Probabilistic sensitivity analysis: contribution to the sample mean plot and moment-independent importance measures

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Outline

Contribution to the sample mean plot

- Contribution to the sample mean plot
- Statistical test for inputs prioritisation

Moment independent sensitivity analysis

- Moment-independent importance measures
- Numerical and computational aspects
- Application examples

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Contribution to the sample mean plot for graphical and numerical sensitivity analysis

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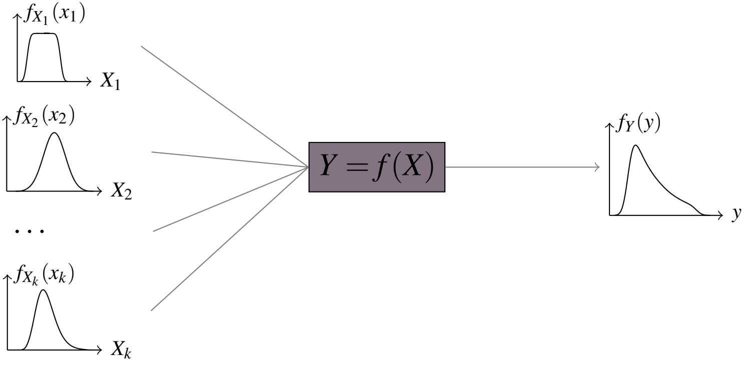
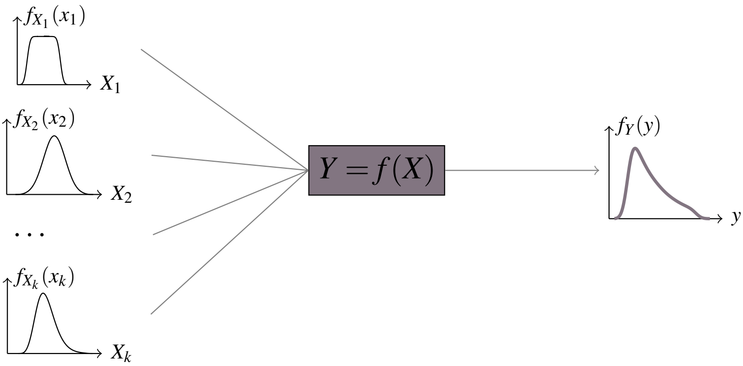


- f a deterministic scalar function
- $X = (X_1, \cdots, X_k)$ and Y random variables
- $x = (x_1, \cdots, x_k)$ realization of the model inputs X
- y realization of the model output Y



- Objective: understand the behaviour of the system with very few model runs

Overview

Context

- *Sinclair, (1993)* investigated how finite changes in inputs pdfs affect the mean and variance of the output
- Contribution to the sample mean (CSM) plot recognized as a general tool for sensitivity analysis

Objectives

- Revive and further develop CSM plot
- Exploit the full potential of this graphical tool
- CSM plot, primary building block of a statistical test for inputs prioritisation

Different steps for the construction of a CSM plot

1. realizations of X_i are sorted, generating $\{{x_i}^{(1)}, {x_i}^{(2)}, \cdots, {x_i}^{(N)}\}$;
2. realizations of Y are sorted accordingly, generating $\{y^{(i,1)}, y^{(i,2)}, \cdots, y^{(i,N)}\}$
3. new variable M_i defined by

$$m_i^{(q)} = \frac{1}{N} \sum_{j=1}^{q} y^{(i,j)} \quad q = 1, \cdots, N$$

4. normalization of the M_i using the sample mean of Y;
5. plot M_i against $F_{X_i}(x_i)$

Underlying features

- For both axes, values lie in the interval $[0,1]$
- $(F_{X_i}(x_i^{(q)}), m_i^q)$: fraction of the output mean due to any given fraction of values of the input X_i.

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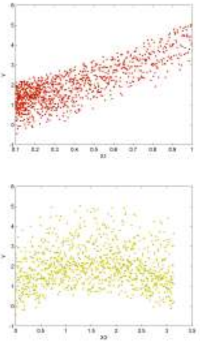
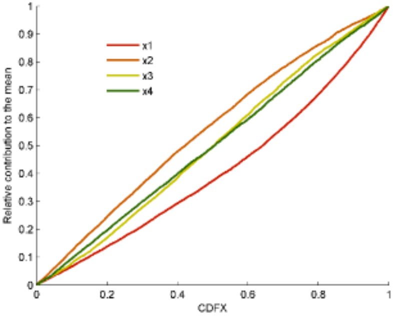
Didactic example

All parameters, single output



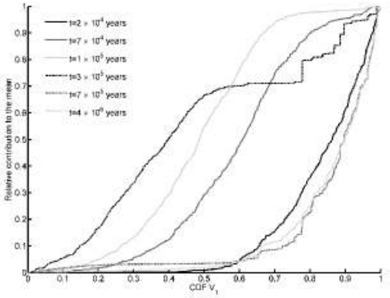


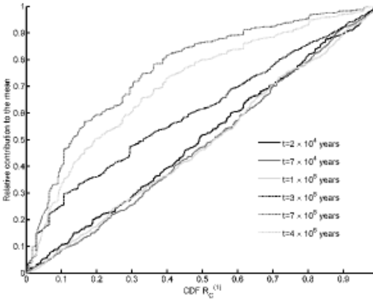
Analytic function

- $Y = 2\exp(X_1) - \exp(X_2) + sin(X_3)$
- X_i, $i = 1,2,4 \sim U(0,1)$, $X_3 \sim U(0,\pi)$

High-level waste repository model (LevelE)

Single parameter, several outputs

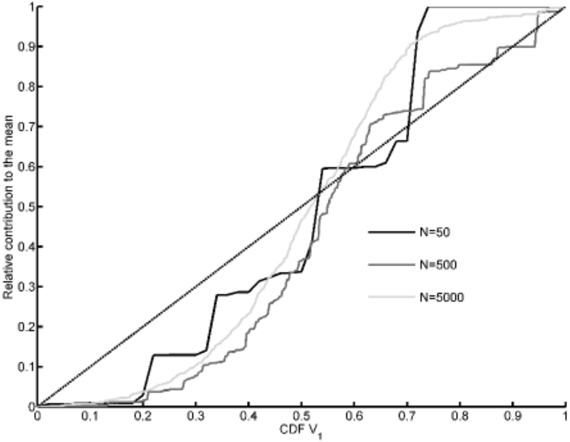




Information provided by the plot

- Effects on the mean of the output of changes in the inputs pdfs
- Underlines the limitations of the sample size/design
- Global importance measures

CSM plot with increasing sample size



Prioritisation of model inputs

CSM plot

- If $F_{X_i}(x_i^{(q)}) \simeq m_i^q \ \forall q$, any quantile range of X_i has a similar influence on the output mean, i.e non-influent model input

Relation with VB

- Variance-based first-order effect

$$S_i = \frac{Var(E[Y|Xi])}{Var(Y)}$$

- CSM plot, variability of $E[Y|Xi < xi^*]$ (rather than $E[Y|Xi = xi^*]$) across the range

Statistical test keynotes

Features

- *Hypotheses* (null hypothesis H_0 and alternative hypothesis H_1):
  - H_0: $f_{Y|X_i}(y|x_i = x_i^*) = f_Y(y)\ \forall x_i^* \in R_i$ (R_i is the support of X_i);
  - H_1: $\exists x_i^*, x_i' \in R_i \quad / \quad f_{Y|X_i}(y|x_i = x_i^*) \neq f_{Y|X_i}(y|x_i = x_i')$.
- *Test statistic*: D_m, the maximum distance to the diagonal

Different steps

1. Empirical distribution of D_m
   - Random permutations of the inputs realizations
   - For each permutation, compute D_m from CSM plot
2. Compute $D_{m\alpha}$, value of the test statistic for the critical level α
3. Estimation of $D_{m_{X_i}}$ v $i = 1, \cdots, k$ from the *original* CSM plot
4. null hypothesis H_0 rejected if $D_{m_{X_i}} > D_{m\alpha}$ (i.e. X_i is an important input))

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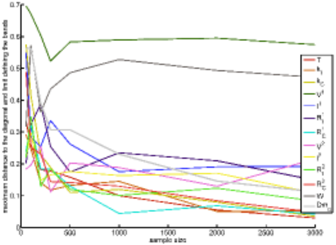
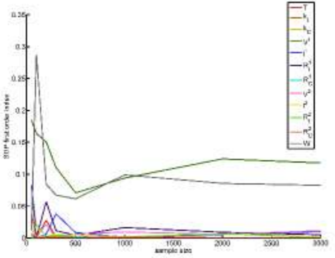
Convergence of importance measures

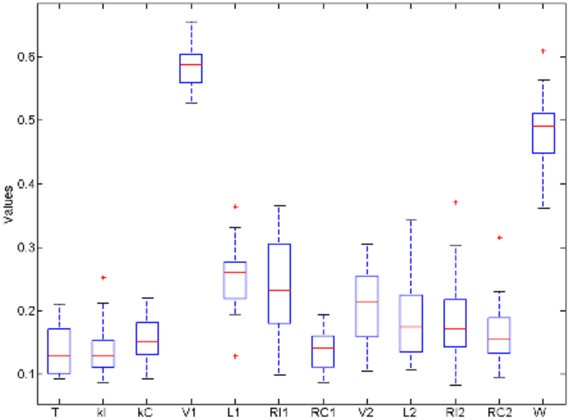
Test statistic and SDP *(Ratto et al, 2007)* first order indices (LHS samples 50-3000)





Robustness of importance measures

Test statistic across 20 LHS replicates of size 500



Conclusions

Potential

- CSM plot: simple, versatile and very informative graphical tool
- Statistical test: identifies important model inputs for very low sample size, no additional model run for robustness analysis

Limitations

- Inputs prioritisation assessment restricted to first order effects
- Statistical test prone to type I error (treating non-influential inputs as important)

To be done ...

- Systematic approach for non-monotonic mappings
- Second order interactions with surfaces
- Investigate the potential of the contribution to the sample variance (CSV) plot

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Relative importance of model inputs on the output probability distribution function

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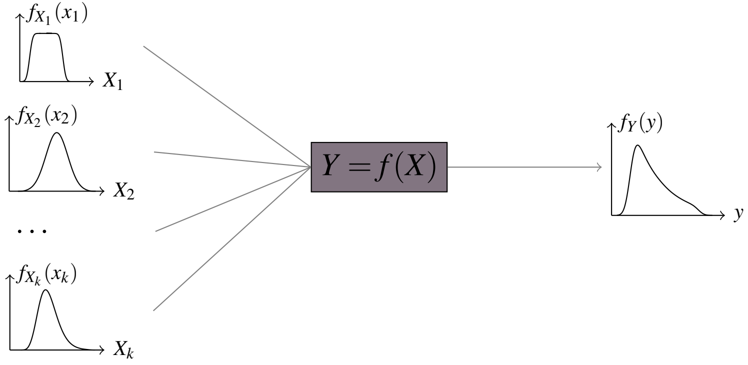

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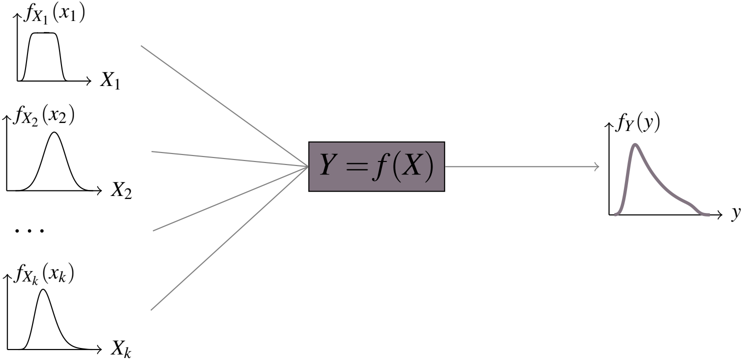

- Variance not necessarily adapted to describe the output variability
- Analysis of the entire output distribution $f_Y(y)$ rather than $V(Y)$

- Conditional variance $V(Y|X_i)$ to be compared with $V(Y)$

$$S_i = \frac{V_i}{V(Y)}$$

$$V_i = V(E(Y|X_i)) = V(Y) - E(V(Y|X_i))$$

$$V(Y) = E(V(Y|X_i)) + V(E(Y|X_i))$$

- Conditional PDF $f_{Y|X_i}(y)$ to be compared with $f_Y(y)$

$$s(X_i) = \int |f_Y(y) - f_{Y|X_i}(y)| dy$$

$$\delta_i = \frac{1}{2} E_{X_i}[s(X_i)]$$

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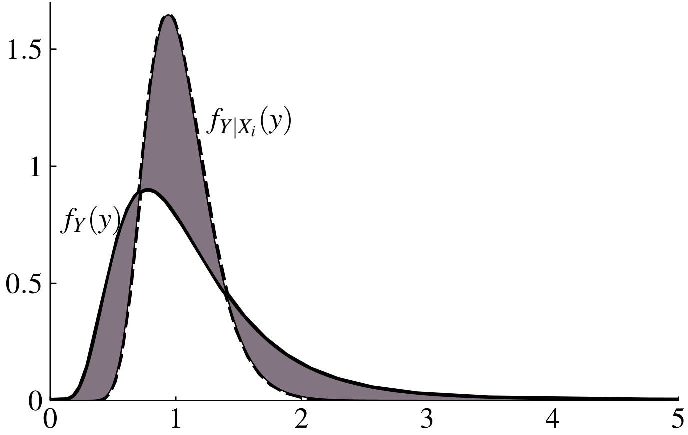

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- Other moment-independent important measures based on CDF (*Park et al, 1994; Chun et al, 2000*),
- The measures proposed by *Borgonovo, (2006)* have interesting normalization properties

Essential properties

✔ Individual importance $0 \leq \delta_i \leq 1$

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Joint importance of X_i and X_j

$$\delta_{ij} = \frac{1}{2} \int f_{X_i,X_j}(x_i, x_j) \left[\int |f_Y(y) - f_{Y|X_i,X_j}(y)| dy \right] dx_i dx_j$$

$\delta_{ij} = \delta_i$ if Y is dependent on X_i but independent on X_j

☞ δ can be extended to any set of inputs (i.e. analysis by groups)

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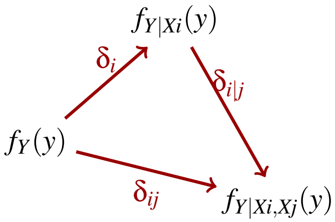
- ✔ Normalization of joint importance $\delta_{1,2,\cdots,k} = 1$
- ✔ Subadditivity $\delta_i \leq \delta_{ij} \leq \delta_i + \delta_{j|i}$

$$\delta_{j|i} = \frac{1}{2} \int f_{X_i,Xj}(x_i, x_j) \times \left[\int |f_{Y|X_i}(y) - f_{Y|X_i,X_j}(y)| dy \right] dx_i dx_j$$



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- All properties hold for dependent inputs
- Proofs are provided in *Borgonovo, (2006; 2007)*

Essential aspects of the computational approach

- Focus on $\delta_i \quad i = 1, \cdots, k$

$$\boxed{\delta_i = \frac{1}{2} \int f_{X_i}(x_i) \left[\int |f_Y(y) - f_{Y|X_i}(y)| dy \right] dx_i}$$

for i=1 to k *Loop on model inputs*

  for j=1 to N *Loop on different values of X_i*

$$s(x_i) = \int |f_Y(y) - f_{Y|X_i}(y|X_i = x_i^{(j)})| dy$$

  endfor

endfor

Key features

1. Sample generation
2. Evaluation of the area $s(X_i)$

Evaluation of $s(X_i)$

1. Discrete model outputs
   - Histograms are perfectly suited
   - Zero width and number of bins calculated from the sample

2. Continuous model outputs
   - Non-parametric estimation of PDFs (e.g. kernel density estimation)
   - Monotonic transformations can be applied without altering δ properties
   - Area calculated from CDFs (Liu and Homma, 2008)

Critical aspects of sample generation

Comparison with variance-based

- Approximation errors are potentially larger when dealing with the entire PDF
- Shortcuts are more difficult to elaborate

1. Shift between $f_Y(y)$ and $f_{Y|X_i}(y)$ should be due to the fact that $X_i = x_i^*$
   i.e. $\int |f_{Y|X_i} - f_Y(y)| dy < \varepsilon$ if X_i is a dummy input
   otherwise type I error (treating non influential inputs as important)
2. A sufficient number of x_i^* should be explored for estimating $E_{X_i}[s(X_i)]$.

Sample generation for independent inputs

- Unconditional sample for $f_X(x)$ and $f_Y(y)$

$$\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_i^{(1)} & \cdots & x_k^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_i^{(2)} & \cdots & x_k^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^{(N-1)} & x_2^{(N-1)} & \cdots & x_i^{(N-1)} & \cdots & x_k^{(N-1)} \\ x_1^{(N)} & x_2^{(N)} & \cdots & x_i^{(N)} & \cdots & x_k^{(N)} \end{pmatrix} = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(N-1)} \\ y^{(N)} \end{pmatrix}$$

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- $X_{\sim i}$ <u>not influenced</u> by the fact that $X_i = x_i^{(j)}$
- Substituted column sampling can be applied

Sample generation for independent inputs

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$$\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_i^{(1)} & \cdots & x_k^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_i^{(2)} & \cdots & x_k^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^{(N-1)} & x_2^{(N-1)} & \cdots & x_i^{(N-1)} & \cdots & x_k^{(N-1)} \\ x_1^{(N)} & x_2^{(N)} & \cdots & x_i^{(N)} & \cdots & x_k^{(N)} \end{pmatrix} = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(N-1)} \\ y^{(N)} \end{pmatrix}$$

- Ex. Conditional sample for $f_{X|X_i}(x|X_i = x_i^{(1)})$ and $f_{Y|X_i}(y|X_i = x_i^{(1)})$

$$\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_i^{(1)} & \cdots & x_k^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_i^{(1)} & \cdots & x_k^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_1^{(N-1)} & x_2^{(N-1)} & \cdots & x_i^{(1)} & \cdots & x_k^{(N-1)} \\ x_1^{(N)} & x_2^{(N)} & \cdots & x_i^{(1)} & \cdots & x_k^{(N)} \end{pmatrix} = \begin{pmatrix} y^{(1')} \\ y^{(2')} \\ \vdots \\ y^{(N-1')} \\ y^{(N')} \end{pmatrix}$$

Sample generation for independent inputs

- N conditional samples of size N required for the calculation of δ_i
- Total number of model evaluations

$$\boxed{COST = N(1 + kN)}$$

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Slightly more efficient calculation strategy

1. Less than N sample points for approaching $f_X(x|X_i = x_i^*)$, i.e. $N_{int} < N$
2. Less than N different values x_i^* of X_i for approaching $E_{X_i}[s(X_i)]$, i.e. $N_{ext} < N$

Sample generation for independent inputs

- N conditional samples of size N

$$\boxed{COST = N(1 + kN)}$$

- N_{ext} conditional samples of size N_{int}

$$\boxed{COST = N + kN_{int}N_{ext}}$$

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- N conditional samples of size N

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- N_{ext} conditional samples of size N_{int}

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- Reducing N_{int} can lead to type I error
- N_{ext} more likely to be reduced given the shape of $s(X_i)$,

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$$\boxed{COST = N(1 + kN)}$$

- N_{ext} conditional samples of size N_{int}

$$\boxed{COST = N + kN_{int}N_{ext}}$$

- No constraints for the design of the unconditional sample
- Efficient sampling strategies like Latin Hypercube Sampling (*McKay, 1979*) or Quasi-Random sampling (ex. *Sobol, 1976*) can be used

Sample generation for dependent inputs

- Unconditional correlated sample for $f_X(x)$ and $f_Y(y)$

$$
\begin{pmatrix}
x_1^{(1)} & x_2^{(1)} & \cdots & x_i^{(1)} & \cdots & x_k^{(1)} \\
x_1^{(2)} & x_2^{(2)} & \cdots & x_i^{(2)} & \cdots & x_k^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
x_1^{(N-1)} & x_2^{(N-1)} & \cdots & x_i^{(N-1)} & \cdots & x_k^{(N-1)} \\
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- $X_{\sim i}$ influenced by the fact that $X_i = x_i^{(j)}$
- Generation of conditional correlated samples for $f_X|X_i(x)$
- Permuted columns sampling plans can be used

Sample generation for dependent inputs

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$$

- Replicated Latin Hypercube Sampling (*McKay, 1995*)
  - r matrices generated trough column permutation
  - Correlations induced trough permutations (*Iman et al, 1987; Stein et al, 1987*)

Approach based on rLHS

Tutorial example for sample generation

- $X_i \quad (i = 1,2,3) \sim U[-\pi - \pi]$
- rLHS sample, number of variables $k = 3$, base sample size $N = 4$,number of replicates $r = 2$

Base sample		
−2.11	−2.38	−2.18
0.79	−0.14	0.53
1.71	0.28	3.03
−0.39	3.09	−0.99
1^{st} *Replicate*		
−2.11	3.09	−2.18
−0.39	−0.14	−0.99
0.79	0.28	3.03
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2^{nd} *Replicate*		
0.79	−2.38	−2.18
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1^{st} *Replicate*		
-2.11	3.09	-2.18
-0.39	-0.14	-0.99
0.79	0.28	3.03
1.71	-2.38	0.53
2^{nd} *Replicate*		
0.79	-2.38	-2.18
-0.39	3.09	-0.99
1.71	-0.14	3.03
-2.11	0.28	0.53

Approach based on rLHS

Tutorial example for sample generation

- $X_i \quad (i = 1,2,3) \sim U[-\pi - \pi]$
- rLHS sample, number of variables $k = 3$, base sample size $N = 4$,number of replicates $r = 2$

$f_X(x)$	*(Base sample)*	
−2.11	−2.38	−2.18
0.79	−0.14	0.53
1.71	0.28	3.03
−0.39	3.09	−0.99
1^{st} *Replicate*		
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0.79	0.28	3.03
1.71	−2.38	0.53
2^{nd} *Replicate*		
0.79	−2.38	−2.18
−0.39	3.09	−0.99
1.71	−0.14	3.03
−2.11	0.28	0.53

Sorting replicates according to values of X_1

−2.11	3.09	−2.18
−2.11	0.28	0.53
−0.39	−0.1409	−0.99
−0.39	3.0980	−0.99
0.79	0.2820	3.03
0.79	−2.3887	−2.18
1.71	−2.38	0.53
1.71	−0.14	3.03

Approach based on rLHS

Tutorial example for sample generation

- $X_i \quad (i = 1,2,3) \sim U[-\pi - \pi]$
- rLHS sample, number of variables $k = 3$, base sample size $N = 4$,number of replicates $r = 2$

$f_X(x)$ *(Base sample)*		
−2.11	−2.38	−2.18
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0.79	0.28	3.03
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2^{nd} *Replicate*		
0.79	−2.38	−2.18
−0.39	3.09	−0.99
1.71	−0.14	3.03
−2.11	0.28	0.53

Conditional samples for $f_{X|X_1}(x)$

$f_{X\|X_1}(x\|X_1 = x_1^{(1)})$		
−2.11	3.09	−2.18
−2.11	0.28	0.53
$f_{X\|X_1}(x\|X_1 = x_1^{(2)})$		
−0.39	−0.1409	−0.99
−0.39	3.0980	−0.99
$f_{X\|X_1}(x\|X_1 = x_1^{(3)})$		
0.79	0.2820	3.03
0.79	−2.3887	−2.18
$f_{X\|X_1}(x\|X_1 = x_1^{(4)})$		
1.71	−2.38	0.53
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Approach based on rLHS

Tutorial example for sample generation

- $X_i \quad (i = 1,2,3) \sim U[-\pi - \pi]$
- rLHS sample, number of variables $k = 3$, base sample size $N = 4$,number of replicates $r = 2$

$f_X(x)$	(Base sample)	
−2.11	−2.38	−2.18
0.79	−0.14	0.53
1.71	0.28	3.03
−0.39	3.09	−0.99
1^{st} Replicate		
−2.11	3.09	−2.18
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0.79	0.28	3.03
1.71	−2.38	0.53
2^{nd} Replicate		
0.79	−2.38	−2.18
−0.39	3.09	−0.99
1.71	−0.14	3.03
−2.11	0.28	0.53

Sorting replicates according to values of X_2

1.71	−2.38	0.53
0.79	−2.38	−2.18
−0.39	−0.14	−0.99
1.71	−0.14	3.03
0.79	0.28	3.03
−2.11	0.28	0.53
−2.11	3.09	−2.18
−0.39	3.09	−0.99

Approach based on rLHS

Tutorial example for sample generation

- $X_i \quad (i = 1, 2, 3) \sim U[-\pi - \pi]$
- rLHS sample, number of variables $k = 3$, base sample size $N = 4$,number of replicates $r = 2$

$f_X(x)$ *(Base sample)*

−2.11	−2.38	−2.18
0.79	−0.14	0.53
1.71	0.28	3.03
−0.39	3.09	−0.99
1^{st} *Replicate*		
−2.11	3.09	−2.18
−0.39	−0.14	−0.99
0.79	0.28	3.03
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2^{nd} *Replicate*		
0.79	−2.38	−2.18
−0.39	3.09	−0.99
1.71	−0.14	3.03
−2.11	0.28	0.53

Conditional samples for $f_{X|X_2}(x)$

$f_{X\|X_2}(x\|X_2 = x_2^{(1)})$		
1.71	−2.38	0.53
0.79	−2.38	−2.18
$f_{X\|X_2}(x\|X_2 = x_2^{(2)})$		
−0.39	−0.14	−0.99
1.71	−0.14	3.03
$f_{X\|X_2}(x\|X_2 = x_2^{(3)})$		
0.79	0.28	3.03
−2.11	0.28	0.53
$f_{X\|X_2}(x\|X_2 = x_2^{(4)})$		
−2.11	3.09	−2.18
−0.39	3.09	−0.99

Approach based on rLHS

Tutorial example for sample generation

- $X_i \quad (i = 1, 2, 3) \sim U[-\pi - \pi]$
- rLHS sample, number of variables $k = 3$, base sample size $N = 4$,number of replicates $r = 2$

$f_X(x)$	(Base sample)	
−2.11	−2.38	−2.18
0.79	−0.14	0.53
1.71	0.28	3.03
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1^{st} Replicate		
−2.11	3.09	−2.18
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0.79	0.28	3.03
1.71	−2.38	0.53
2^{nd} Replicate		
0.79	−2.38	−2.18
−0.39	3.09	−0.99
1.71	−0.14	3.03
−2.11	0.28	0.53

Sorting replicates according to values of X_3

−2.11	3.0980	−2.18
0.79	−2.3887	−2.18
−0.39	−0.1409	−0.99
−0.39	3.0980	−0.99
1.71	−2.3887	0.53
−2.11	0.2820	0.53
0.79	0.28	3.03
1.71	−0.14	3.03

Approach based on rLHS

Tutorial example for sample generation

- $X_i \quad (i = 1, 2, 3) \sim U[-\pi - \pi]$
- rLHS sample, number of variables $k = 3$, base sample size $N = 4$,number of replicates $r = 2$

$f_X(x)$ *(Base sample)*

−2.11	−2.38	−2.18
0.79	−0.14	0.53
1.71	0.28	3.03
−0.39	3.09	−0.99
1^{st} *Replicate*		
−2.11	3.09	−2.18
−0.39	−0.14	−0.99
0.79	0.28	3.03
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2^{nd} *Replicate*		
0.79	−2.38	−2.18
−0.39	3.09	−0.99
1.71	−0.14	3.03
−2.11	0.28	0.53

Conditional samples for $f_{X|X_3}(x)$

$f_{X\|X_3}(x\|X_3 = x_3^{(1)})$		
−2.11	3.0980	−2.18
0.79	−2.3887	−2.18
$f_{X\|X_3}(x\|X_3 = x_3^{(2)})$		
−0.39	−0.1409	−0.99
−0.39	3.0980	−0.99
$f_{X\|X_3}(x\|X_3 = x_3^{(3)})$		
1.71	−2.3887	0.53
−2.11	0.2820	0.53
$f_{X\|X_3}(x\|X_3 = x_3^{(4)})$		
0.79	0.28	3.03
1.71	−0.14	3.03

Sample generation for dependent inputs

- Replicated Latin Hypercube Sampling (*McKay, 1995*)
  - r matrices of size N generated trough column permutation
  - Correlations induced trough permutations (*Iman et al, 1987; Stein et al, 1987*)

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- Sample size used for approaching $f_{X|X_i}(x)$ (i.e. N_{int}) is given by the number of replicates r
- Number of values of x_i explored (i.e. N_{ext}) for estimation $E_{X_i}[s(X_i)]$ given by the base sample size N

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$$\boxed{COST = r * N}$$

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Sample generation for dependent inputs

- Replicated Latin Hypercube Sampling (*McKay, 1995*)
  - r matrices of size N generated trough column permutation
  - Correlations induced trough permutations (*Iman et al, 1987; Stein et al, 1987*)

$$\boxed{COST = r * N}$$

- r should be close to N in order to ensure that $\int |f_{Y|X_i} - f_Y(y)| dy < \varepsilon$ if X_i is a dummy input

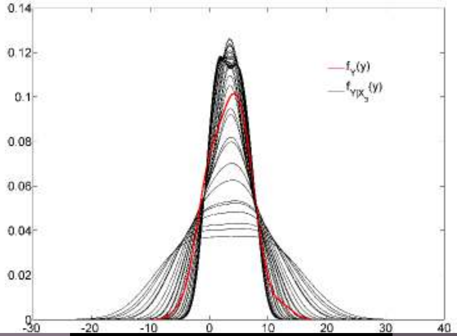
$$\boxed{COST \sim N^2}$$

☞ Number of model evaluations independent from k

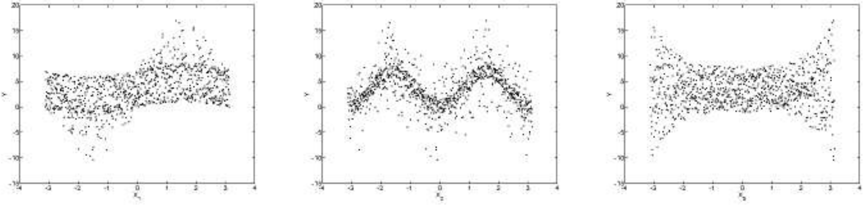
Brute force approach

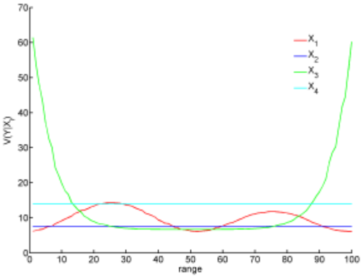
- $X_i \quad (i = 1, \cdots 4) \sim U[-\pi - \pi]$
- Ishigami function $f(Y) = \sin X_1 + 7 \sin^2 X_2 + 0.1 X_3^4 \sin X_1$
- X_4 is a dummy input

	X_1	X_2	X_3	X_4
S_i	0.3139	0.4424	0.	0.
δ_i	0.2110	0.4073	0.1568	0.

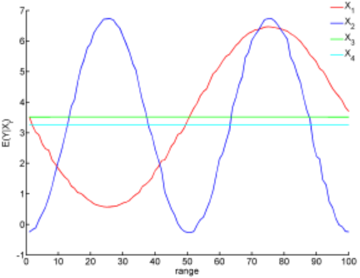


Brute force approach



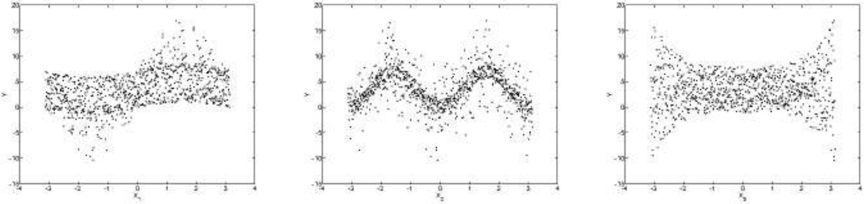


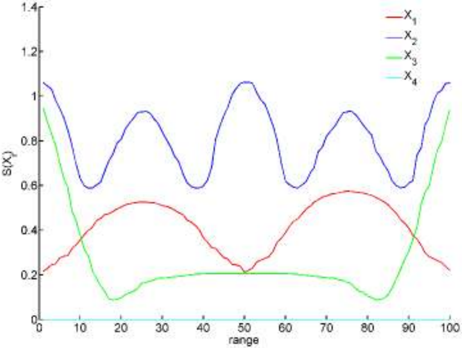
$V(Y|X_i)$



$E(Y|X_i)$

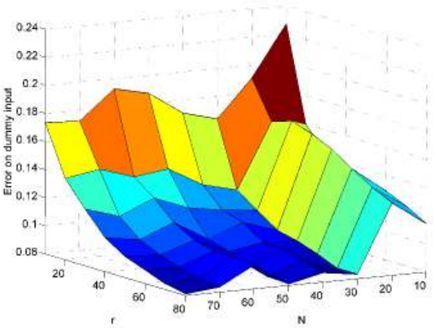
Brute force approach



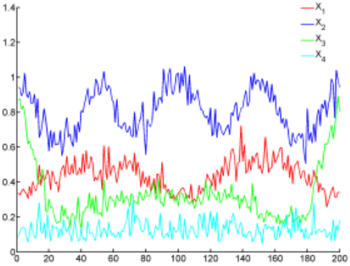


$s(X_i)$

- Number of replicates essential in order to ensure that $\int |f_{Y|X_i} - f_Y(y)|dy < \varepsilon$ for dummy input factors



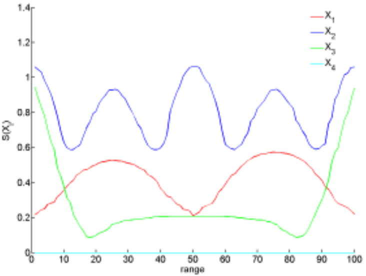
- Lack of correspondence for $X_{\sim i}$ lead to approximation error for $s(Xi)$
- Reasonable accuracy for δ_i estimates



rLHS approach

$\delta_1 = 0.21$, $\delta_2 = 0.40$

$\delta_2 = 0.16$, $\delta_4 = 0.06$



Brute force approach

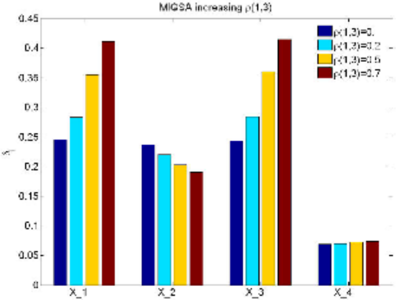
$\delta_1 = 0.20$, $\delta_2 = 0.40$

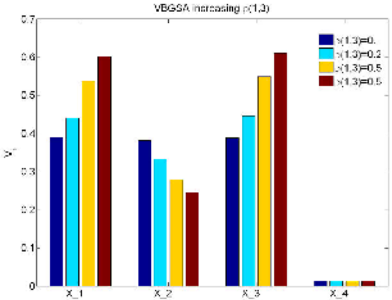
$\delta_2 = 0.13$, $\delta_4 = 0.$

rLHS approach

Effect of dependence among inputs

- $X_i \quad (i = 1, \cdots 4) \sim U[01]$
- $f(Y) = X_1 + X_2 + X_3$
- X_4 is a dummy input

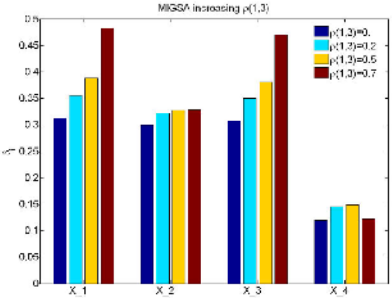


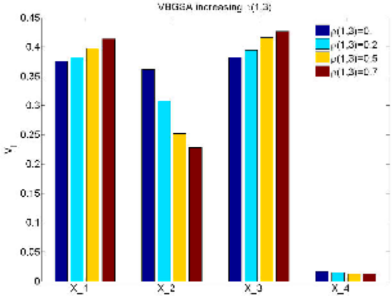


rLHS approach

Effect of dependence among inputs

- $X_i \quad (i = 1, \cdots 4) \sim U[01]$
- $f(Y) = X_1.X_2.X_3$
- X_4 is a dummy input





rLHS approach

Effect of dependence among inputs

- The presence of correlations or/and interactions increases the approximation error

- The additional terms generated by dependence create a non-null effect for a dummy factor
- Correlations increase the importance of the correlated parameters for both VB and MI
- In the presence of interactions, influence on other factors can be different

Conclusions

- Moment-independent importance measures with interesting properties
- Any shortcut is prone to substantial approximation errors when dealing with the entire PDF
- Computationally intensive assessment
- Calculation methods for independent and dependent model inputs, other sampling plans to be investigated

☞ Rather than the entire PDF, a specific portion might be of interest

Ex. Focus on the variability of extremes

- Relative importance of model inputs in determining the variability of extremes
  $$s(x_i) = \int |f_Y(y|Y > y^{90\%}) - f_{Y|X_i}(y|X_i, Y > y^{90\%})|dy$$

Monte Carlo Filtering

- Select the sample points verifying $Y > y^{90\%}$
- Induced correlation structure for $f_X(x|Y > y^{90\%})$
- Conditional samples generation (i.e. $f_{Y|X_i}(y|X_i, Y > y^{90\%})$) might be difficult

Adaptation of importance measure

- Restrict the area calculation to the targeted region of the model output
  $$s(x_i) = \int_\Omega |f_Y(y) - f_{Y|X_i}(y|X_i)|dy$$

Merci de votre attention ...

Q?