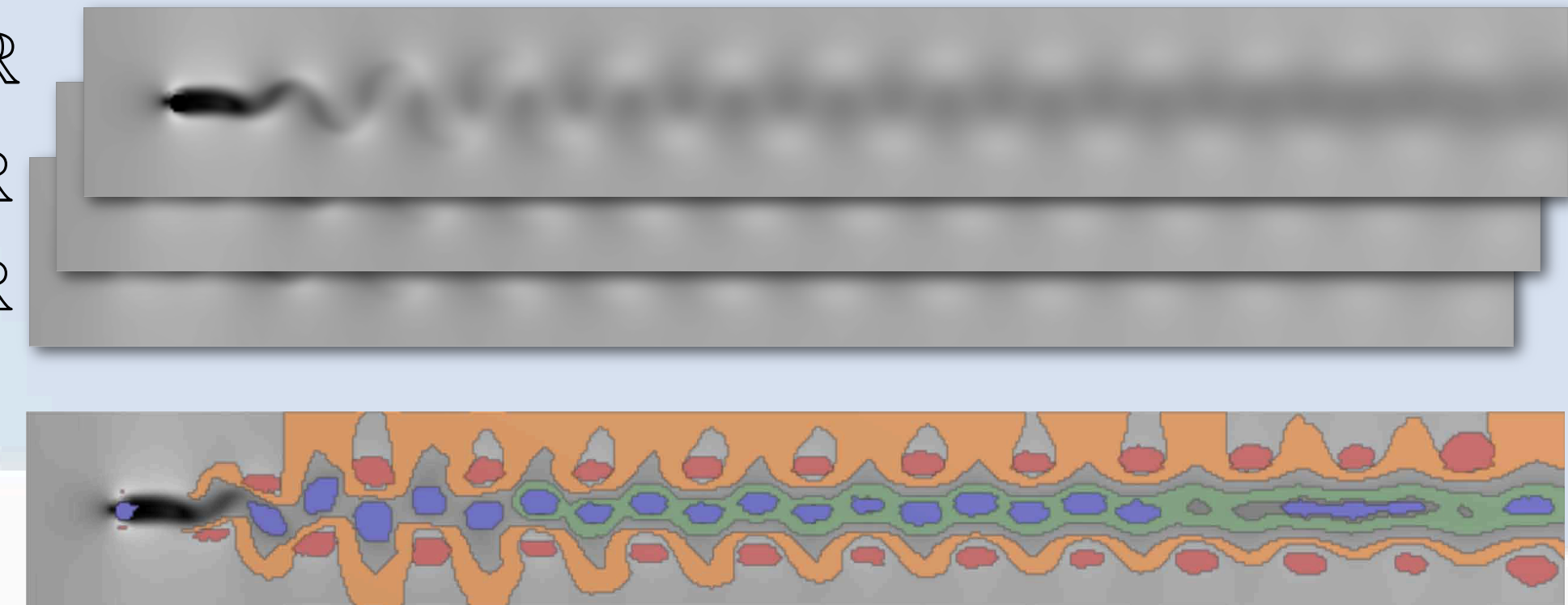
Critical Components

- Minima
- Join-saddles
- Split-saddles
- Maxima

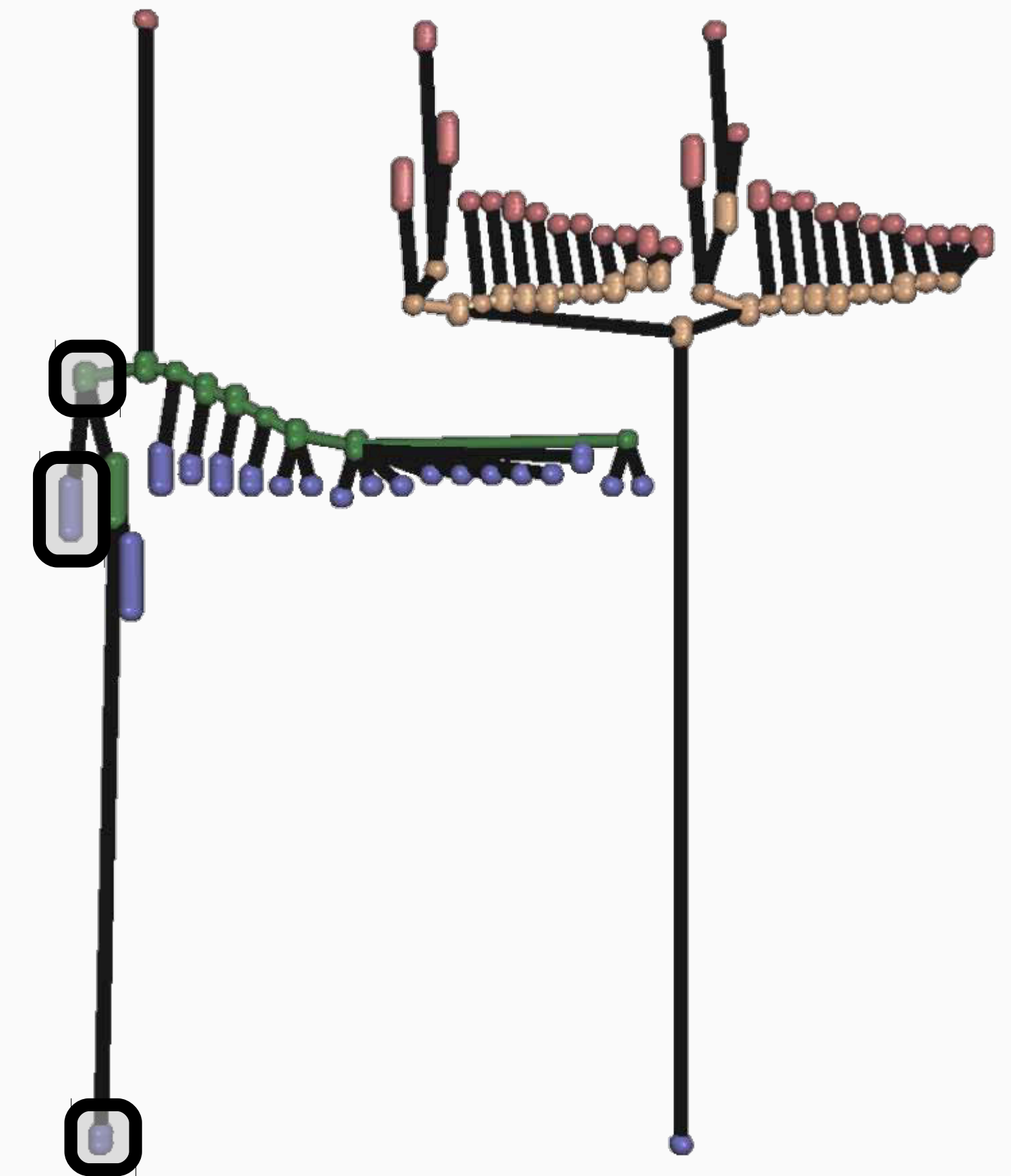


Experimental Verification

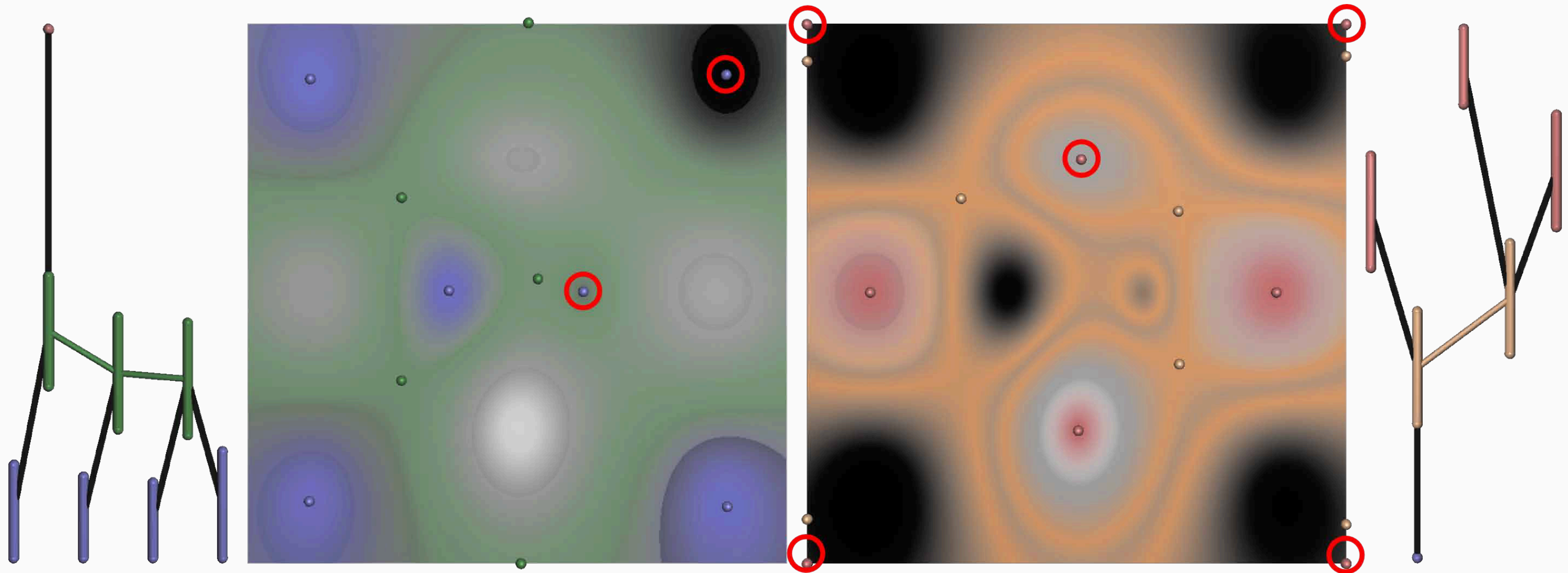
$$\begin{aligned} f_n &: \mathcal{D} \rightarrow \mathbb{R} \\ \vdots \\ f_1 &: \mathcal{D} \rightarrow \mathbb{R} \\ f_0 &: \mathcal{D} \rightarrow \mathbb{R} \end{aligned}$$



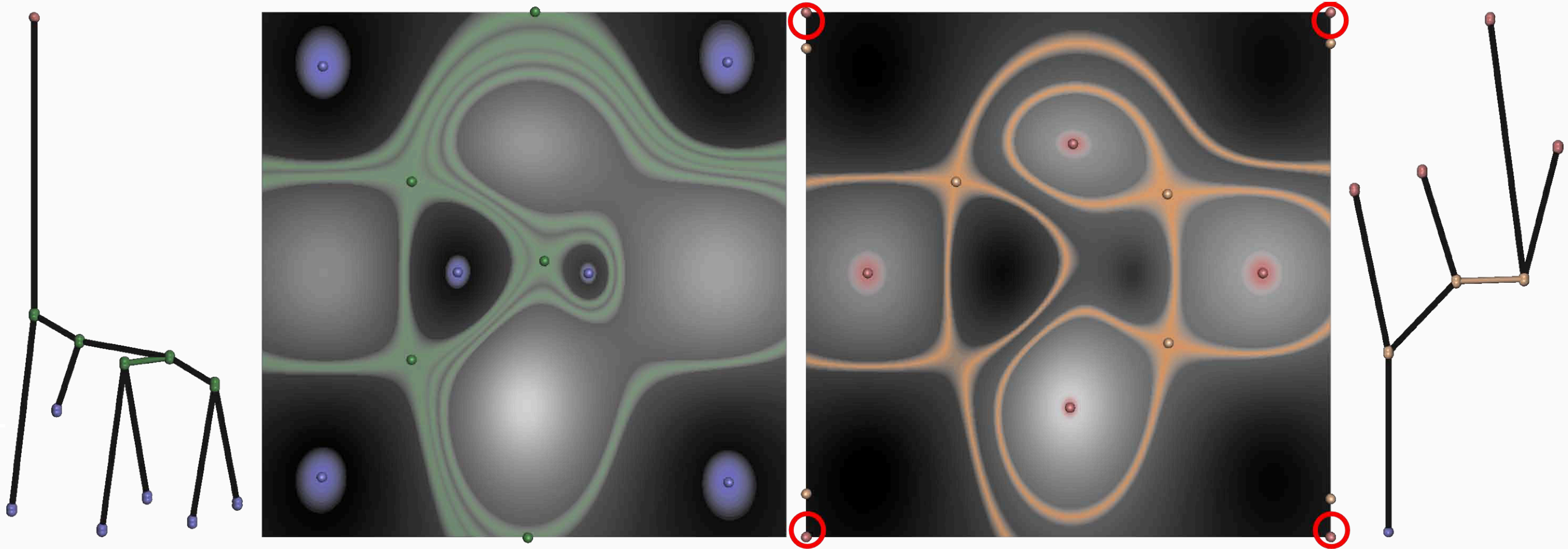
- 5000 random realization fields for each data-set
- Mandatory critical point verification
 - At least one critical point?
 - In the predicted component **ok!**
 - In the predicted interval **ok!**
- Mandatory join tree verification
 - Given a triplet (M_1, M_2, S)
 - Predicted triplet (m_1, m_2, s) ? **ok!**



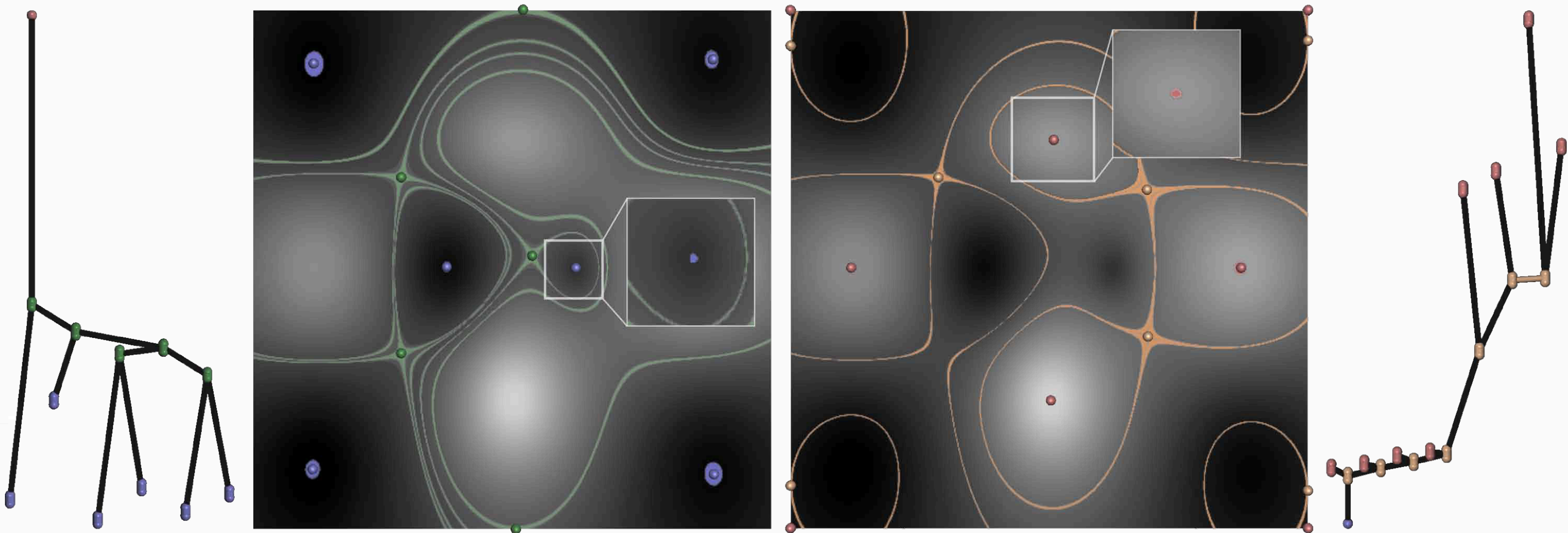
Empirical Convergence



Empirical Convergence

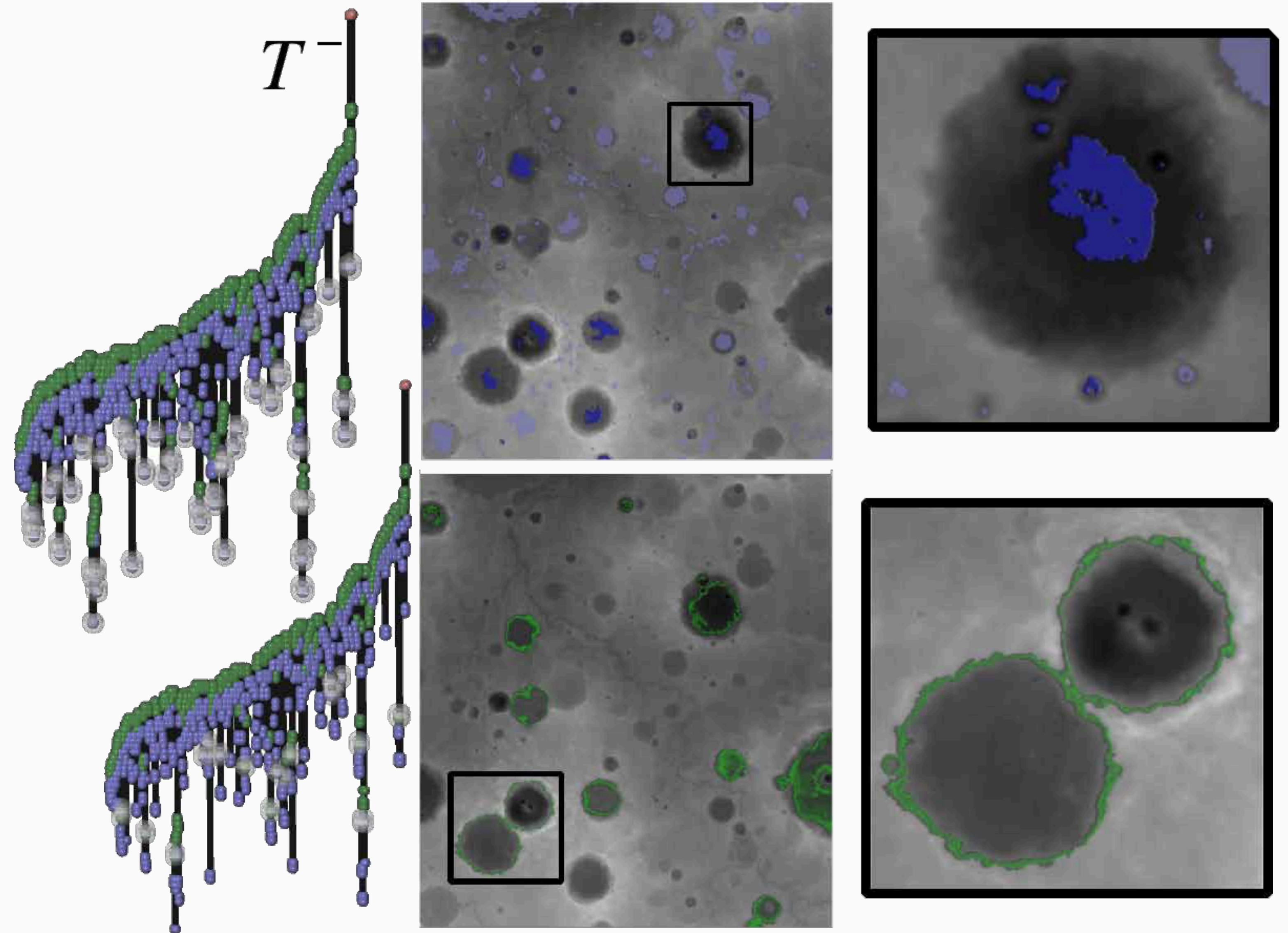


Empirical Convergence

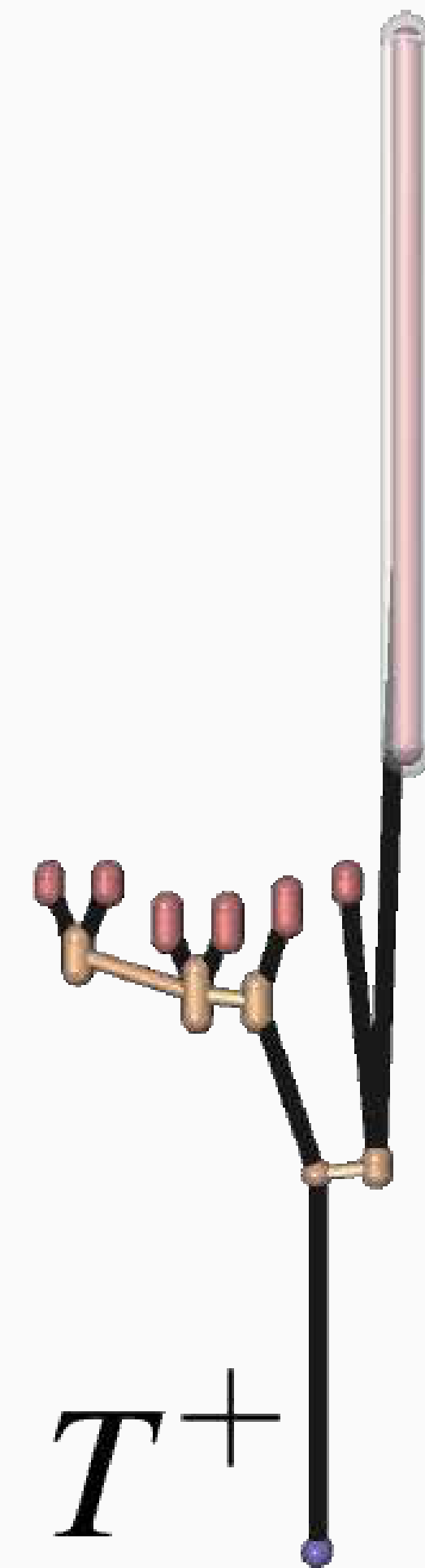
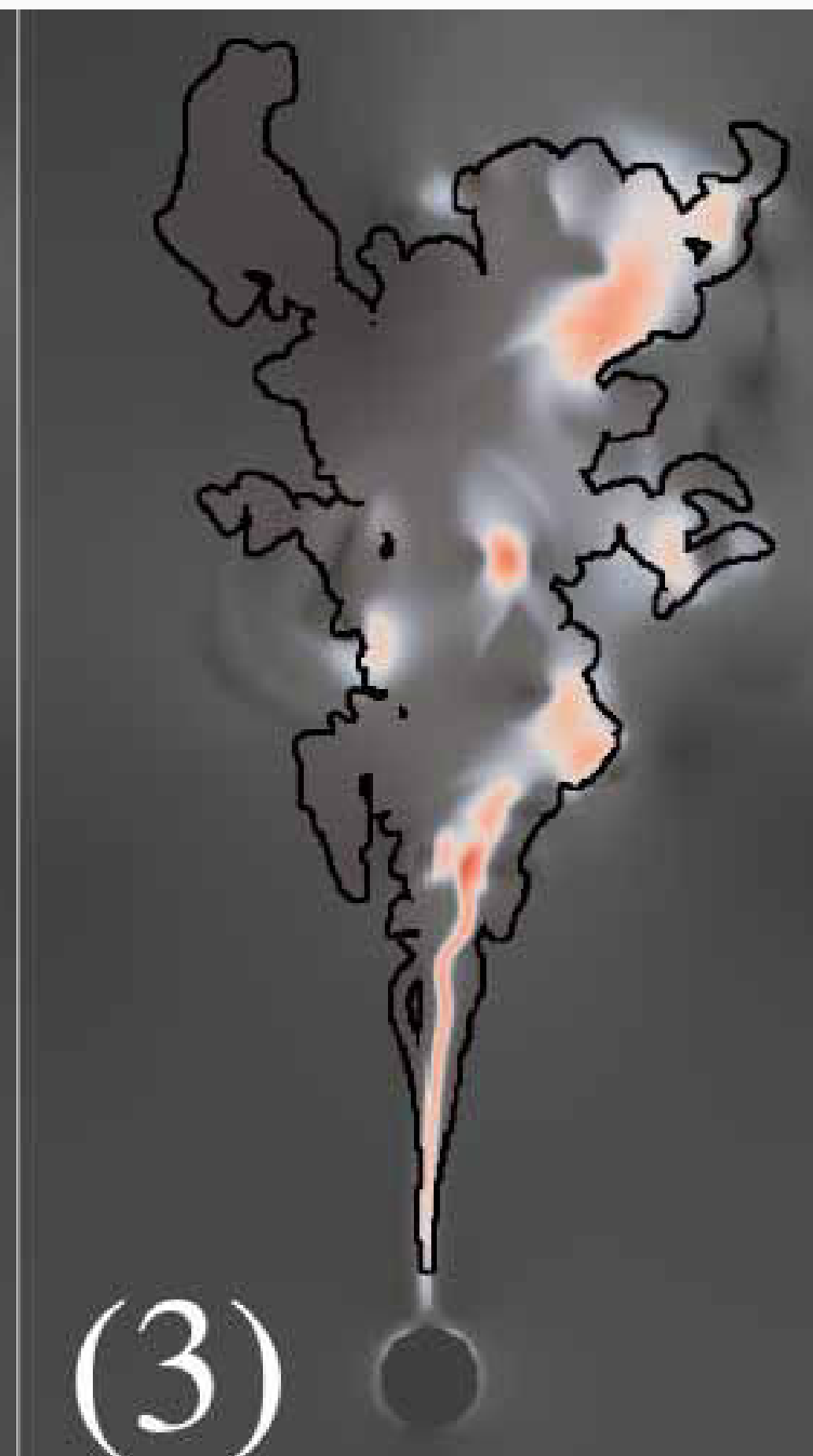
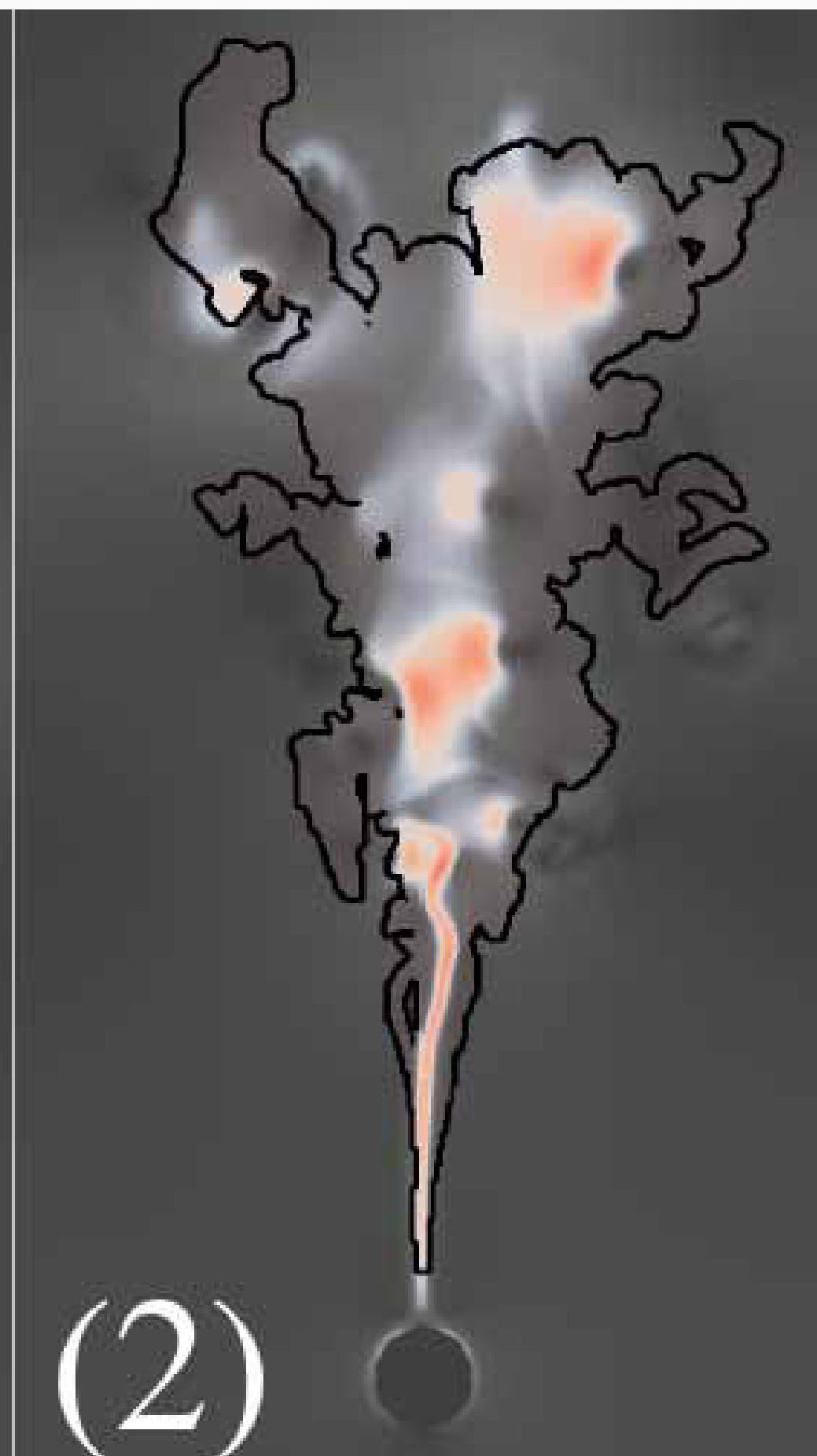
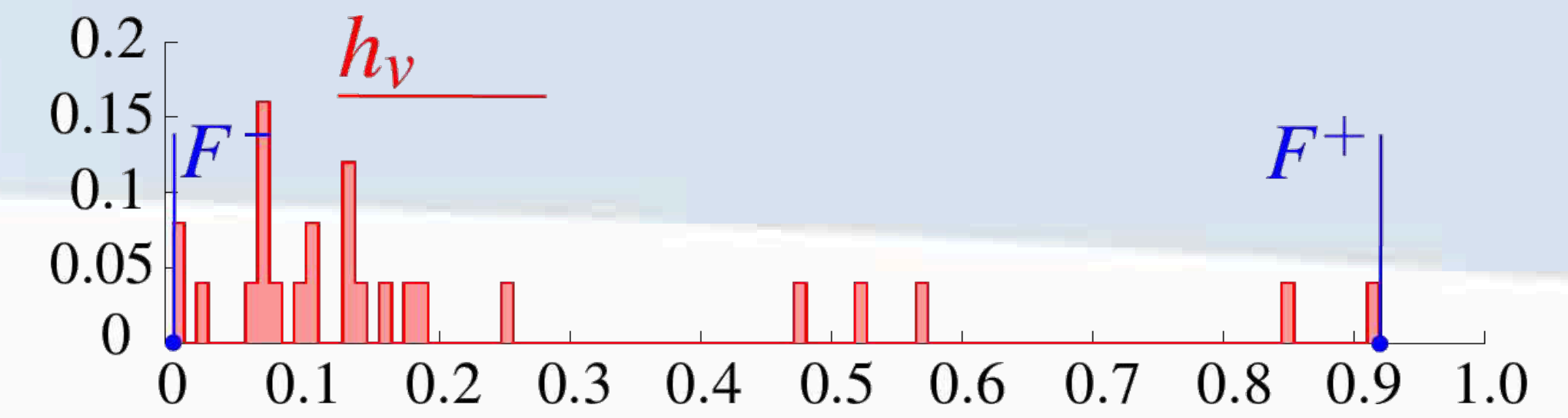


Experiments: Martian Elevation Map

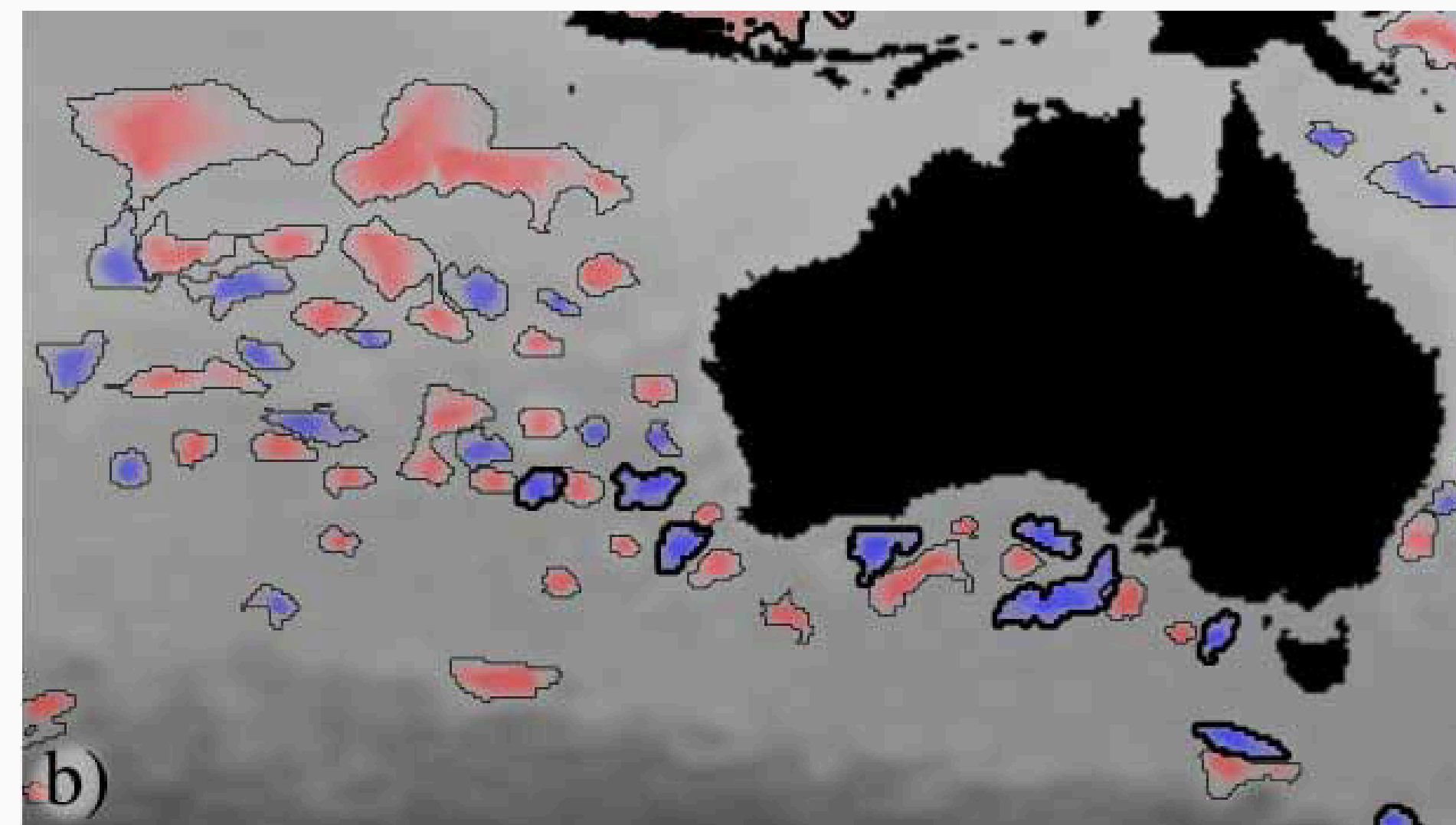
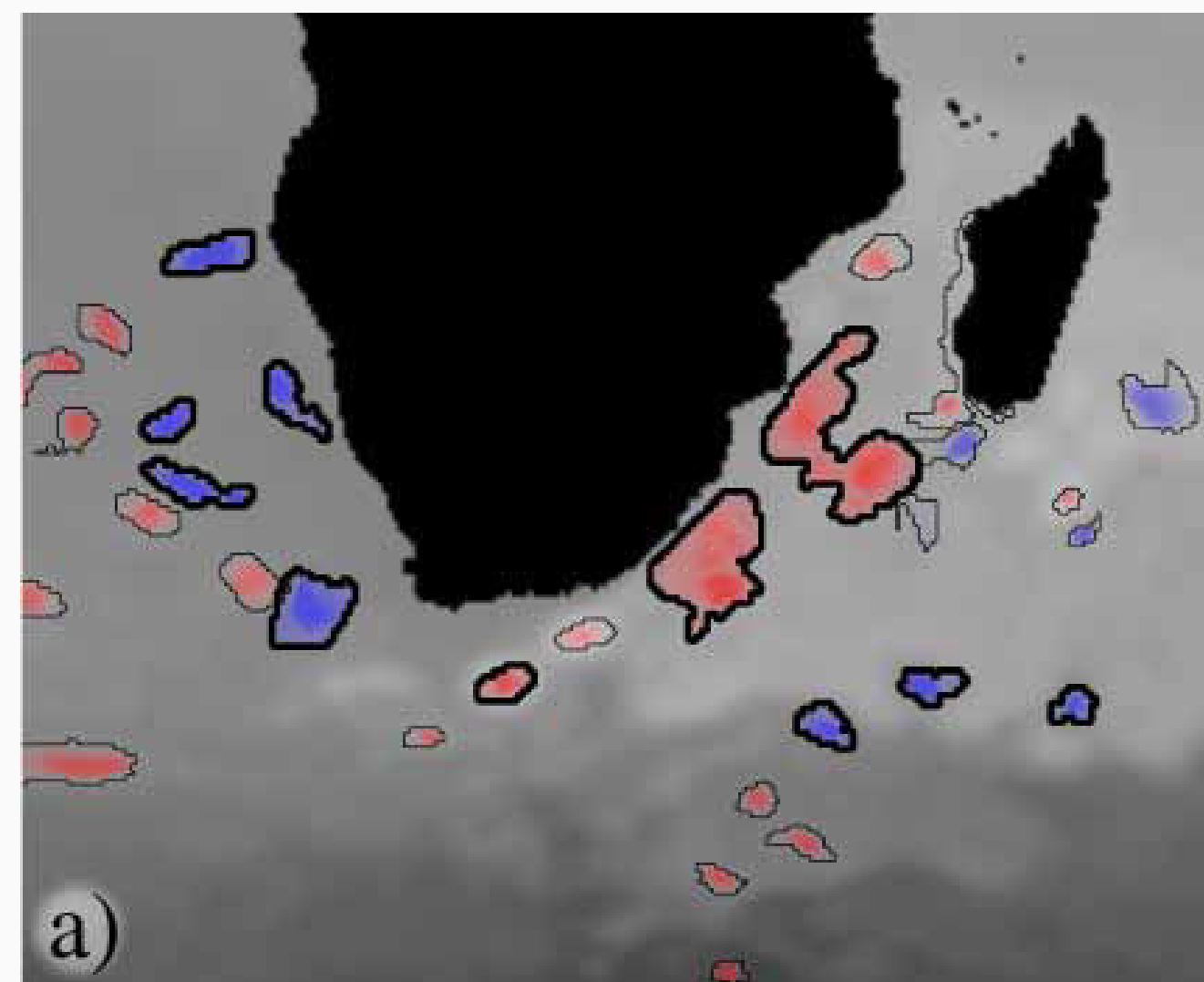
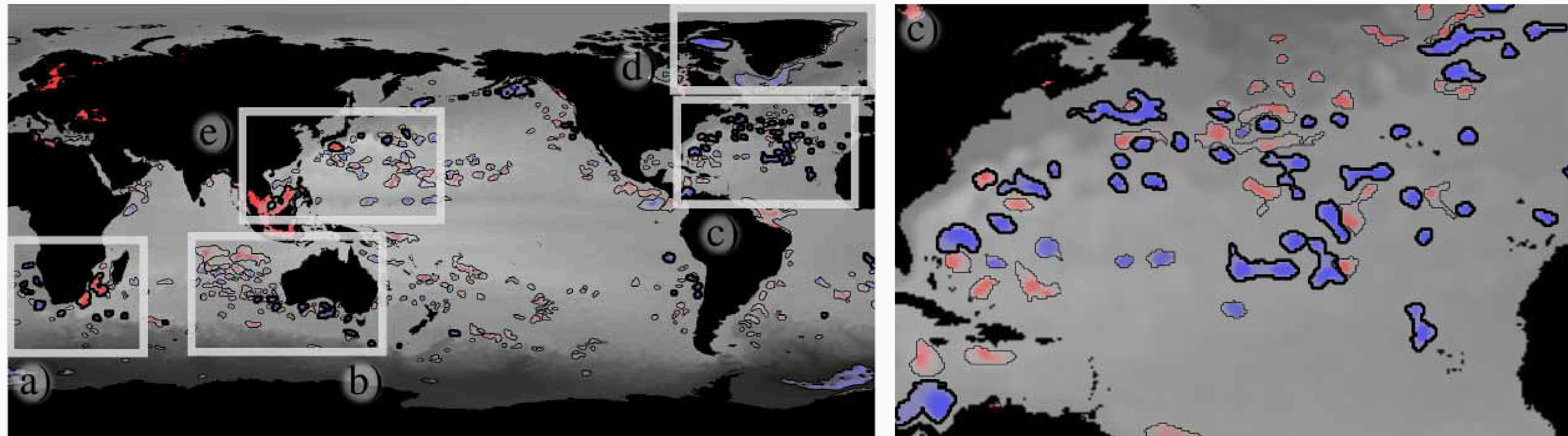
- Elevation error:
 - 40m (1%)
- Positional uncertainty
 - Craters
 - Rims



Experiments: Heated Cylinder



Experiments: Sea Surface Height



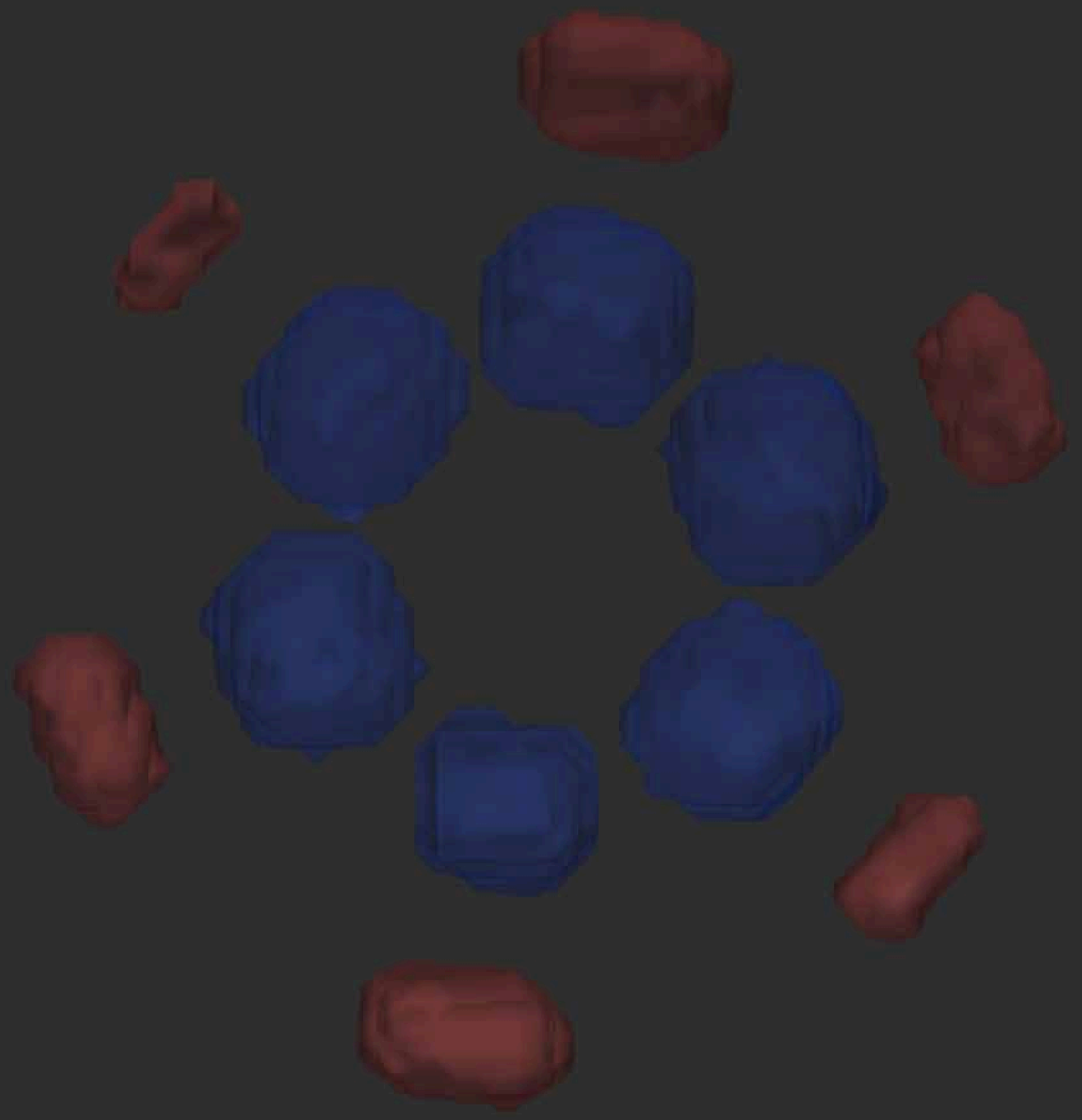
Time Performance

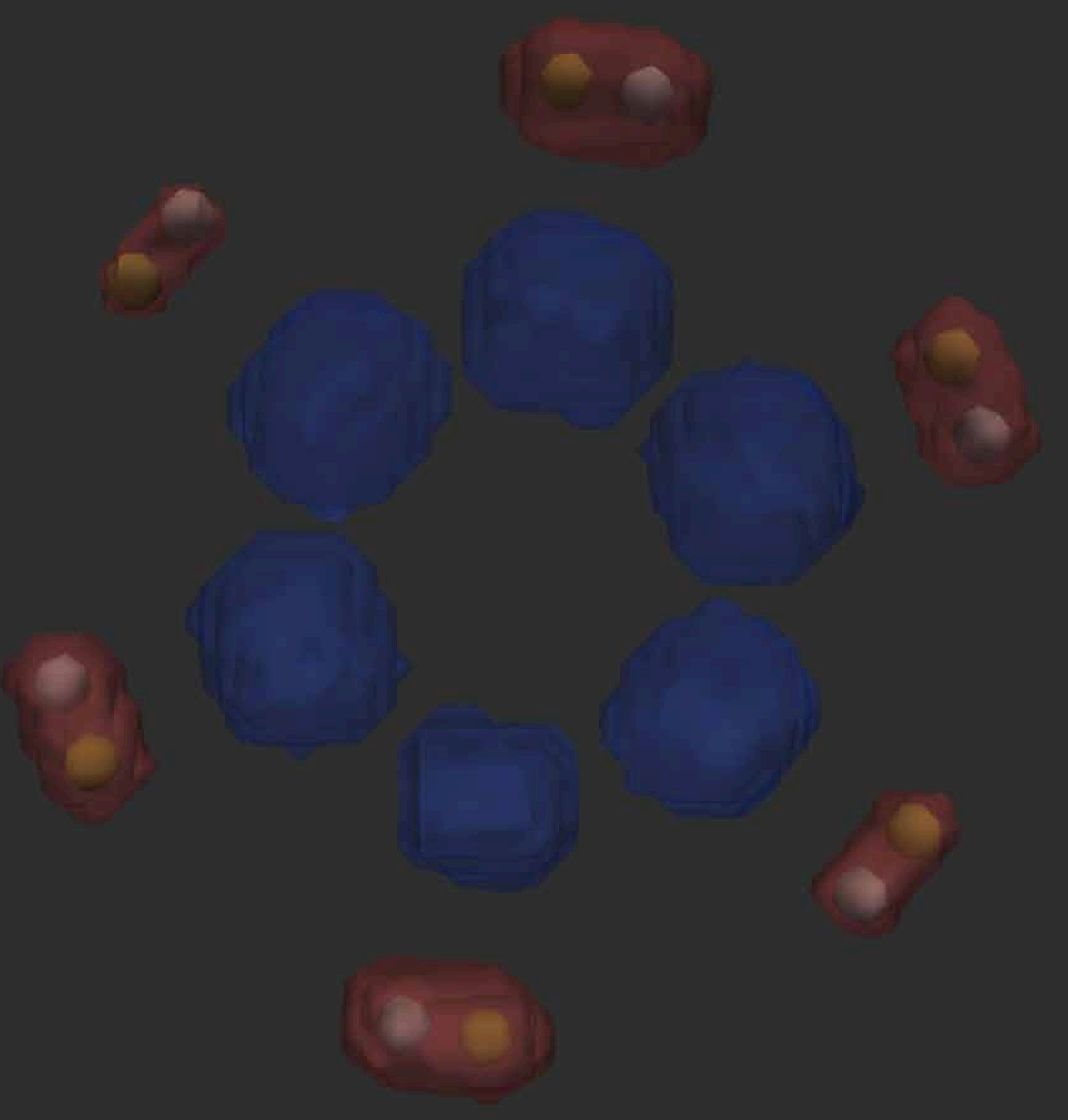
- Time complexity
 - $O(ne^2)$

Data set	n	N	e	s	f^-, f^+ (sec)	E.C. (sec)	S.C. (sec)	Overall (sec)	O.V. (sec)	R.V. (sec)
Vortex Street	33,345	100	47	35	0.1	< 0.1	< 0.1	0.1	0.2	0.5
Heated Cylinder	197,633	25	8	5	0.4	0.1	< 0.1	0.5	0.5	4.7
Synthetic	262,144		14	12	0.5	0.9	0.1	1.5	0.9	0.9
Sea Surface Height	1,036,800	31	860	235	2.1	0.6	18.0	20.7	10,073.2	76,760.7
Mars	1,476,993		995	784	4.0	1.5	130.3	135.8	81,480.6	> 86,400.0

Limitations

- Key assumption
 - Compact support with reliable estimation
- Time complexity
- Probable critical points?
- Volumetric data?





Conclusion

- Overview of Topological Data Analysis for Visualization
 - Critical Points
 - Persistence
 - Structural representations
- Extension to the uncertain setting
- Perspectives: inSitu, multi-variate, vectors and tensors

Questions?

- Thanks!

Related publications

- V. Pascucci, X. Tricoche, H. Hagen, **J. Tierny**, (Editors) “*Topological Data Analysis and Visualization: Theory, Algorithms and Applications*”, **Springer**, 2010.
- A. Gyulassy, D. Guenther, J. Levine, **J. Tierny**, V. Pascucci, “*Conforming Morse-Smale Complexes*”, **IEEE Transactions on Visualization and Computer Graphics (Proc. Of IEEE VIS)**, 2014
- D. Guenther, R. Alvarez-Boto, J. Contreras, J.P. Piquemal, **J. Tierny**, “*Characterizing Molecular Interactions in Chemical Systems*”, **IEEE Transactions on Visualization and Computer Graphics (Proc. Of IEEE VIS)**, 2014
- D. Guenther, J. Salmon, **J. Tierny**, “*Mandatory Critical Points of 2D Uncertain Scalar Fields*”, **Computer Graphics Forum (Proc. Of EuroVis)**, 2014
- **J. Tierny**, V. Pascucci, “*Generalized Topological Simplification of Scalar Fields on Surfaces*”, **IEEE Transactions on Visualization and Computer Graphics (Proc. Of IEEE VIS)**, 2012
- **J. Tierny**, J. Daniels, G. Nonato, V. Pascucci, C. Silva, “*Interactive Quadrangulation with Reeb Atlases and Connectivity Textures*”, **IEEE Transactions on Visualization and Computer Graphics**, 2011
- T. Etienne, G. Nonato, C. Scheidegger, **J. Tierny**, T. Peters, V. Pascucci, M. Kirby, C. Silva, “*Topology Verification for Isosurface Extraction*”, **IEEE Transactions on Visualization and Computer Graphics (presented at IEEE VIS)**, 2011
- T. Bremer, G. Weber, **J. Tierny**, V. Pascucci, M. Day, J. Bell, “*Interactive Exploration and Analysis of Large-Scale Simulations Using Topology-based Data Segmentation*”, **IEEE Transactions on Visualization and Computer Graphics**, 2010
- **J. Tierny**, A. Gyulassy, E. Simon, V. Pascucci, “*Loop Surgery for Volumetric Meshes: Reeb Graphs Reduced to Contour Trees*”, **IEEE Transactions on Visualization and Computer Graphics (Proc. Of IEEE VIS)**, 2009