Variance Reduction and Applications to Finance

Ahmed KEBAIER

Université Paris 13

19 Mars 2009

Introduction

General Framework

- Let $X \in \mathbb{R}^d$ be solution of

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^d,$$

  - $W = (W^1, \ldots, W^q)$ a B.M.
  - $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times q}$ of $\mathcal{C}^1$ such that
  - $\exists\ C_T > 0\,;\ \ \forall x, y \in \mathbb{R}^d$, such that

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq C_T|y - x|.$$

General Framework

- Let $X \in \mathbb{R}^d$ be solution of

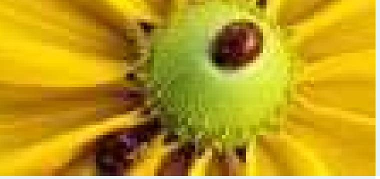
$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^d,$$

  - $W = (W^1, \ldots, W^q)$ a B.M.
  - $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times q}$ of $\mathcal{C}^1$ such that
  - $\exists\ C_T > 0\,;\ \ \forall x, y \in \mathbb{R}^d$, such that

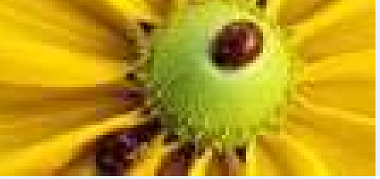
$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq C_T|y - x|.$$

- Let X^n the Euler scheme of step number $\delta = T/n$

$$dX_t^n = b(X_{\eta_n(t)})dt + \sigma(X_{\eta_n(t)})dW_t, \quad \eta_n(t) = [t/\delta]\delta.$$



On the discretisation error



Discretisation Error

- For a given f, we set

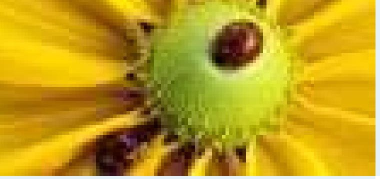
$$\varepsilon_n := \mathbb{E}f(X_T^n) - \mathbb{E}f(X_T)$$

Discretisation Error

- For a given f, we set

$$\varepsilon_n := \mathbb{E}f(X_T^n) - \mathbb{E}f(X_T)$$

- **Case I** (without nondegeneracy conditions)



Discretisation Error

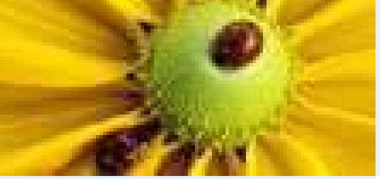
- For a given f, we set

$$\varepsilon_n := \mathbb{E}f(X_T^n) - \mathbb{E}f(X_T)$$

- **Case I** (without nondegeneracy conditions)

- Talay & Tubaro (90)

Si f est $\mathcal{C}^6$ alors $\varepsilon_n \simeq 1/n$



Discretisation Error

- For a given f, we set

$$\varepsilon_n := \mathbb{E}f(X_T^n) - \mathbb{E}f(X_T)$$

- **Case I** (without nondegeneracy conditions)

- Talay & Tubaro (90)

Si f est $\mathcal{C}^6$ alors $\varepsilon_n \simeq 1/n$

- Kurtz & Protter (99)

If f of $\mathcal{C}^3$ then $\varepsilon_n \simeq 1/n$

Useful Notation

- We set

$$\varphi(X_t) = \begin{pmatrix} b_1(X_t) & \sigma_{11}(X_t) & \dots & \sigma_{1q}(X_t) \\ b_2(X_t) & \sigma_{21}(X_t) & \dots & \sigma_{2q}(X_t) \\ \vdots & \vdots & & \vdots \\ b_d(X_t) & \sigma_{d1}(X_t) & \dots & \sigma_{dq}(X_t) \end{pmatrix}, dY_t = \begin{pmatrix} dt \\ dW_t^1 \\ \vdots \\ dW_t^q \end{pmatrix}$$

Useful Notation

- We set

$$\varphi(X_t) = \begin{pmatrix} b_1(X_t) & \sigma_{11}(X_t) & \dots & \sigma_{1q}(X_t) \\ b_2(X_t) & \sigma_{21}(X_t) & \dots & \sigma_{2q}(X_t) \\ \vdots & \vdots & & \vdots \\ b_d(X_t) & \sigma_{d1}(X_t) & \dots & \sigma_{dq}(X_t) \end{pmatrix}, dY_t = \begin{pmatrix} dt \\ dW_t^1 \\ \vdots \\ dW_t^q \end{pmatrix}$$

- The S.D.E can be written $dX_t = \varphi(X_t)dY_t$.

Useful Notation

- We set

$$\varphi(X_t) = \begin{pmatrix} b_1(X_t) & \sigma_{11}(X_t) & \dots & \sigma_{1q}(X_t) \\ b_2(X_t) & \sigma_{21}(X_t) & \dots & \sigma_{2q}(X_t) \\ \vdots & \vdots & & \vdots \\ b_d(X_t) & \sigma_{d1}(X_t) & \dots & \sigma_{dq}(X_t) \end{pmatrix}, dY_t = \begin{pmatrix} dt \\ dW_t^1 \\ \vdots \\ dW_t^q \end{pmatrix}$$

- The S.D.E can be written $dX_t = \varphi(X_t)dY_t$.
- The Euler Scheme X^n of step $\delta = T/n$ is given by:

$$dX_t^n = \varphi(X_{\eta_n(t)})dY_t, \quad \eta_n(t) = [t/\delta]\delta.$$

Useful Notation

- We set

$$\varphi(X_t) = \begin{pmatrix} b_1(X_t) & \sigma_{11}(X_t) & \dots & \sigma_{1q}(X_t) \\ b_2(X_t) & \sigma_{21}(X_t) & \dots & \sigma_{2q}(X_t) \\ \vdots & \vdots & & \vdots \\ b_d(X_t) & \sigma_{d1}(X_t) & \dots & \sigma_{dq}(X_t) \end{pmatrix}, dY_t = \begin{pmatrix} dt \\ dW_t^1 \\ \vdots \\ dW_t^q \end{pmatrix}$$

- The S.D.E can be written $dX_t = \varphi(X_t)dY_t$.
- The Euler Scheme X^n of step $\delta = T/n$ is given by:

$$dX_t^n = \varphi(X_{\eta_n(t)})dY_t, \quad \eta_n(t) = [t/\delta]\delta.$$

- **Theorem**

$$\sqrt{n}U^n =: \sqrt{n}(X^n - X) \Rightarrow^{stably} U,$$

Useful Notation

- With U solution to

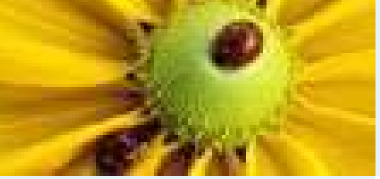
$$dU_t = \sum_{j=1}^{q+1} \varphi'^{j}\left(X_t\right)\left[U_t dY_t^j - \varphi\left(X_t\right) dN_t^j\right], \qquad U_0 = 0$$

Useful Notation

- With U solution to

$$dU_t = \sum_{j=1}^{q+1} \varphi^{,j}(X_t)\left[U_t dY_t^j - \varphi(X_t)\, dN_t^j\right], \quad U_0 = 0$$

- $\varphi^{,j} \in \mathbb{R}^{d \times d}$ with $(\varphi^{,j})_{ik} = \varphi_k^{,ij}$.



Useful Notation

- With U solution to

$$dU_t = \sum_{j=1}^{q+1} \varphi^{,j}(X_t)\left[U_t dY_t^j - \varphi(X_t)\, dN_t^j\right], \qquad U_0 = 0$$

- $\varphi^{,j} \in \mathbb{R}^{d\times d}$ with $(\varphi^{,j})_{ik} = \varphi_k^{,ij}$.

- $N \in \mathbb{R}^{q+1\times q+1}$:

$$\begin{cases} N^{1i} & = 0, \quad 1 \leq i \leq q+1, \\ N^{j1} & = 0, \quad 1 \leq j \leq q+1, \\ N^{ij} & = \frac{B_{ij}}{\sqrt{2}}, \quad 2 \leq i,j \leq q+1, \end{cases}$$

Useful Notations

- **Proposition**
  With the above notations we have

$$\tilde{\mathbb{E}}(U_T/\mathcal{F}_T) = 0.$$

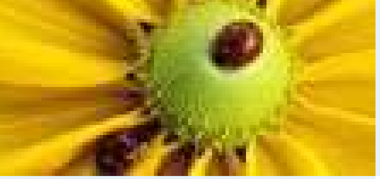
Useful Notations

- **Proposition**
  With the above notations we have

$$\tilde{\mathbb{E}}(U_T/\mathcal{F}_T) = 0.$$

- We recall that

$$U_T = \int_0^T \sum_{j=1}^{q+1} \varphi^{,j}\left(X_t\right)\left[U_t dY_t^j - \varphi\left(X_t\right) dN_t^j\right],$$



Useful notations

- $(\mathcal{H}_{\mathbf{f}}) \qquad |f(x) - f(y)| \leq C(1 + |x|^p + |y|^p)|x - y|$

Useful notations

- $(\mathcal{H}_{\mathbf{f}}) \qquad |f(x) - f(y)| \leq C(1 + |x|^p + |y|^p)|x - y|$
- $\mathcal{D}_f := \{x \in \mathbb{R}^d \mid f$ is differentiable on $x\}$

Useful notations

- $(\mathcal{H}_{\mathbf{f}}) \qquad |f(x) - f(y)| \leq C(1 + |x|^p + |y|^p)|x - y|$
- $\mathcal{D}_f := \{x \in \mathbb{R}^d \mid f$ is differentiable on $x\}$
- $\mathcal{C}_f = \Big\{ f \,/\, f$ satisfies$(\mathcal{H}_{\mathbf{f}})$ and $\mathbb{P}(X_T \notin \mathcal{D}_f) = 0 \Big\}$

Useful notations

- $(\mathcal{H}_{\mathbf{f}}) \qquad |f(x) - f(y)| \leq C(1 + |x|^p + |y|^p)|x - y|$
- $\mathcal{D}_f := \{x \in \mathbb{R}^d \mid f$ is differentiable on $x\}$
- $\mathcal{C}_f = \Big\{ f \, / \, f$ satisfies$(\mathcal{H}_{\mathbf{f}})$ and $\mathbb{P}(X_T \notin \mathcal{D}_f) = 0 \Big\}$

- **Proposition**

  For $f \in \mathcal{C}_f$ we have

Useful notations

- $(\mathcal{H}_{\mathbf{f}}) \qquad |f(x) - f(y)| \leq C(1 + |x|^p + |y|^p)|x - y|$
- $\mathcal{D}_f := \{x \in \mathbb{R}^d \mid f$ is differentiable on $x\}$
- $\mathcal{C}_f = \Big\{ f \,/\, f$ satisfies$(\mathcal{H}_{\mathbf{f}})$ and $\mathbb{P}(X_T \notin \mathcal{D}_f) = 0 \Big\}$

- **Proposition**

  For $f \in \mathcal{C}_f$ we have

  - $$\lim_{n\to\infty} \sqrt{n}\,\varepsilon_n := \lim_{n\to\infty} \sqrt{n}\left(\mathbb{E}f(X_T^n) - \mathbb{E}f(X_T)\right) = 0.$$

Useful notations

- $(\mathcal{H}_{\mathbf{f}}) \qquad |f(x) - f(y)| \leq C(1 + |x|^p + |y|^p)|x - y|$
- $\mathcal{D}_f := \{x \in \mathbb{R}^d \mid f$ is differentiable on $x\}$
- $\mathcal{C}_f = \left\{ f \,/\, f \text{ satisfies} (\mathcal{H}_{\mathbf{f}}) \text{ and } \mathbb{P}(X_T \notin \mathcal{D}_f) = 0 \right\}$

- **Proposition**

  For $f \in \mathcal{C}_f$ we have

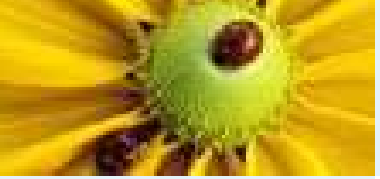
  - $$\lim_{n \to \infty} \sqrt{n}\, \varepsilon_n := \lim_{n \to \infty} \sqrt{n} \left(\mathbb{E} f(X_T^n) - \mathbb{E} f(X_T) \right) = 0.$$
  - $\forall \alpha \in [1/2, 1]$, $\exists f \in \mathcal{C}_f$ and $\exists X$ such that
    $$n^\alpha \varepsilon_n \to C_f(T, \alpha) \neq 0$$

On the statistical Error

CLT for Monte Carlo

- **Theorem**
  let $f \in \mathcal{C}_f$ s.t. for a given $\alpha \in [1/2, 1]$ we have

$$(\mathcal{H}_{\varepsilon_{\mathbf{n}}}) \qquad \lim_{n \to \infty} n^{\alpha} \varepsilon_n = C_f(T, \alpha).$$



CLT for Monte Carlo

- **Theorem**
  let $f \in \mathcal{C}_f$ s.t. for a given $\alpha \in [1/2, 1]$ we have

  $$(\mathcal{H}_{\varepsilon_{\mathbf{n}}}) \qquad \lim_{n\to\infty} n^{\alpha}\varepsilon_n = C_f(T,\alpha).$$

  - then

  $$n^{\alpha}\Big(\frac{1}{n^{2\alpha}}\sum_{i=1}^{n^{2\alpha}} f(X_{T,i}^{n}) - \mathbb{E}\, f(X_T)\Big) \Rightarrow \sigma\bar{G} + C_f(T,\alpha),$$

  with $\sigma^2 = Var(f(X_T))$.

- **Theorem**
  let $f \in \mathcal{C}_f$ s.t. for a given $\alpha \in [1/2, 1]$ we have

$$(\mathcal{H}_{\varepsilon_{\mathbf{n}}}) \qquad \lim_{n \to \infty} n^{\alpha} \varepsilon_n = C_f(T, \alpha).$$

  - then

$$n^{\alpha}\Big(\frac{1}{n^{2\alpha}} \sum_{i=1}^{n^{2\alpha}} f(X_{T,i}^n) - \mathbb{E}\, f(X_T)\Big) \Rightarrow \sigma \bar{G} + C_f(T, \alpha),$$

  with $\sigma^2 = Var(f(X_T))$.

- Complexity

$$\mathbf{C_{MC} = C \times n^{2\alpha+1}}$$

Statistical Romberg Algorithm

- We fix $m << n$ and set two Euler schemes X_T^n and X_T^m of step T/n and T/m.

Statistical Romberg Algorithm

- We fix $m << n$ and set two Euler schemes X_T^n and X_T^m of step T/n and T/m.
- Let

$$E_m = \mathbb{E}f\left(X_T^m\right).$$

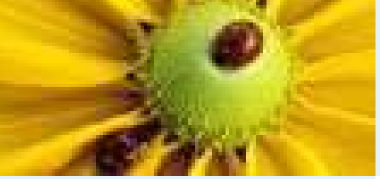
Statistical Romberg Algorithm

- We fix $m << n$ and set two Euler schemes X_T^n and X_T^m of step T/n and T/m.
- Let

$$E_m = \mathbb{E} f\left(X_T^m\right).$$

- We set

$$Q = f\left(X_T^n\right) - f\left(X_T^m\right) + E_m$$

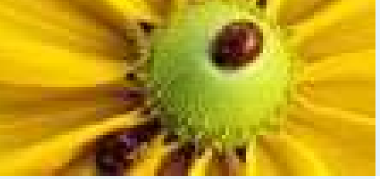


Statistical Romberg Algorithm

- We fix $m << n$ and set two Euler schemes X_T^n and X_T^m of step T/n and T/m.
- Let
$$E_m = \mathbb{E} f\left(X_T^m\right).$$
- We set
$$Q = f\left(X_T^n\right) - f\left(X_T^m\right) + E_m$$
- Note that
$$\mathbb{E}(Q) = \mathbb{E} f(X_T^n)$$

Statistical Romberg Algorithm

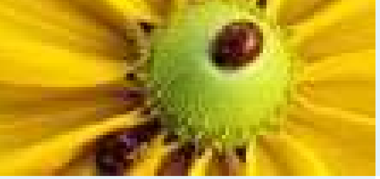
- We fix $m << n$ and set two Euler schemes X_T^n and X_T^m of step T/n and T/m.
- Let
$$E_m = \mathbb{E} f\left(X_T^m\right).$$
- We set
$$Q = f\left(X_T^n\right) - f\left(X_T^m\right) + E_m$$
- Note that
$$\mathbb{E}(Q) = \mathbb{E} f(X_T^n)$$
- **Proposition 1:**
$$\sigma_Q^2 =: Var(Q) = O\left(\frac{1}{m}\right).$$



Algorithm

- To evaluate E_m we construct $(f(\hat{X}^m_{T,i}))_{1\leq i\leq N_m}$ a sample of size N_m and compute:

$$\overline{E}_m = \frac{1}{N_m}\sum_{i=1}^{N_m} f(\hat{X}^m_{T,i}).$$



Algorithm

- To evaluate E_m we construct $(f(\hat{X}^m_{T,i}))_{1\le i\le N_m}$ a sample of size N_m and compute:

$$\overline{E}_m = \frac{1}{N_m}\sum_{i=1}^{N_m} f(\hat{X}^m_{T,i}).$$

- We construct $\big(f(X^n_{T,i}) - f(X^m_{T,i})\big)_{1\le i\le N_{n,m}}$ a sample of size $N_{n,m}$ s.t.:

Algorithm

- To evaluate E_m we construct $(f(\hat{X}^m_{T,i}))_{1\leq i\leq N_m}$ a sample of size N_m and compute:

$$\overline{E}_m = \frac{1}{N_m}\sum_{i=1}^{N_m} f(\hat{X}^m_{T,i}).$$

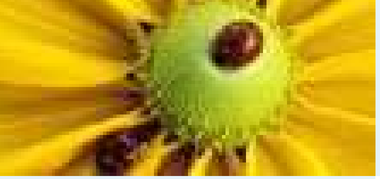
- We construct $\left(f(X^n_{T,i}) - f(X^m_{T,i})\right)_{1\leq i\leq N_{n,m}}$ a sample of size $N_{n,m}$ s.t.:
  - X^n_T and X^m_T are constructed using the same B.M.

Algorithm

- To evaluate E_m we construct $(f(\hat{X}^m_{T,i}))_{1\leq i\leq N_m}$ a sample of size N_m and compute:

$$\overline{E}_m = \frac{1}{N_m}\sum_{i=1}^{N_m} f(\hat{X}^m_{T,i}).$$

- We construct $\left(f(X^n_{T,i}) - f(X^m_{T,i})\right)_{1\leq i\leq N_{n,m}}$ a sample of size $N_{n,m}$ s.t.:
  - X^n_T and X^m_T are constructed using the same B.M.
  - $(X^n_T, X^m_T) \coprod \hat{X}^m_T$



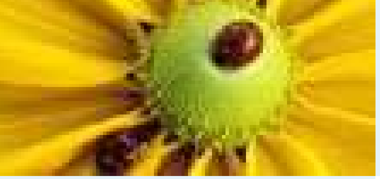
Algorithm

- To evaluate E_m we construct $(f(\hat{X}^m_{T,i}))_{1\leq i\leq N_m}$ a sample of size N_m and compute:

$$\overline{E}_m = \frac{1}{N_m}\sum_{i=1}^{N_m} f(\hat{X}^m_{T,i}).$$

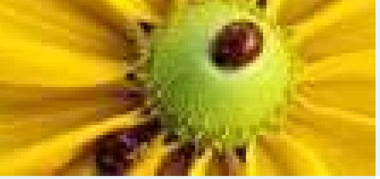
- We construct $\left(f(X^n_{T,i}) - f(X^m_{T,i})\right)_{1\leq i\leq N_{n,m}}$ a sample of size $N_{n,m}$ s.t.:
  - X^n_T and X^m_T are constructed using the same B.M.
  - $(X^n_T, X^m_T) \coprod \hat{X}^m_T$
- We compute

$$\overline{E}_{n,m} = \frac{1}{N_{n,m}}\sum_{i=1}^{N_{n,m}} f(X^n_{T,i}) - f(X^m_{T,i})$$



Problem

- How to choose n, m, N_n, $N_{n,m}$?



Problem

- How to choose n, m, N_n, $N_{n,m}$?
- We fix n and we set
  - $m = n^\beta, \quad 0 < \beta < 1$
  - $N_m = n^{\gamma_1}, \quad \gamma_1 > 0$
  - $N_{n,m} = n^{\gamma_2}, \quad \gamma_2 > 0$

Problem

- How to choose n, m, N_n, $N_{n,m}$?
- We fix n and we set
  - $m = n^{\beta}, \quad 0 < \beta < 1$
  - $N_m = n^{\gamma_1}, \quad \gamma_1 > 0$
  - $N_{n,m} = n^{\gamma_2}, \quad \gamma_2 > 0$
- Statistical Romberg method

$$V_n := \frac{1}{n^{\gamma_1}} \sum_{i=1}^{n^{\gamma_1}} f(\hat{X}_{T,i}^{n^{\beta}}) + \frac{1}{n^{\gamma_2}} \sum_{i=1}^{n^{\gamma_2}} f(X_{T,i}^{n}) - f(X_{T,i}^{n^{\beta}}),$$

Key

- Theorem (Jacod & Protter 1998)

For

$$U_t^n := X_t^n - X_t$$

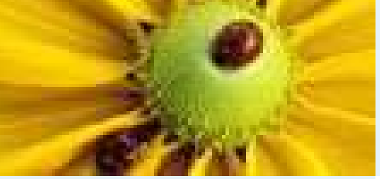
we have

$$\sqrt{n}U^n = \sqrt{n}(X_t^n - X_t) \Longrightarrow^{stable} U$$

CLT for Statistical Romberg method

- **Theorem**
  Let $f \in \mathcal{C}_f$ s.t. for a given $\alpha \in [1/2, 1]$ we have

$$(\mathcal{H}_{\varepsilon_{\mathbf{n}}}) \qquad \lim_{n\to\infty} n^{\alpha}\varepsilon_n = C_f(T,\alpha).$$



CLT for Statistical Romberg method

Theorem

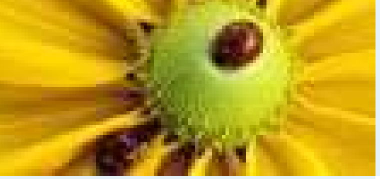
Let $f \in \mathcal{C}_f$ s.t. for a given $\alpha \in [1/2, 1]$ we have

$$(\mathcal{H}_{\varepsilon_{\mathbf{n}}}) \qquad \lim_{n \to \infty} n^{\alpha} \varepsilon_n = C_f(T, \alpha).$$

- for $\gamma_1 = 2\alpha$ and $\gamma_2 = 2\alpha - \beta$

$$n^{\alpha}\big(V_n - \mathbb{E} f(X_T)\big) \Rightarrow \sigma_2 \tilde{G} + C_f(T, \alpha),$$

with $\sigma_2^2 = Var(f(X_T)) + \tilde{V}ar\big(\nabla f(X_T).U_T\big)$



CLT for Statistical Romberg method

Theorem

Let $f \in \mathcal{C}_f$ s.t. for a given $\alpha \in [1/2, 1]$ we have

$$(\mathcal{H}_{\varepsilon_{\mathbf{n}}}) \qquad \lim_{n\to\infty} n^\alpha \varepsilon_n = C_f(T,\alpha).$$

- for $\gamma_1 = 2\alpha$ and $\gamma_2 = 2\alpha - \beta$

$$n^\alpha \big(V_n - \mathbb{E}\, f(X_T)\big) \Rightarrow \sigma_2 \tilde{G} + C_f(T,\alpha),$$

with $\sigma_2^2 = Var(f(X_T)) + \tilde{V}ar\big(\nabla f(X_T).U_T\big)$

$$\mathbf{C_{RS}} = \mathbf{C} \times \left(\mathbf{n}^{\beta+2\alpha} + \mathbf{n}^{1+2\alpha-\beta}\right)$$

CLT for Statistical Romberg method

- **Theorem**
  Let $f \in \mathcal{C}_f$ s.t. for a given $\alpha \in [1/2, 1]$ we have

$$(\mathcal{H}_{\varepsilon_{\mathbf{n}}}) \qquad \lim_{n\to\infty} n^\alpha \varepsilon_n = C_f(T,\alpha).$$

  - for $\gamma_1 = 2\alpha$ and $\gamma_2 = 2\alpha - \beta$

$$n^\alpha \big(V_n - \mathbb{E}\, f(X_T)\big) \Rightarrow \sigma_2 \tilde{G} + C_f(T,\alpha),$$

  with $\sigma_2^2 = Var(f(X_T)) + \tilde{V}ar\big(\nabla f(X_T).U_T\big)$

$$\mathbf{C_{RS} = C \times \left(n^{\beta+2\alpha} + n^{1+2\alpha-\beta}\right)}$$

$$\mathbf{C^\star_{RS} = C \times n^{2\alpha+1/2} < C_{MC} = C \times n^{2\alpha+1}}$$

umerical Results

- Consider the bi-dimensional diffusion

$$Z_t = \big(\cos(\theta + W_t), \sin(\theta + W_t)\big), \quad \theta \in [0, 2\pi],$$

and the map

$$f_\alpha : z = (x, y) \mapsto \big||z|^2 - 1\big|^{2\alpha} + x, \quad \alpha \in [1/2, 1],$$

umerical Results

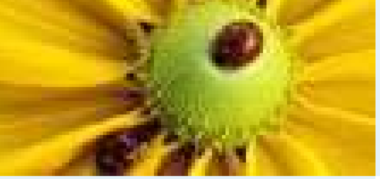
- Consider the bi-dimensional diffusion

$$Z_t = \big(\cos(\theta + W_t), \sin(\theta + W_t)\big), \qquad \theta \in [0, 2\pi],$$

and the map

$$f_\alpha : z = (x, y) \mapsto \left| |z|^2 - 1 \right|^{2\alpha} + x, \qquad \alpha \in [1/2, 1],$$

- We use $M = 200$ Random Values of $Z_0 = (X_0, Y_0)$



umerical Results

- Consider the bi-dimensional diffusion

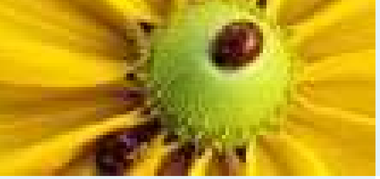
$$Z_t = \big(\cos(\theta + W_t), \sin(\theta + W_t)\big), \qquad \theta \in [0, 2\pi],$$

and the map

$$f_\alpha : z = (x, y) \mapsto \big||z|^2 - 1\big|^{2\alpha} + x, \qquad \alpha \in [1/2, 1],$$

- We use $M = 200$ Random Values of $Z_0 = (X_0, Y_0)$

- $RMS = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\text{Real value} - \text{ Simulated value})^2}$,



umerical Results

- Consider the bi-dimensional diffusion

$$Z_t = \Big(\cos(\theta + W_t), \sin(\theta + W_t)\Big), \quad \theta \in [0, 2\pi],$$

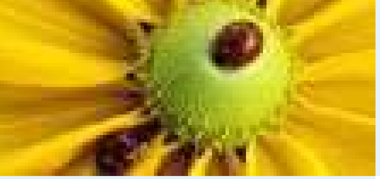
and the map

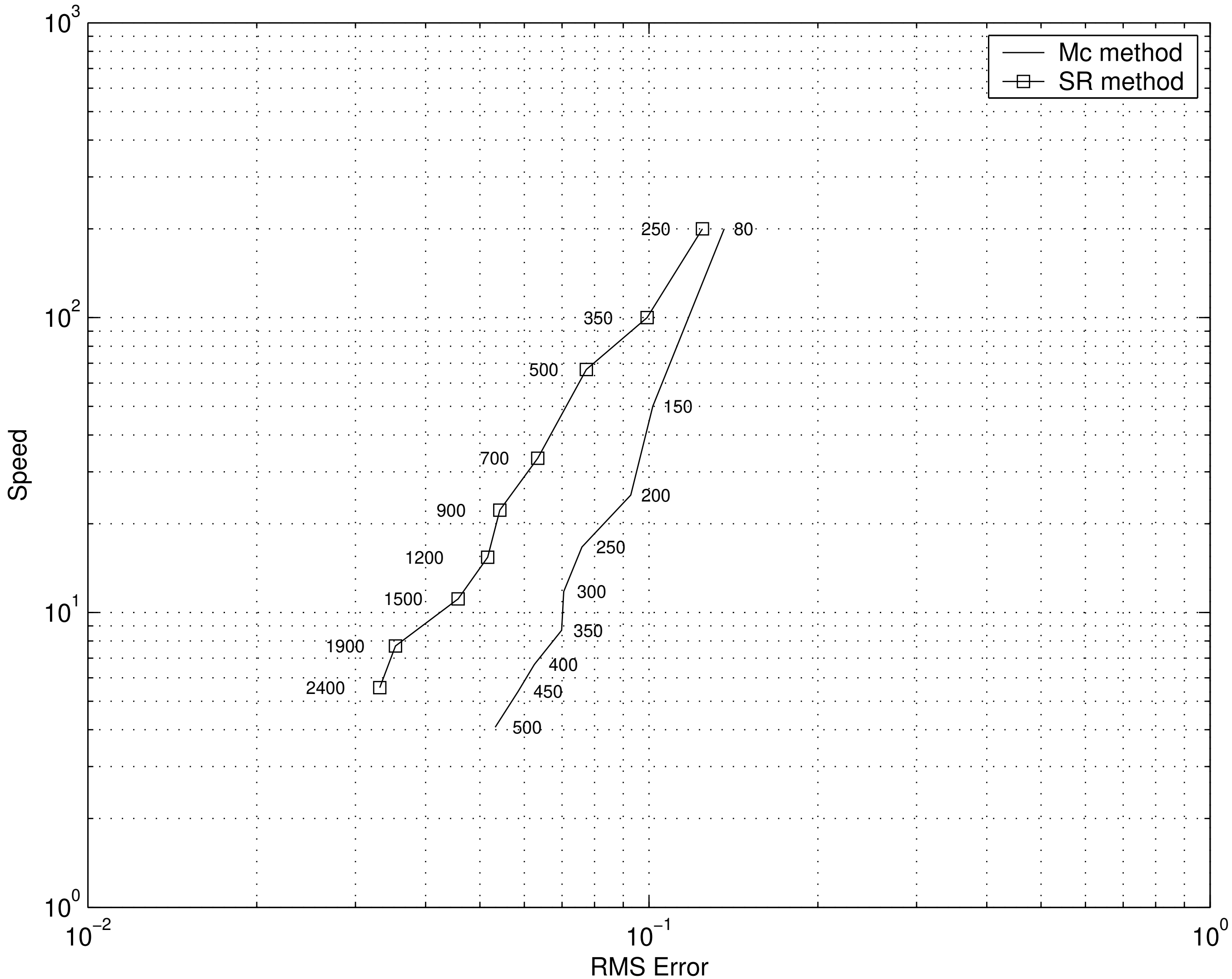
$$f_\alpha : z = (x, y) \mapsto \big||z|^2 - 1\big|^{2\alpha} + x, \quad \alpha \in [1/2, 1],$$

- We use $M = 200$ Random Values of $Z_0 = (X_0, Y_0)$

- $RMS = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\text{Real value} - \text{ Simulated value})^2}$,

- $Speed =$ Computation time$/M$





Mc method
SR method
Speed
RMS Error
10^0
10^1
10^2
10^3
10^{-2}
10^{-1}
10^0
250
350
500
700
900
1200
1500
1900
2400
80
150
200
250
300
350
400
450
500

RMS error	MC method Speed	SR method Speed
10^{-1}	45.405	102.718
9.10^{-2}	23.599	85.876
8.10^{-2}	18.647	70.712
7.10^{-2}	9.219	49.179
6.10^{-2}	5.883	29.234

Table 1

Complexity reduction for $\alpha = 1/2$

Application to Finance

- We consider the Blak & Schles model given by

$$dS_t = rS_t dt + \sigma dW_t, \quad t \in [0,T], \; T > 0.$$

The SR method for pricing Asian options

- We consider the Blak & Schles model given by

$$dS_t = rS_t dt + \sigma dW_t, \quad t \in [0,T],\ T > 0.$$

- we set $I_t = \dfrac{1}{T}\displaystyle\int_0^t S_u du.$

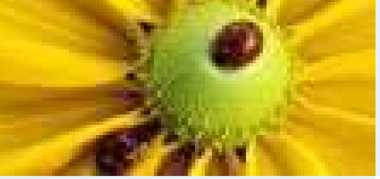
The SR method for pricing Asian options

- We consider the Blak & Schles model given by

$$dS_t = rS_t dt + \sigma dW_t, \quad t \in [0,T],\ T > 0.$$

- we set $I_t = \dfrac{1}{T}\displaystyle\int_0^t S_u du.$
- Aim:

$$\Pi = e^{-rT}\mathbb{E}f(I_T, S_T)$$



The SR method for pricing Asian options

- We consider the Blak & Schles model given by

$$dS_t = rS_t dt + \sigma dW_t, \quad t \in [0,T],\ T > 0.$$

- we set $I_t = \dfrac{1}{T}\displaystyle\int_0^t S_u du.$
- Aim:

$$\Pi = e^{-rT}\mathbb{E} f(I_T, S_T)$$

- Riemann scheme: $I_T \simeq \dfrac{h}{T}\displaystyle\sum_{k=0}^{n-1} S_{t_k}.$

Algorithm

- trapezoidal scheme:

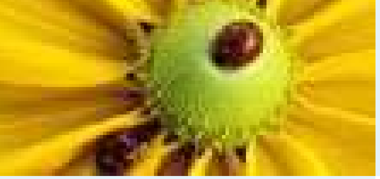
$$I_T^n = \frac{h}{T} \sum_{k=0}^{n-1} S_{t_k} \left(1 + \frac{rh}{2} + \sigma \frac{W_{t_{k+1}} - W_{t_k}}{2} \right).$$

Algorithm

- trapezoidal scheme:

$$I_T^n = \frac{h}{T} \sum_{k=0}^{n-1} S_{t_k} \left(1 + \frac{rh}{2} + \sigma \frac{W_{t_{k+1}} - W_{t_k}}{2} \right).$$

- The SR method approximate Π by: $\Pi_{RS} = \bar{E}_1 + \bar{E}_2,$ with

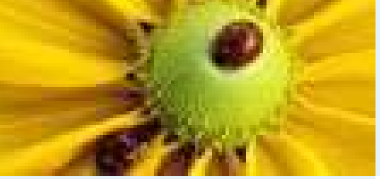


Algorithm

- trapezoidal scheme:

$$I_T^n = \frac{h}{T}\sum_{k=0}^{n-1} S_{t_k}\left(1 + \frac{rh}{2} + \sigma\frac{W_{t_{k+1}} - W_{t_k}}{2}\right).$$

- The SR method approximate Π by: $\Pi_{RS} = \bar{E}_1 + \bar{E}_2,$ with
- $\bar{E}_1 = \frac{e^{-rT}}{n^{\alpha_1}}\sum_{i=1}^{n^{\alpha_1}} f(\hat{I}_{T,i}^{n^\beta})$



Algorithm

- trapezoidal scheme:

$$I_T^n = \frac{h}{T}\sum_{k=0}^{n-1} S_{t_k}\left(1 + \frac{rh}{2} + \sigma\frac{W_{t_{k+1}} - W_{t_k}}{2}\right).$$

- The SR method approximate Π by: $\Pi_{RS} = \bar{E}_1 + \bar{E}_2,$ with
- $\bar{E}_1 = \dfrac{e^{-rT}}{n^{\alpha_1}} \displaystyle\sum_{i=1}^{n^{\alpha_1}} f(\hat{I}_{T,i}^{n^\beta})$
- $\bar{E}_2 = \dfrac{e^{-rT}}{n^{\alpha_2}} \displaystyle\sum_{i=1}^{n^{\alpha_2}} f(\bar{I}_{T,i}^{n}) - f(\bar{I}_{T,i}^{n^\beta})$

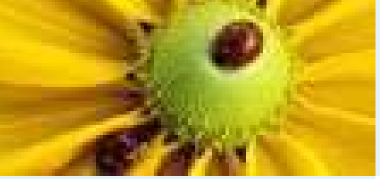
Algorithm

- trapezoidal scheme:

$$I_T^n = \frac{h}{T} \sum_{k=0}^{n-1} S_{t_k} \left(1 + \frac{rh}{2} + \sigma \frac{W_{t_{k+1}} - W_{t_k}}{2} \right).$$

- The SR method approximate Π by: $\Pi_{RS} = \bar{E}_1 + \bar{E}_2,$ with
- $\bar{E}_1 = \dfrac{e^{-rT}}{n^{\alpha_1}} \displaystyle\sum_{i=1}^{n^{\alpha_1}} f(\hat{I}_{T,i}^{n^\beta})$
- $\bar{E}_2 = \dfrac{e^{-rT}}{n^{\alpha_2}} \displaystyle\sum_{i=1}^{n^{\alpha_2}} f(\bar{I}_{T,i}^{n}) - f(\bar{I}_{T,i}^{n^\beta})$

- **Problem: how to choose α_1, α_2 et β ?**



The error distribution on the trapezodal scheme

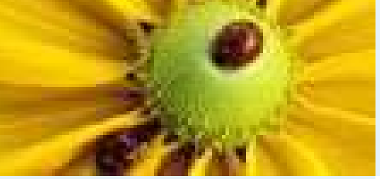
- **Theorem:**
  Using the above notation

$$n(I - I^n) \Rightarrow^{stably} \chi$$

where

$$\chi_t := \frac{\sigma}{2\sqrt{3}} \int_0^t S_u dB'_s$$

with B' a standard B.M independent of W.



The SR method parameters

- **Theorem:**
  Let f be a given function s.t.

  $$(\mathcal{H}'_{\mathbf{f}})\ |f(x,y_1) - f(x,y_2)| \leq C(1 + |x|^p + |y_1|^p + |y_2|^p)|y_1 - y_2|,$$

  Suppose that $\mathbb{P}\big((S_T, I_T) \notin \mathscr{D}_f\big) = 0$, with

  $$\mathscr{D}_f := \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid \partial_2 f(x,y) \text{ exists}\}.$$

  Then for all $\beta \in (0,1)$, if $\gamma_1 = 2$ and $\gamma_2 = 2 - 2\beta$ we have

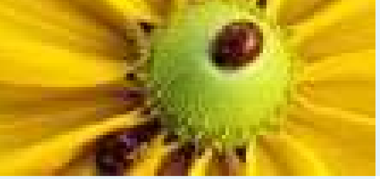
  $$n(E_n - \mathbb{E}\, f(S_T, I_T)) \Rightarrow \hat{\sigma}_2 \hat{G} + \hat{\mathbb{E}}\, (\partial_2 f(S_T, I_T)\chi_T)\,,$$

  avec $\hat{\sigma}_2^2 = Var(f(S_T, I_T)) + \hat{V}ar(\partial_2 f(S_T, I_T)\chi_T)$.

Complexity ananlysis

- The SR method complexity

$$C_{SR} \simeq n^{\alpha_1}.n^{\beta} + n^{\alpha 2}.(n + n^{\beta})$$
$$\simeq n^{\beta+2} + (n + n^{\beta})n^{2-2\beta}$$



Complexity ananlysis

- The SR method complexity

$$C_{SR} \simeq n^{\alpha_1}.n^{\beta} + n^{\alpha 2}.(n + n^{\beta})$$
$$\simeq n^{\beta+2} + (n + n^{\beta})n^{2-2\beta}$$

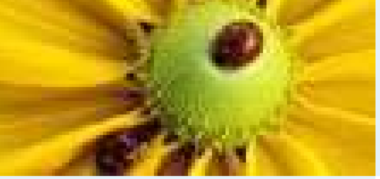
- $\beta^{\star} = n^{1/3}$ which gives:

$$C_{SR} \simeq n^{7/3}$$

Remark

- A byproduct of the above CLT

$$\lim_{n\to\infty} n\mathbb{E}\left(f(S_T, I_T^n) - f(S_T, I_T)\right) = \hat{\mathbb{E}}\big(\partial_2 f(S_T, I_T)\chi_T\big).$$



Remark

- A byproduct of the above CLT

$$\lim_{n\to\infty} n\mathbb{E}\left(f(S_T, I_T^n) - f(S_T, I_T)\right) = \hat{\mathbb{E}}\big(\partial_2 f(S_T, I_T)\chi_T\big).$$

- which is an alternative to the result of B.Lapeyre & E.Temam (2001)

Controle variate of Kemna & Vorst

- We approximate $\frac{1}{T}\int_0^T S_u du$ by:

$$\exp(\underbrace{\frac{1}{T}\int_0^T \log\left(S_u\right) du}_{=Z})$$

Controle variate of Kemna & Vorst

- We approximate $\frac{1}{T}\int_0^T S_u du$ by:

$$\exp(\underbrace{\frac{1}{T}\int_0^T \log\left(S_u\right) du}_{=Z})$$

- we obtain

$$\Pi = e^{-rT}\mathbb{E}\left[(I_T - K)_+ - (\exp(Z) - K)_+\right] + \underbrace{e^{-rT}\mathbb{E}\left(\exp(Z) - K\right)_+}_{=A}$$

MC method

- T-MC method(No variance reduction):

$$\bar{\Pi}_{MC} = \frac{e^{-rT}}{n^2} \sum_{i=1}^{n^2} \left(\bar{I}^n_{T,i} - K\right)_+ .$$

- T-KV-MC method (With variance reduction) :

$$\bar{\Pi}_{KV-MC} = \frac{e^{-rT}}{n^2} \sum_{i=1}^{n^2} \left(\bar{I}^n_{T,i} - K\right)_+ - \left(\bar{Z}^n_{T,i} - K\right)_+$$

$$+A$$

SR method(No variance reduction)

- T-SR method:

$$\Pi_{RS} = \bar{E}_1 + \bar{E}_2$$

$$\bar{E}_1 = \frac{e^{-rT}}{n^2} \sum_{i=1}^{n^2} \left(\widehat{I}_{T,i}^{n^{1/3}} - K \right)_+$$

$$\bar{E}_2 = \frac{e^{-rT}}{n^{4/3}} \sum_{i=1}^{n^{4/3}} \left(\bar{I}_{T,i}^{n} - K \right)_+ - \left(\bar{I}_{T,i}^{n^{1/3}} - K \right)_+ .$$

SR method (With variance reduction)

- T-KV-RS method:

$$\Pi_{KV-RS} = \bar{F}_1 + \bar{F}_2$$

SR method (With variance reduction)

- T-KV-RS method:

$$\Pi_{KV-RS} = \bar{F}_1 + \bar{F}_2$$

- $\bar{F}_1 = \frac{e^{-rT}}{n^2} \sum_{i=1}^{n^2} \left(\widehat{I}_{T,i}^{n^{1/3}} - K \right)_+ - \left(\widehat{Z}_{T,i}^{n^{1/3}} - K \right)_+ + A$

SR method (With variance reduction)

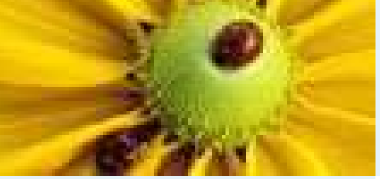
- T-KV-RS method:

$$\Pi_{KV-RS} = \bar{F}_1 + \bar{F}_2$$

- $\bar{F}_1 = \dfrac{e^{-rT}}{n^2} \displaystyle\sum_{i=1}^{n^2} \left(\widehat{I}_{T,i}^{n^{1/3}} - K\right)_+ - \left(\widehat{Z}_{T,i}^{n^{1/3}} - K\right)_+ + A$

- $\bar{F}_2 = \dfrac{e^{-rT}}{n^{\frac{4}{3}}} \displaystyle\sum_{i=1}^{n^{\frac{4}{3}}} \left(\bar{I}_{T,i}^{n} - K\right)_+ - \left(\bar{Z}_{T,i}^{n} - K\right)_+$

$$+ \left(\bar{I}_{T,i}^{n^{1/3}} - K\right)_+ - \left(\bar{Z}_{T,i}^{n^{1/3}} - K\right)_+ .$$

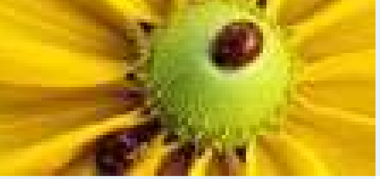


Performances (without variance reduction)

RMS	T-MC Speed	T-SR Speed
3.10^{-2}	481.05	777.82
10^{-2}	55.57	129.92
7.10^{-3}	21.76	67.28
5.10^{-3}	7.95	35.97
3.10^{-3}	1.66	10.97

Table 2

Complexity reduction for the Asian Call



Performances (avec réduction de variance)

RMS	T-KV-MC Speed	T-KV-SR Speed
2.10^{-3}	79.3994	122.1907
10^{-3}	12.6353	30.2952
8.10^{-4}	7.4484	20.9045
6.10^{-4}	3.2687	11.6519
3.10^{-4}	0.1460	1.97

Table 3

Complexity reduction for the Asian Call

with the control variate of Kemna & Vorst