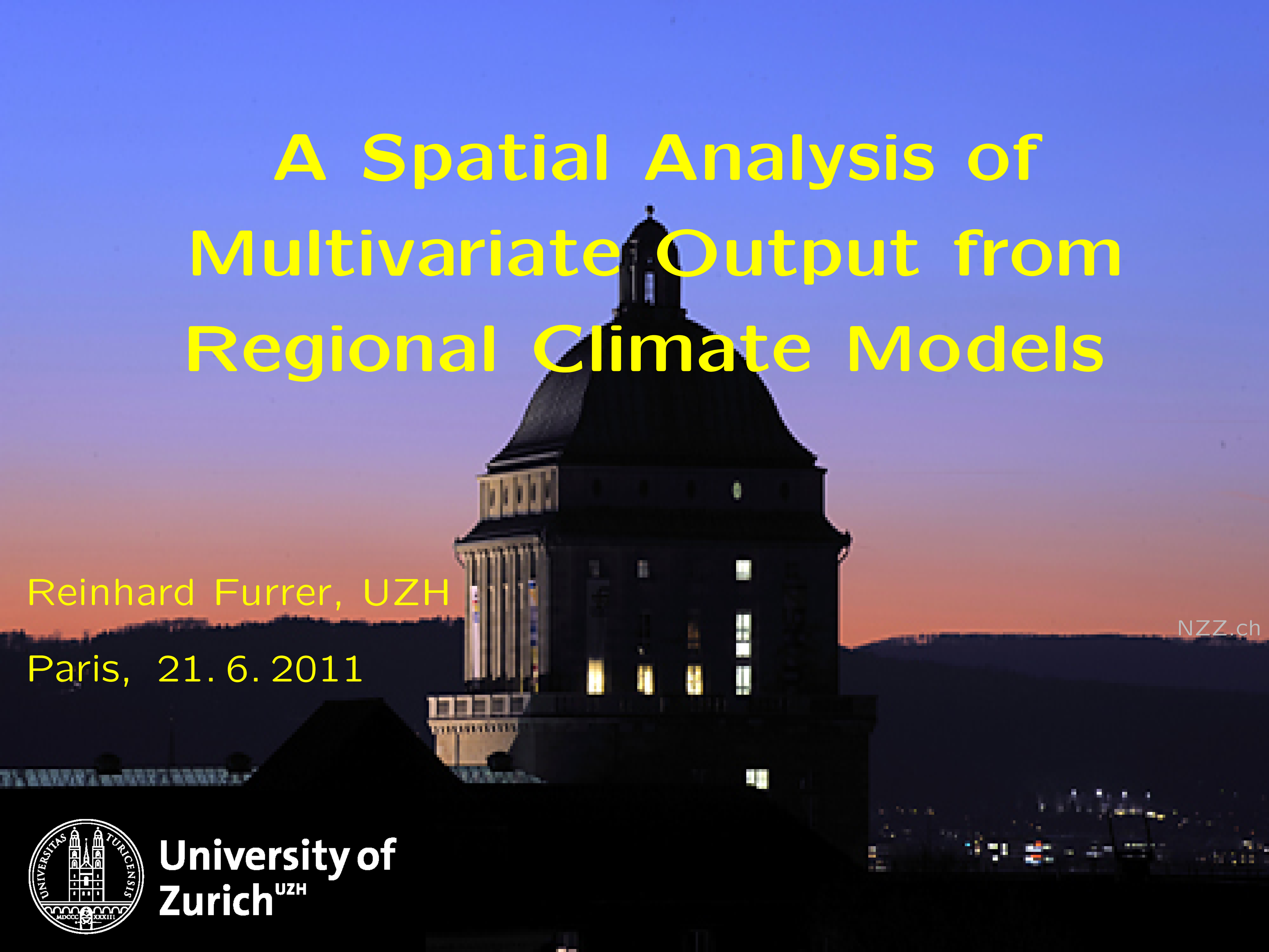
A Spatial Analysis of Multivariate Output from Regional Climate Models

Reinhard Furrer, UZH

Paris, 21. 6. 2011



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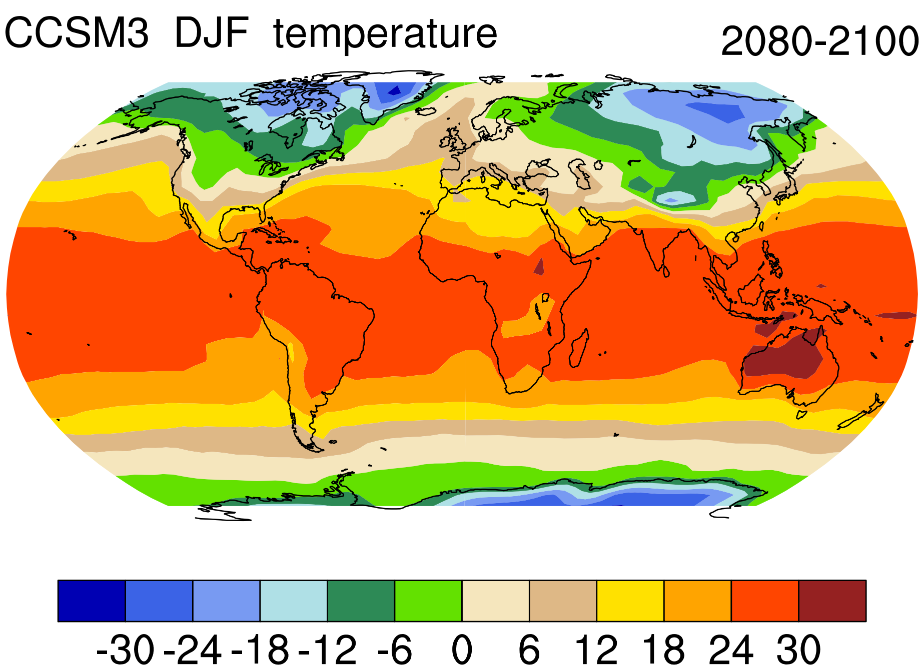
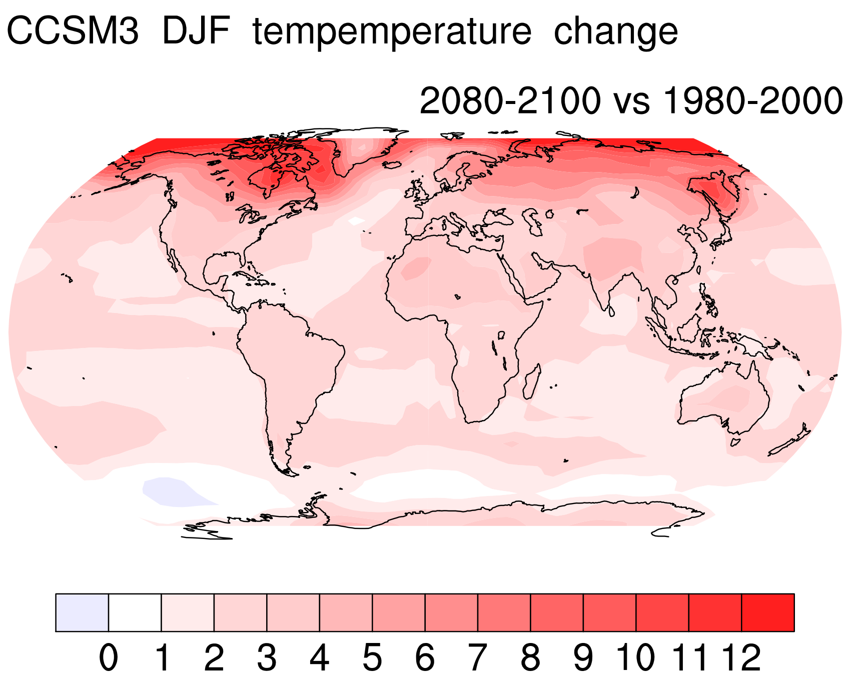
Outline

- Examples of spatial climate data
- Uni-/Multivariate spatial models
- General approaches
- Markovian modeling
- Example: Regional temperature & precipitation change
- Outlook

Spatial data

General circulation model data





Source: www.cisl.ucar.edu

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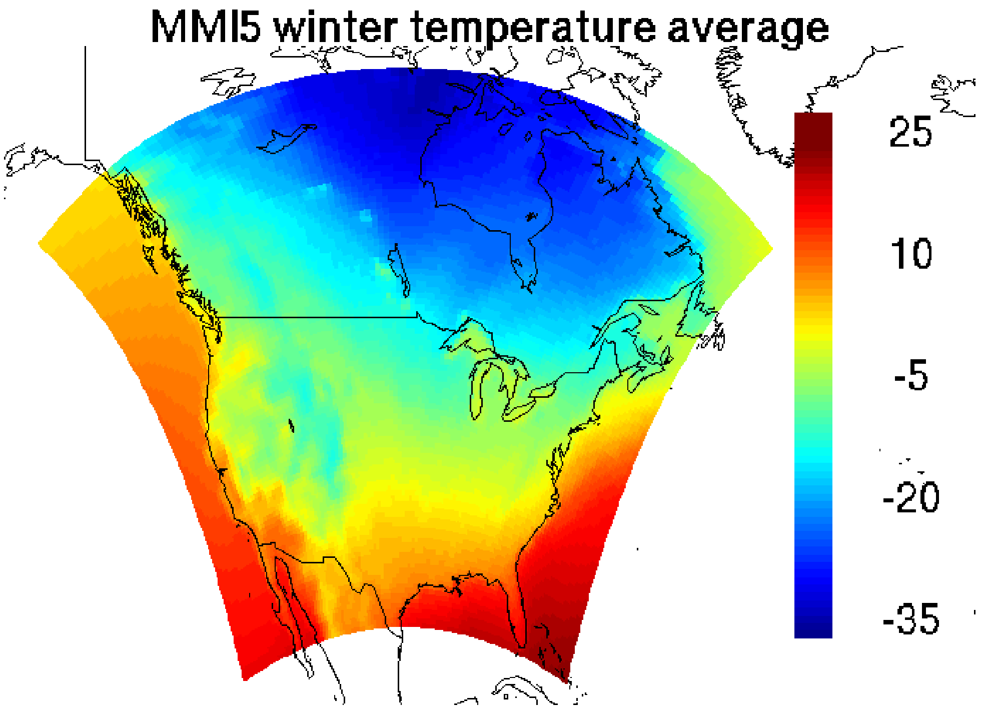
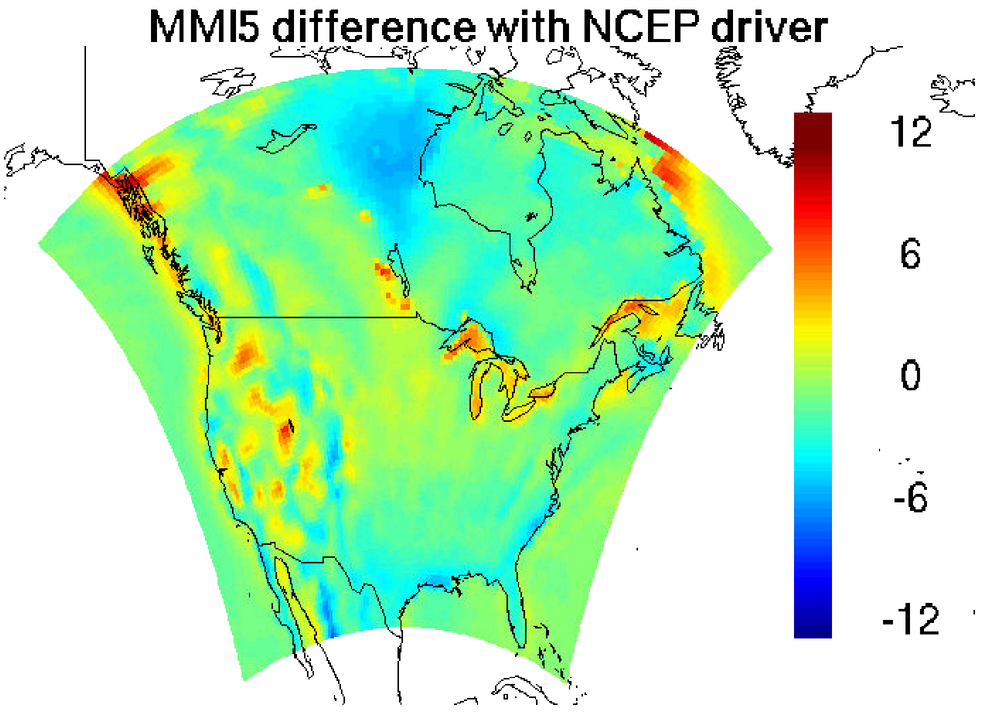
Spatial data

Regional climate model data





Spatial data

Typical features

- Large datasets
- Complex nonstationarities
- Unknown dependencies
- Difficulty visualizing the results

Univariate modeling: setting

Spatial, additive mixed effects model:

$$\begin{aligned} \text{data} &= \text{signal} + \text{noise} \\ &= \text{fixed effects} + \text{trend} + \text{spatial term} + \text{noise} \end{aligned}$$

Univariate modeling: setting

Spatial, additive mixed effects model:

$$\begin{aligned} \text{data} &= \text{signal} + \text{noise} \\ &= \text{fixed effects} + \text{trend} + \text{spatial term} + \text{noise} \end{aligned}$$

or

$$Y(\mathbf{s}) = \mathbf{X}\beta + \alpha(\mathbf{s}) + \gamma(\mathbf{s}) + \varepsilon(\mathbf{s}) \qquad \mathbf{s} \in \mathcal{D} \subset \mathbb{R}^d, \ d \geq 1$$

with

- $Y(\mathbf{s})$: observations
- $\mathbf{X}\beta$: fixed effects and trend
- $\alpha(\mathbf{s})$: spline component (trend)
- $\gamma(\mathbf{s})$: zero mean spatial Gaussian process
- $\varepsilon(\mathbf{s})$: iid Gaussian noise, orthogonal to $\gamma(\mathbf{s})$

Univariate modeling: setting

Parameters $\boldsymbol{\theta} = \left(\boldsymbol{\beta}^\mathsf{T}, {\boldsymbol{\theta}_\alpha}^\mathsf{T}, \lambda_\alpha, {\boldsymbol{\theta}_\gamma}^\mathsf{T}, \sigma^2\right)^\mathsf{T}$:

$\mathbf{X}\boldsymbol{\beta}$:	coefficients $\boldsymbol{\beta}$
$\alpha(\mathbf{s})$:	basis function coefficients $\boldsymbol{\theta}_\alpha$; smoothing parameter λ_α
$\gamma(\mathbf{s})$:	parameters $\boldsymbol{\theta}_\gamma$ decribing the covariance function
$\varepsilon(\mathbf{s})$:	variance σ^2

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Statistical tasks:

- estimation of $\boldsymbol{\theta}$
- smoothing or prediction
- uncertainty assessment
- model validation

Univariate modeling: setting

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$$
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Y_1(\mathbf{s}) &= \mathbf{X}_1\boldsymbol{\beta}_1 + \alpha_1(\mathbf{s}) + \gamma_1(\mathbf{s}) + \varepsilon_1(\mathbf{s}) \\
&\vdots \\
Y_p(\mathbf{s}) &= \mathbf{X}_p\boldsymbol{\beta}_p + \alpha_p(\mathbf{s}) + \gamma_p(\mathbf{s}) + \varepsilon_p(\mathbf{s}) \qquad \mathbf{s} \in \mathcal{D} \subset \mathbb{R}^d,\ d \geq 1
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with

- $Y_i(\mathbf{s})$: observations
- $\mathbf{X}_i\boldsymbol{\beta}_i$: fixed effects and trends
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Multivariate modeling: setting

Modeling the spatial processes themselves:

		Random field:	
		GRF	GMRF
Dependency:	common process(es)	①	②
	cross-correlation model	③	④

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Inlet: dependency modeling

Common process(es):

$$\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}_\mathbf{X}, \Sigma_\mathbf{X}) \qquad \mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \Sigma_\mathbf{Z})$$
$$\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}_\mathbf{Y}, \Sigma_\mathbf{Y})$$

$$\rightsquigarrow \quad \begin{pmatrix} \mathbf{X}+\mathbf{Z} \\ \mathbf{Y}+\mathbf{Z} \end{pmatrix} \sim \mathcal{N}_{2n}\left(\begin{pmatrix} \mu_\mathbf{X} \\ \mu_\mathbf{Y} \end{pmatrix}, \begin{pmatrix} \Sigma_\mathbf{X}+\Sigma_\mathbf{Z} & \Sigma_\mathbf{Z} \\ \Sigma_\mathbf{Z} & \Sigma_\mathbf{Y}+\Sigma_\mathbf{Z} \end{pmatrix}\right)$$

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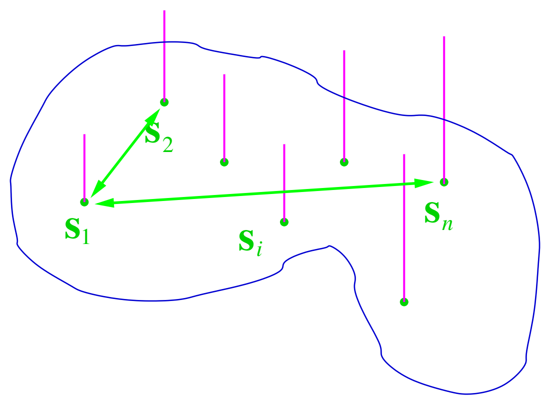
Cross-correlation model:

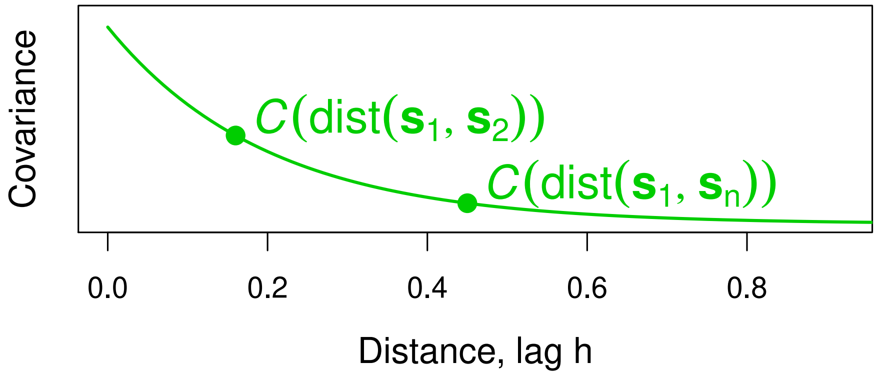
$$\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}_{\mathbf{X}}, \Sigma_{\mathbf{X}}) \qquad \mathrm{Cov}(\mathbf{X}, \mathbf{Y}) = \Sigma_{\mathbf{XY}}$$
$$\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}_{\mathbf{Y}}, \Sigma_{\mathbf{Y}})$$

$$\rightsquigarrow \quad \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \sim \mathcal{N}_{2n}\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{X}} \\ \boldsymbol{\mu}_{\mathbf{Y}} \end{pmatrix}, \begin{pmatrix} \Sigma_{\mathbf{X}} & \Sigma_{\mathbf{XY}} \\ \Sigma_{\mathbf{XY}}{}^{\mathsf{T}} & \Sigma_{\mathbf{Y}} \end{pmatrix} \right)$$

Inlet: random field modeling

Spatial process (GRF):





Covariance matrix: Σ

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Inlet: random field modeling

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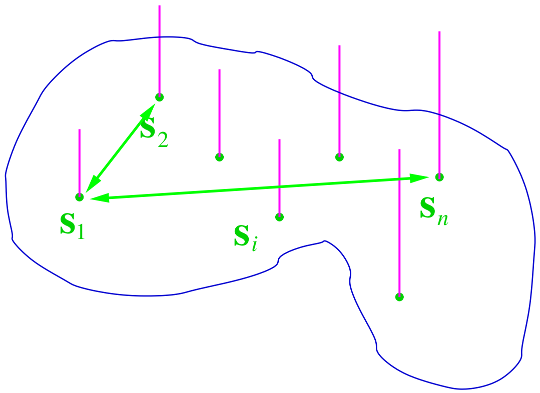
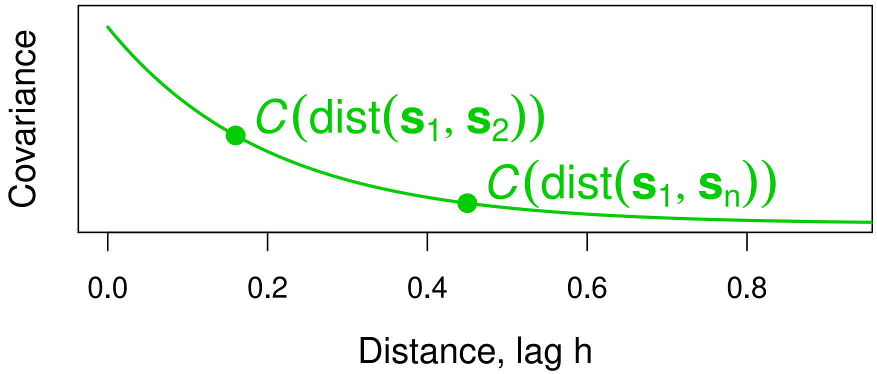
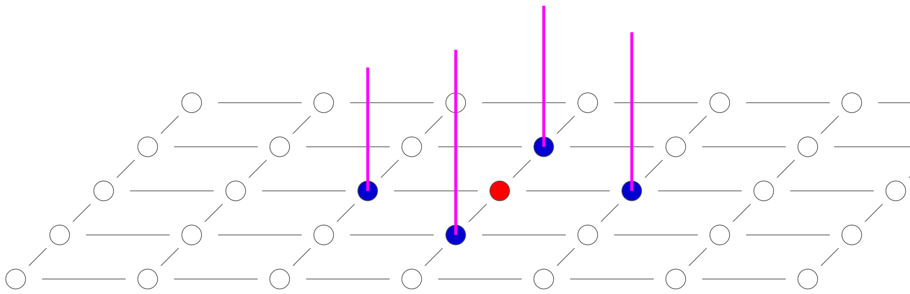




Covariance matrix: $\boldsymbol{\Sigma}$

Lattice data (GMRF):



$$\mathrm{E}[y_i|y_{-i}] = \beta \sum_{j \text{ neighbor of } i} y_j$$

$$\mathrm{Var}[y_i|y_{-i}] = \tau^2$$

Gaussianity and regularity conditions:

$$\boldsymbol{\Sigma} = \tau^2(\mathbf{I} - \mathbf{B})^{-1}$$

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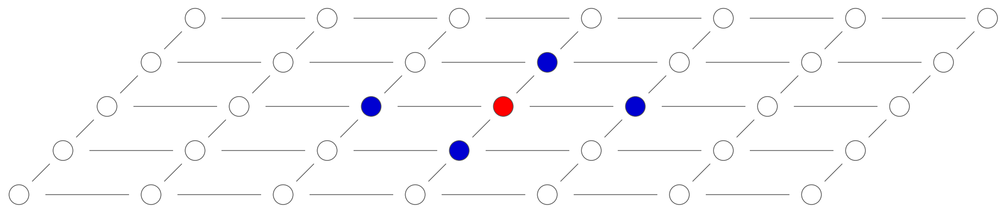
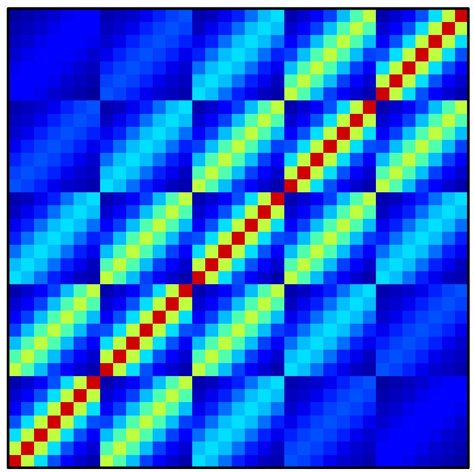
Inlet: random field modeling

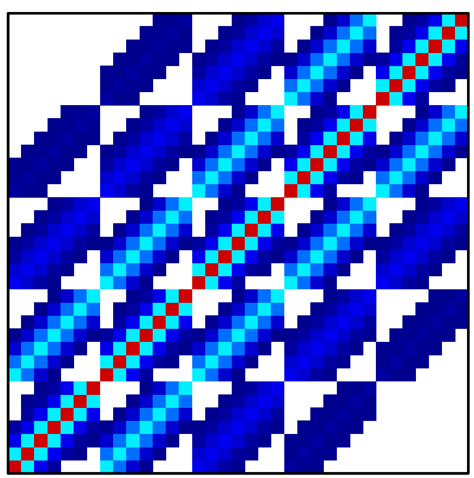


Spatial process:

Σ

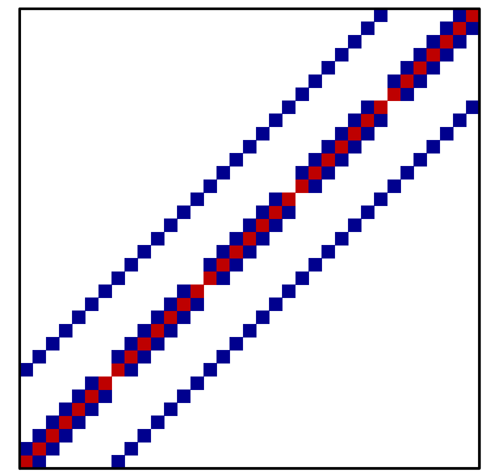


Σ_{tapered}

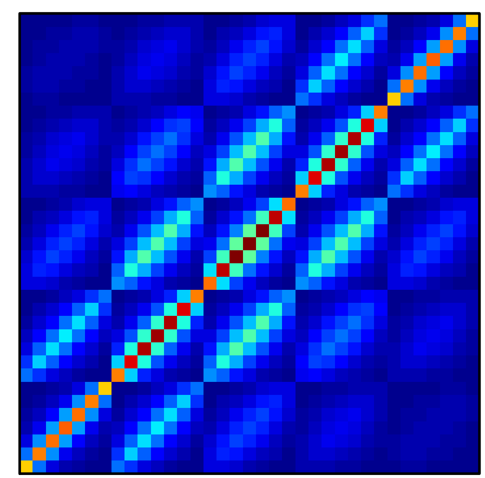


Lattice data:

Σ^{-1}



Σ



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Multivariate modeling: approaches

- Iterative approaches
  - \+ Flexible, numerically feasible
  - − Uncertainties
- Maximum likelihood
  - \+ Uncertainties, asymptotics
  - − Numerical issues
- Bayesian hierarchical models
  - \+ Flexible, uncertainties
  - − MCMC

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Backfitting: ① ② ③ ④ ◄

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Tapering: ① ③; ② ④ ◀

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Tapering: ① ③; ② ④ ◀

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Dimension-reduction: ④ ◀

Multivariate modeling: backfitting

Recall:

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\end{aligned}
$$

Multivariate modeling: backfitting

Recall:

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Extending the ‘classical’ backfitting approach to dependent data:

```
repeat until convergence
    repeat until convergence
        estimate fixed effects
    for all ‘stochastic’ effects
        estimate parameters
        predict smooth field
```

Multivariate modeling: backfitting

- Intuitive, stable.
- Computationally easy to implement, handles very large datasets.
  Limitation of handling is one single $\alpha_i(\mathbf{s})$ or $\gamma_i(\mathbf{s})$ field.
- Known covariance structure:
  Equivalence after convergence and convergence.
- Unknown covariance structure:
  'Nothing' can be said.
- Uncertainties ...

See Furrer, Sain (2009) StCo; Heersink, Furrer (subm.) LAA.

Multivariate modeling: tapering

Univariate case: tapering based on infill asymptotics
and equivalent Gaussian measures.

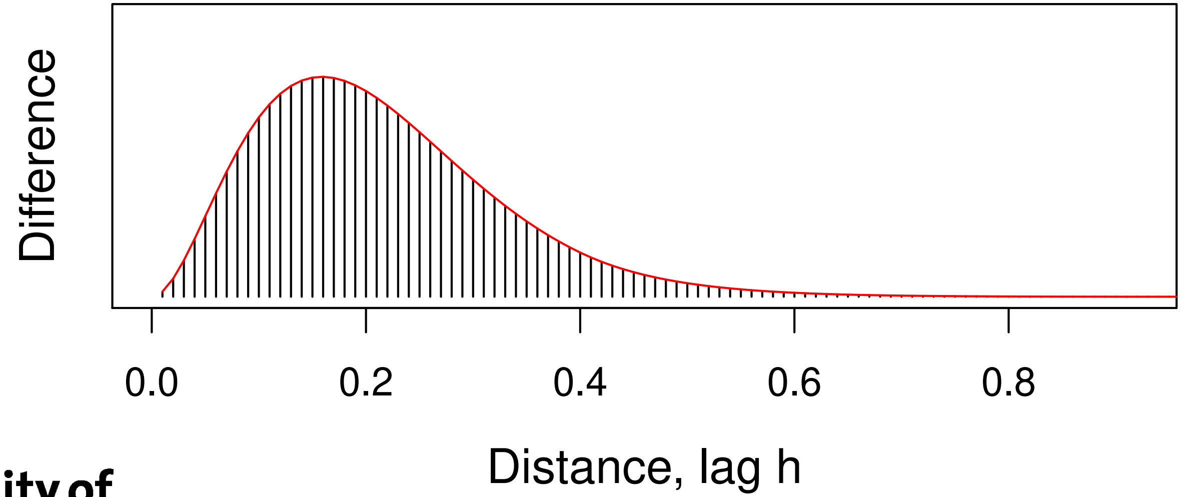
Multivariate modeling: tapering

Univariate case: tapering based on infill asymptotics
and equivalent Gaussian measures.

Idea:

choose an asymptotic framework such that original and tapered covariance matrix are asymptotically equivalent.

Then the difference in the likelihoods tends to zero almost surely.



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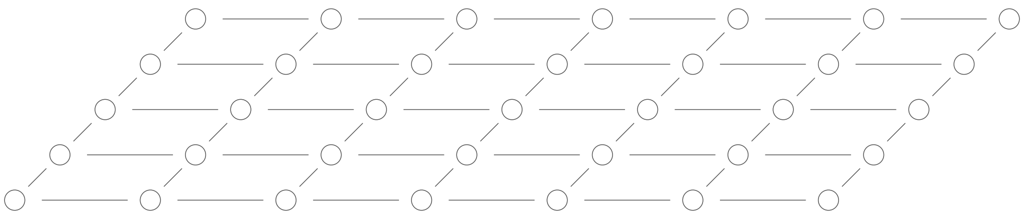
Multivariate modeling: dim-reduction

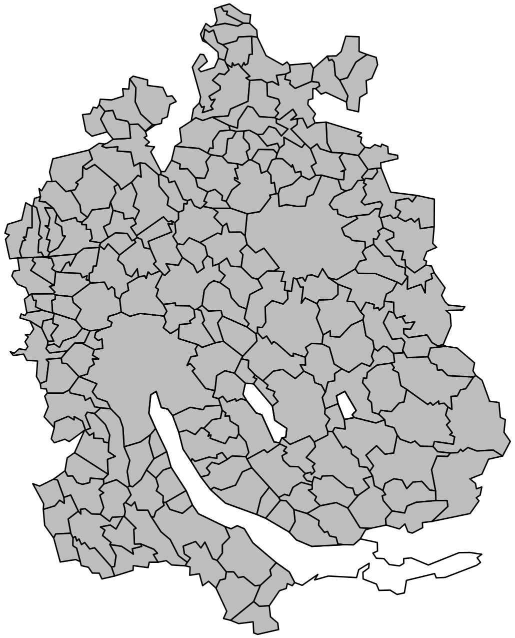
Difficulty in modeling flexibly joint multivariate processes

Idea for lattice data:

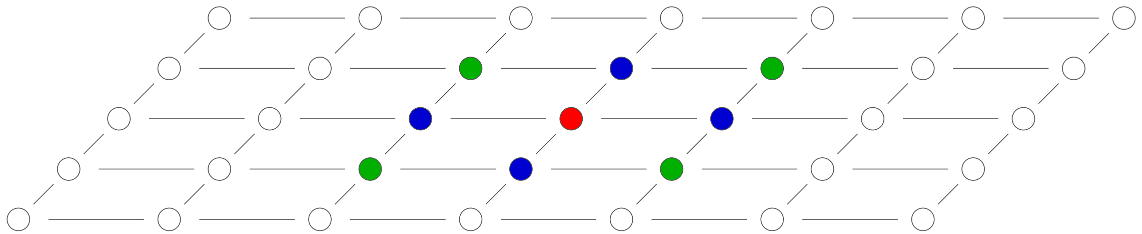
> For conditional autoregressive models (CARs), consider one three-dimensional lattice instead of several two-dimensional lattices.

See Sain, Furrer, Cressie (2011) AOAS.





Multivariate modeling: dim-reduction



- Univariate CAR:

$$\mathsf{E}[y_i|y_{-i}] = \mu_i + \sum_{j\neq i} \beta_{ij}(y_j - \mu_j) \qquad \mathsf{Var}[y_i|y_{-i}] = \tau_i^2$$

+ regularity conditions

- Multivariate CAR:

$$\mathsf{E}[\mathbf{y}_i|\mathbf{y}_{-i}] = \boldsymbol{\mu}_i + \sum_{j\neq i} \mathbf{B}_{ij}(\mathbf{y}_j - \boldsymbol{\mu}_j) \qquad \mathsf{Var}[\mathbf{y}_i|\mathbf{y}_{-i}] = \mathbf{T}_i$$

+ regularity conditions

- Multivariate CAR, alternative formulation:
  (following slide)

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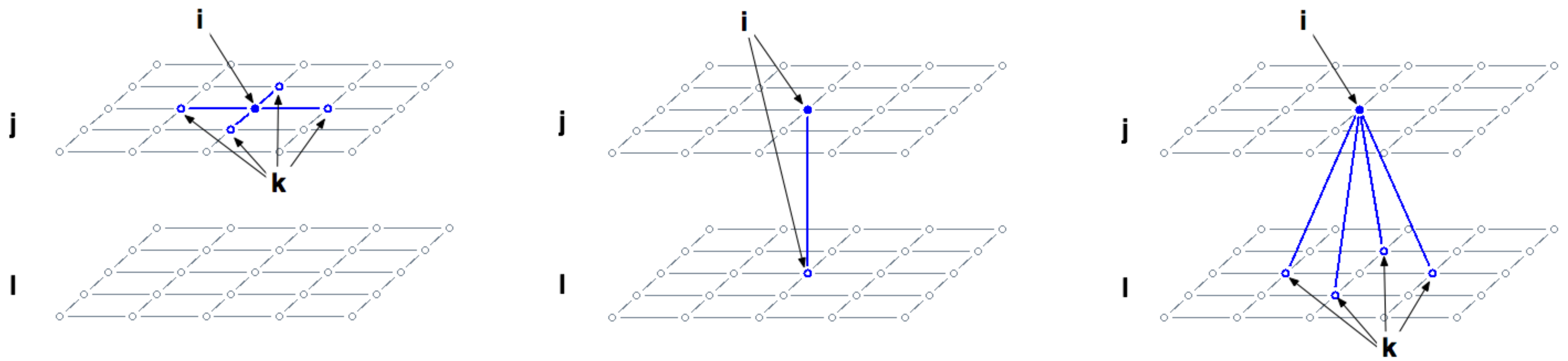
Multivariate modeling: dim-reduction



$$E[y_{ij}|y_{-\{ij\}}] = \mu_{ij} + \sum_{k \neq i} \beta_{ijkj}(y_{kj} - \mu_{kj})$$

$$+ \sum_{\ell \neq j} \beta_{iji\ell}(y_{i\ell} - \mu_{i\ell})$$

$$+ \sum_{k,\ell \neq i,j} \beta_{ijk\ell}(y_{k\ell} - \mu_{k\ell})$$

$$\text{Var}[y_{ij}|y_{-\{ij\}}] = \tau_{ij}^2$$

+ regularity conditions

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Multivariate modeling: dim-reduction

Overparameterized! Simplify to:

- constant variance
- constant dependencies + symmetry

Multivariate modeling: dim-reduction

Overparameterized! Simplify to:

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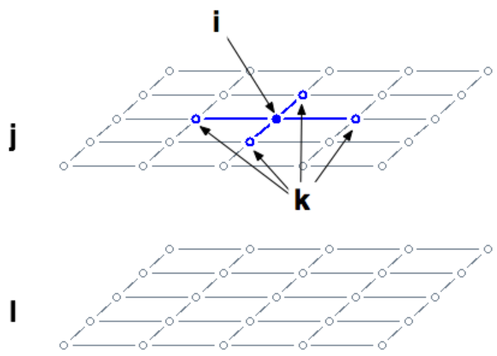
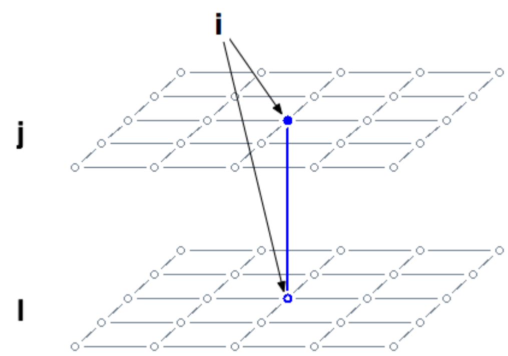
For example: $\quad b_{ijil} = \rho_{j\ell}\tau_j\tau_\ell \quad b_{ijk\ell} = \phi_{j\ell}\tau_j\tau_\ell \quad$ results in:

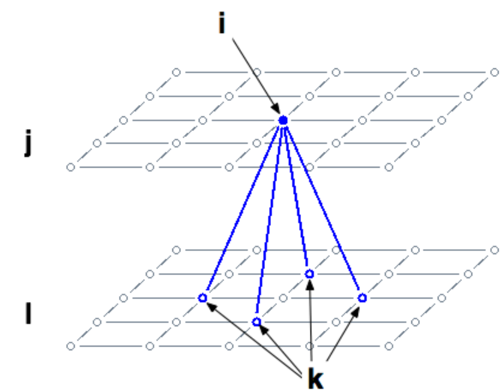
$\{\phi_{jj}\}$:



$\{\rho_{j\ell}\}$:



$\{\phi_{j\ell}\}$



plus p variance parameters: $\{\tau_j\}$

Multivariate modeling: dim-reduction

- Falls within the framework of a unidimensional lattice model
- Guarantees sparse precision matrices
- Flexibly modeling multivariate spatial dependencies
- MCMC is (often) required and may be difficult to tame
- Possibility to implement asymmetric cross-dependencies, ...

Example: temp. & precip. change

"Multivariate" RCM experiment:

- NCAR/DOE Parallel Climate Model to drive the NCAR/Penn State Mesoscale Model (MM5)
- One control run from 1995–2015 and three future runs (ensemble members) from 2040–2060
(1% annual increase in the amount of greenhouse gases)
- Difference between future and control twenty-year winter (DJF) and summer (JJA) average temperature and average total precipitation
- Spatial fields with $44 \times 56 = 2464$ grid boxes

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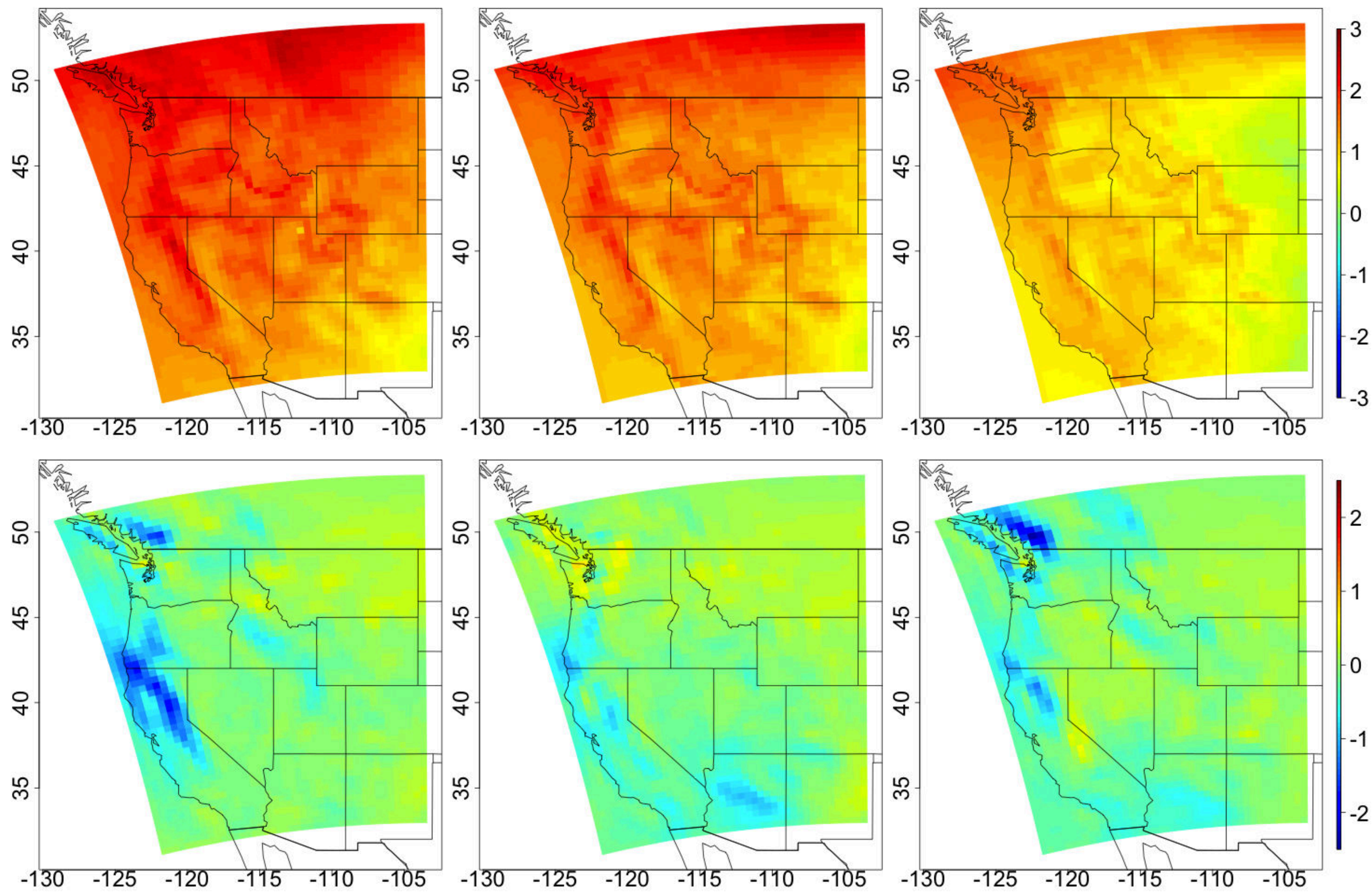
Example: temp. & precip. change



Differences in DJF temperature (°K) and in total precipitation (in).

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Example: temp. & precip. change

Hierarchical model:

Data level

$$\text{data}_{\text{var, run}} = \text{fixed effects}_{\text{var}} + \text{random effects}_{\text{var, run}} + \text{error}$$

Process level

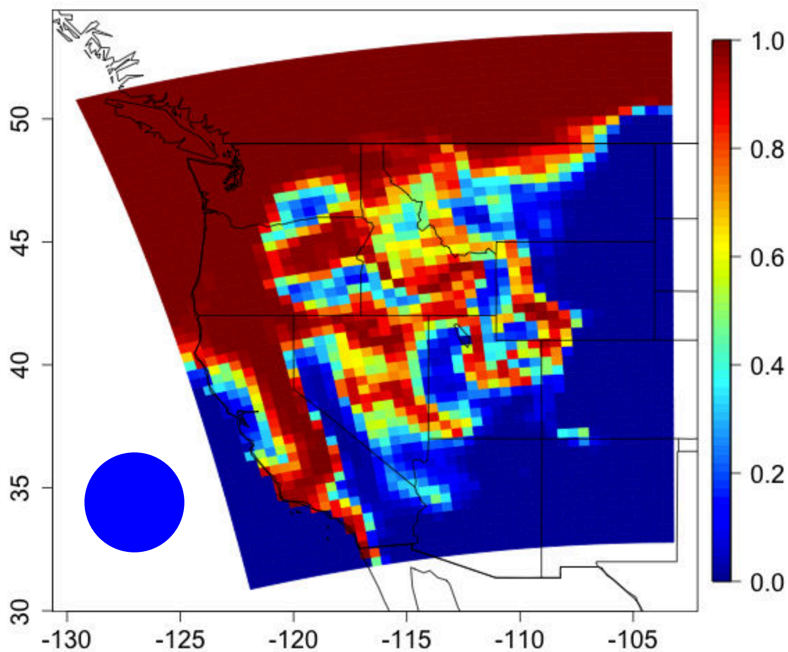
$$\text{fixed effects}_{\text{var}} = \text{lat}_{\text{var}} + \text{lon}_{\text{var}} + \text{elevation}_{\text{var}}$$
$$\text{random effects}_{\text{var, run}} = \text{intercept}_{\text{var, run}} + \text{MGMRF}_{\text{var, run}}$$

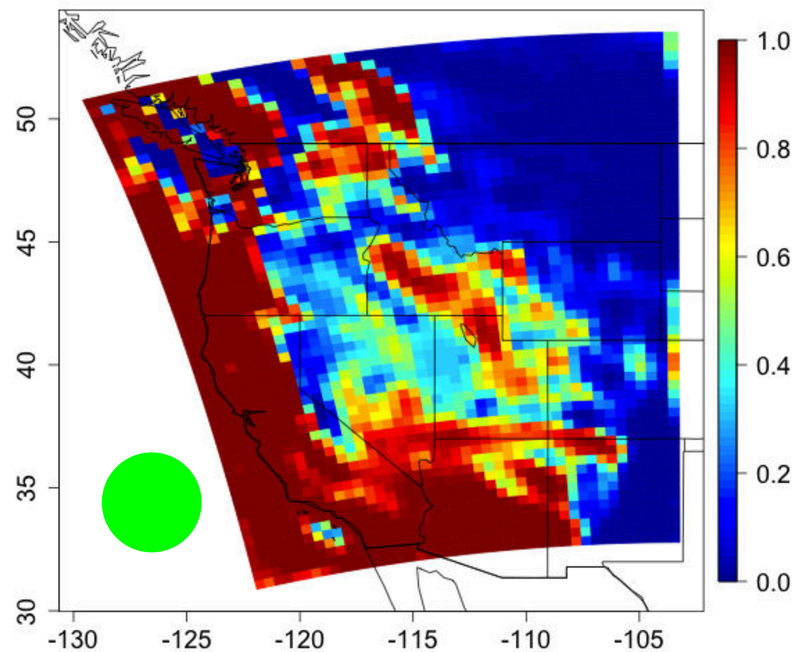
Prior level

conjugate or uniform over valid parameter range

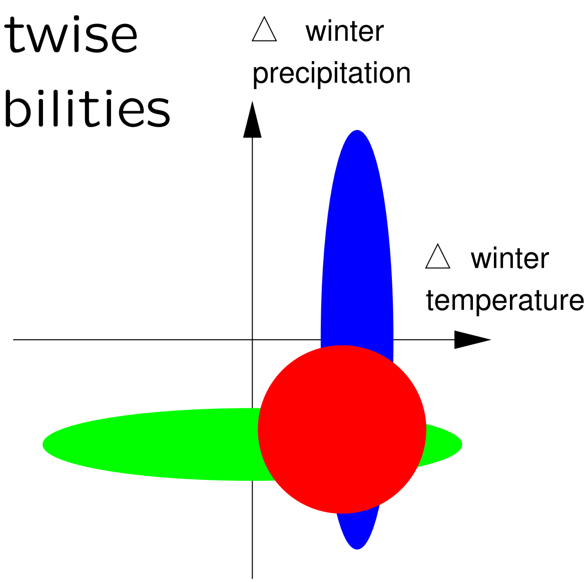
$\rightsquigarrow$ run MCMC beast

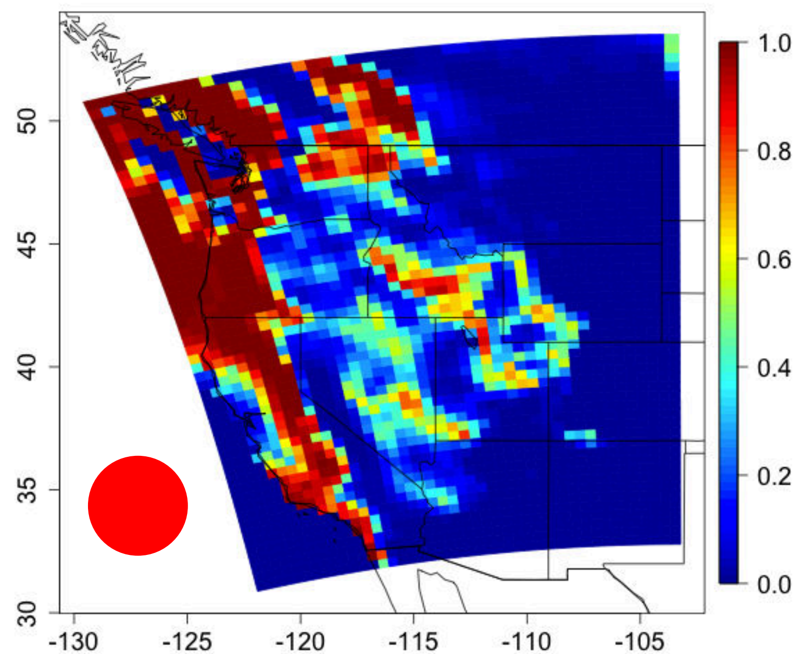
Example: temp. & precip. change





Estimated pointwise posterior probabilities





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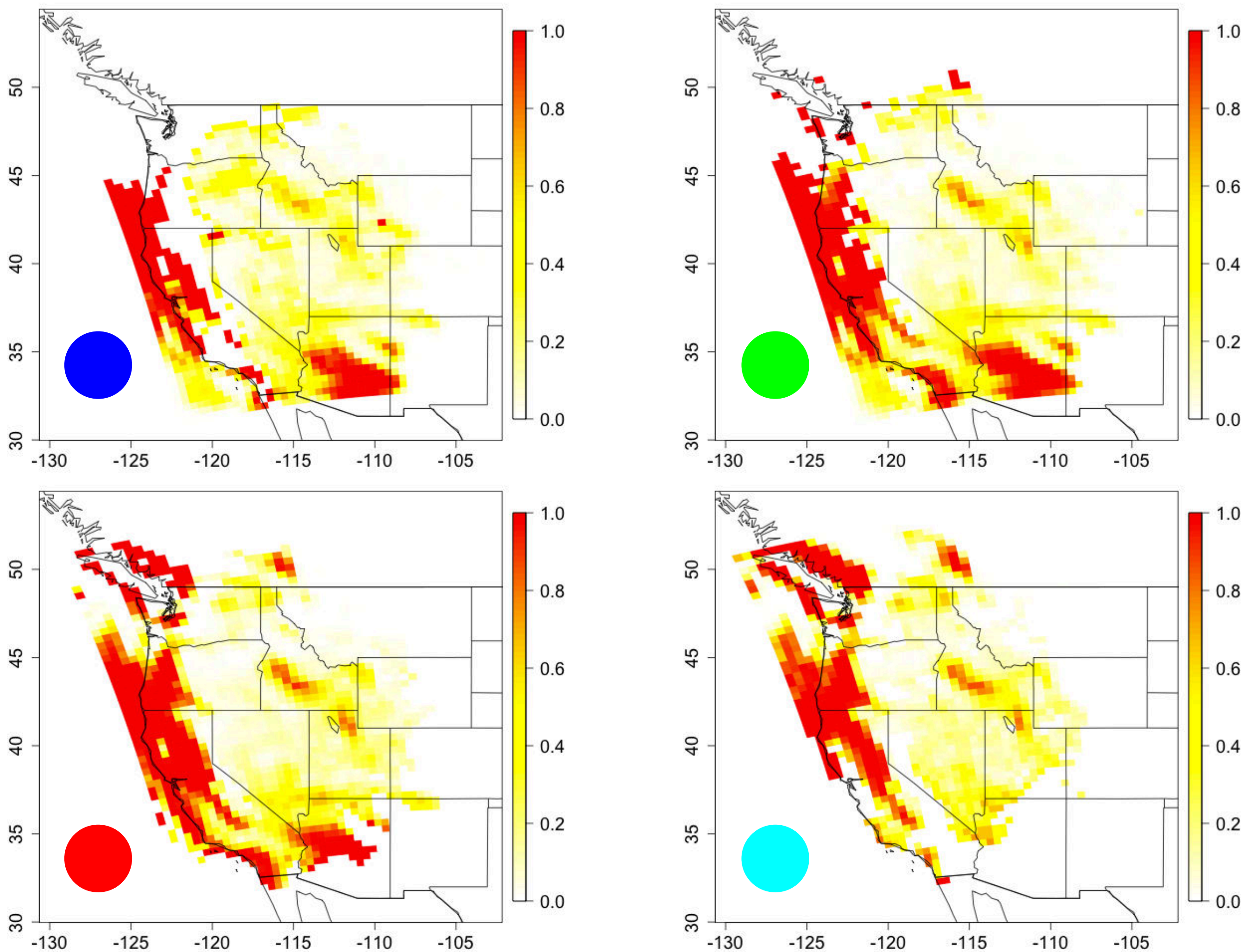
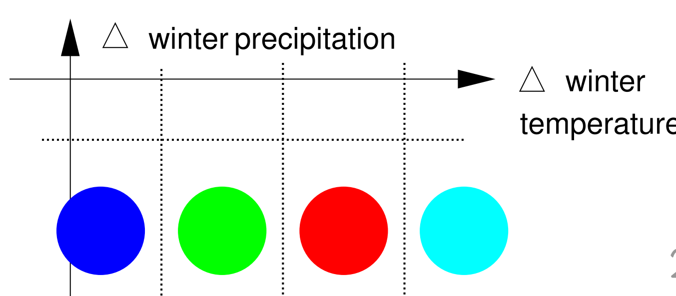
Example: temp. & precip. change



Probability of large decrease in winter precipitation conditional on increase in temperature.



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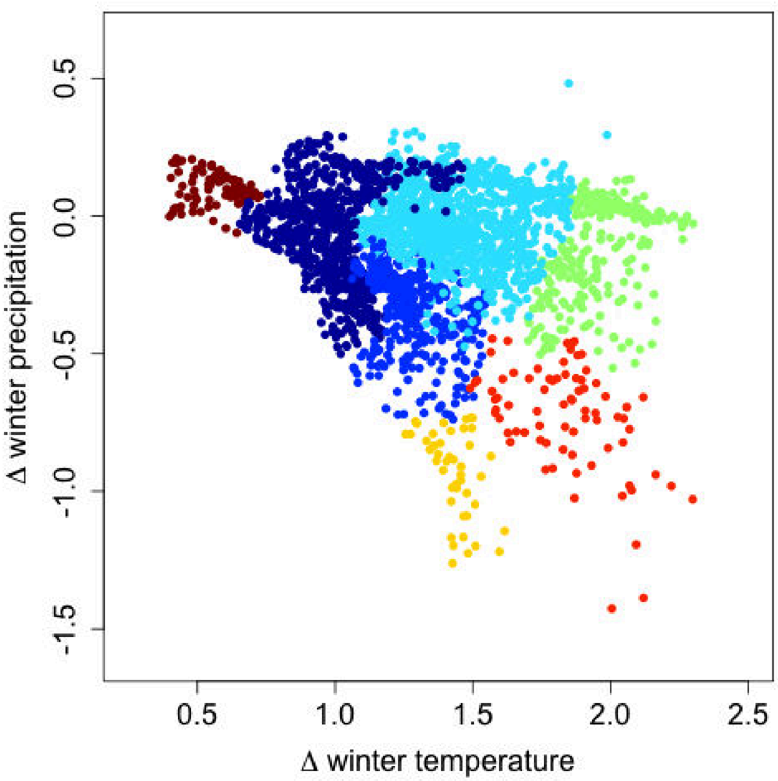
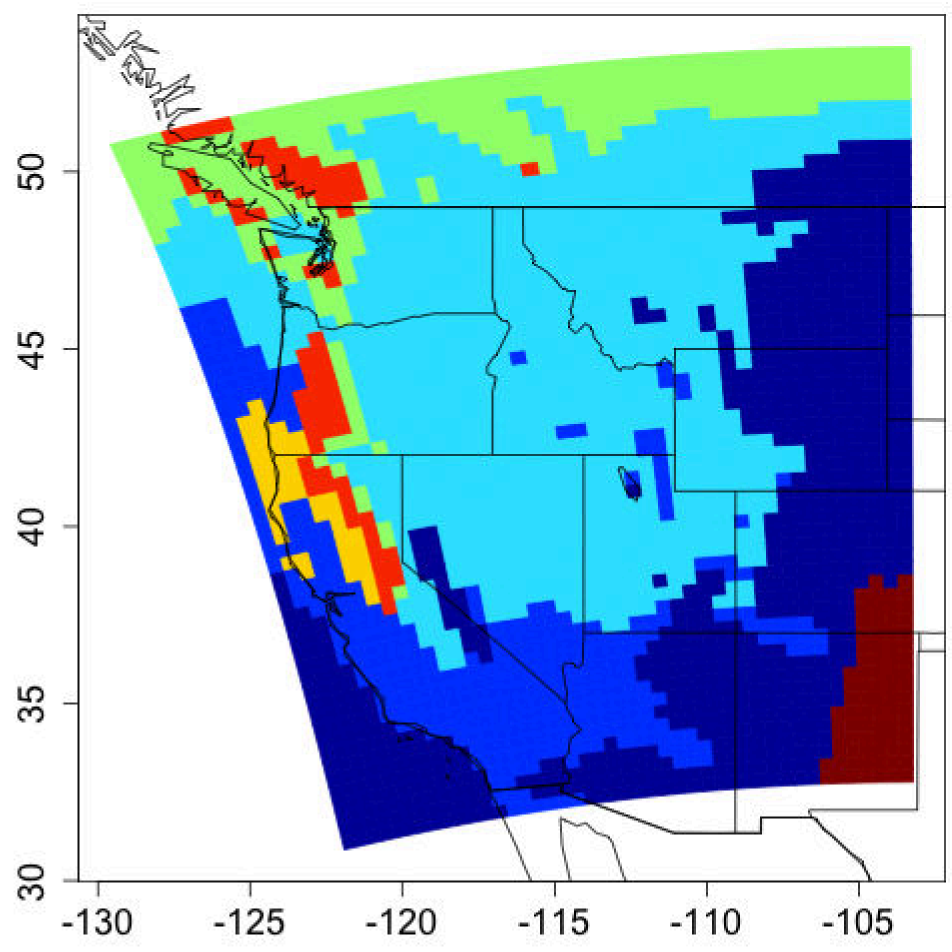


Clustering based on the posterior distribution



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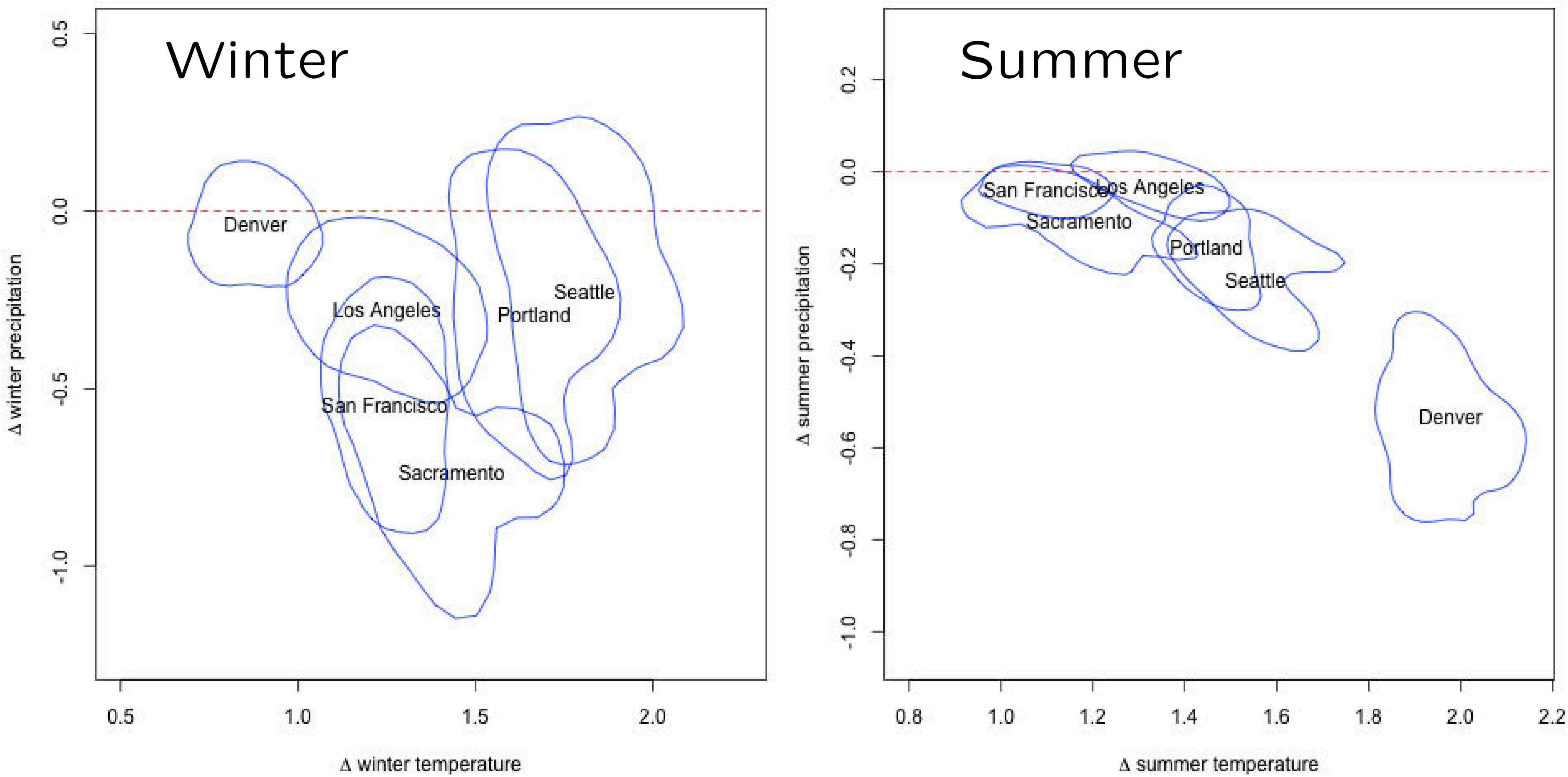
Example: temp. & precip. change



Approximate 95% contours for the average change in temperature and precipitation for five consolidated metropolitan areas.

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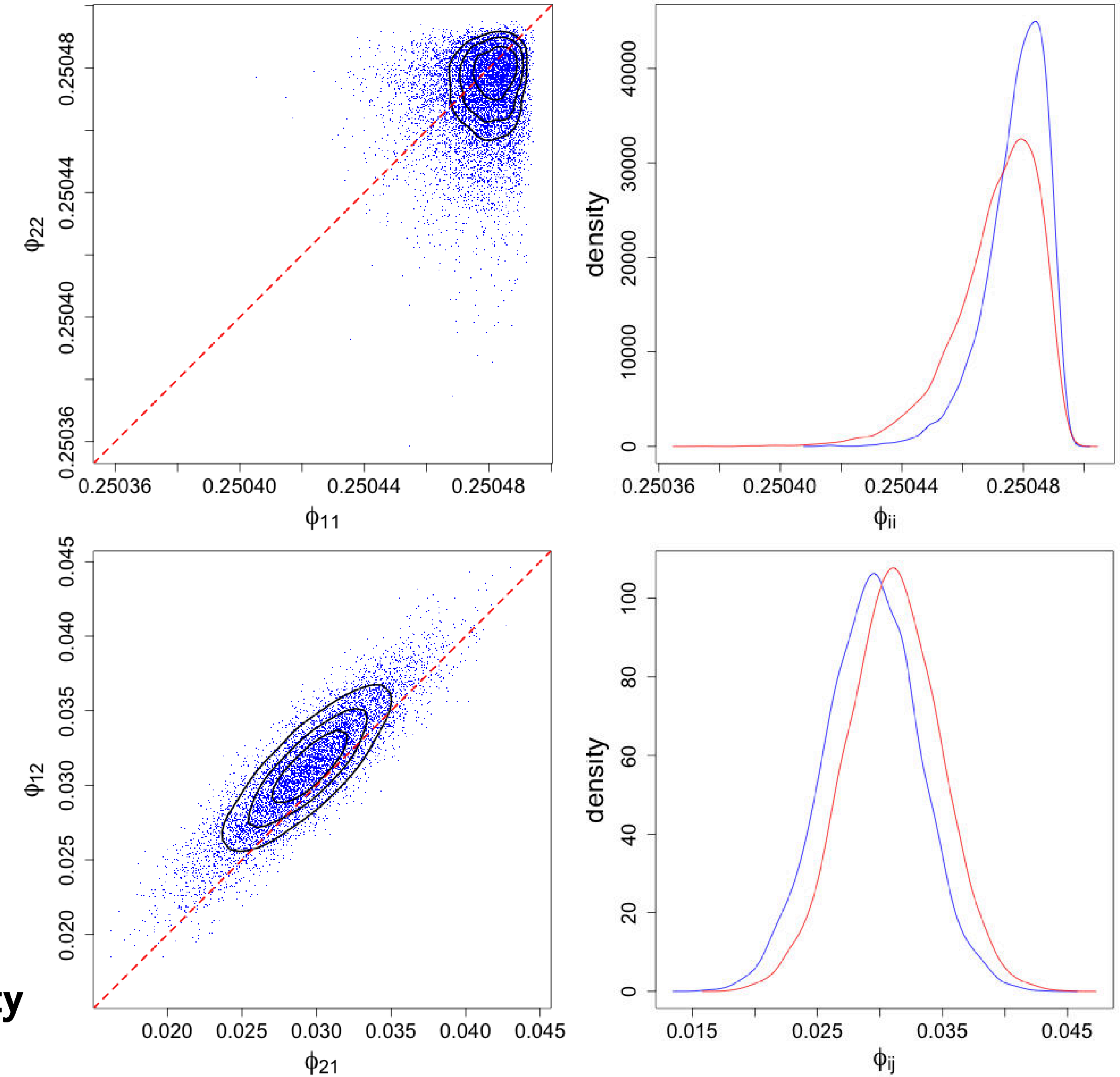
Example: temp. & precip. change



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Outlook

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  - \+ Flexible, numerically feasible
  - − Uncertainties — Backfitting: ① ② ③ ④ ◀
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  - − Numerical issues — Tapering: ① ③; ② ④ ◀
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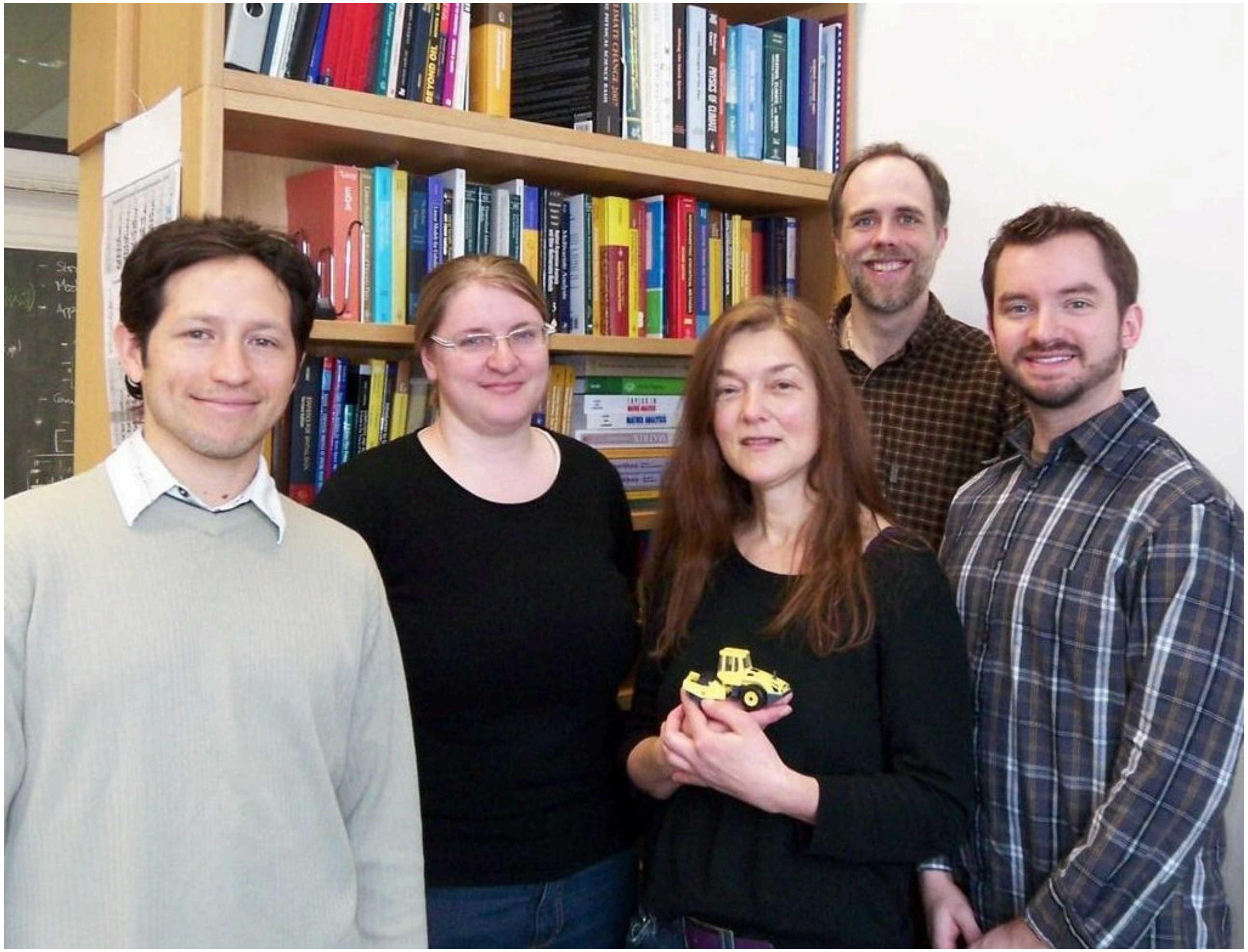
  - \+ Flexible, uncertainties
  - − MCMC — Dimension-reduction: ④ ◀



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...

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