Correlated Bernoulli Process using De Bruijn Graphs

LOUISE KIMPTON

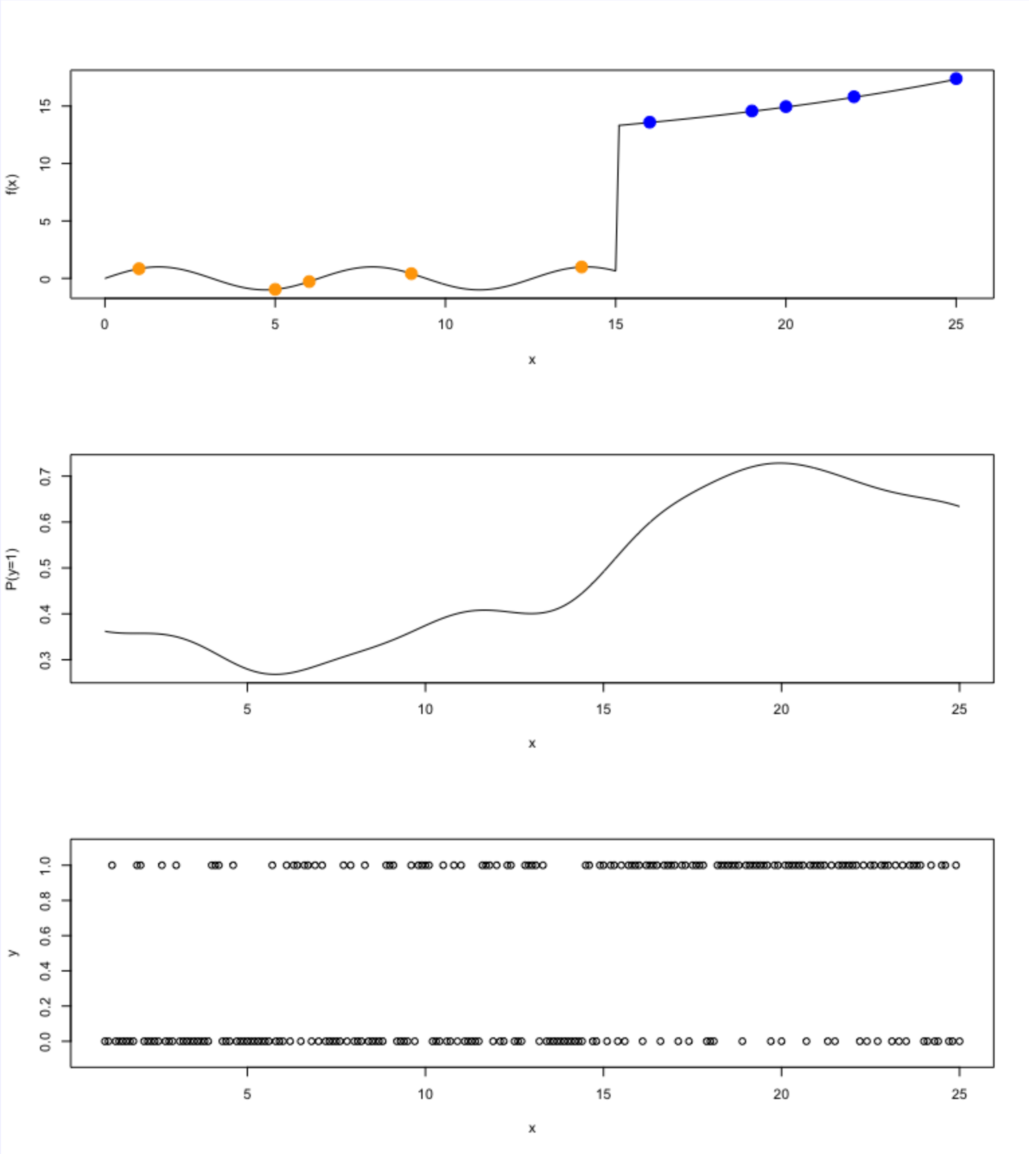
EMAIL: LMK212@EXETER.AC.UK

Introduction

- Want to create chains of 0’s and 1’s that cluster or stick together
- Put structure into a Bernoulli distribution to make a correlated Bernoulli process
- We do this using de Bruijn graphs

Motivation

- Improvement on logistic regression and classification
- Need to include correlation between points
- Look for a clean boundary with no drop-outs



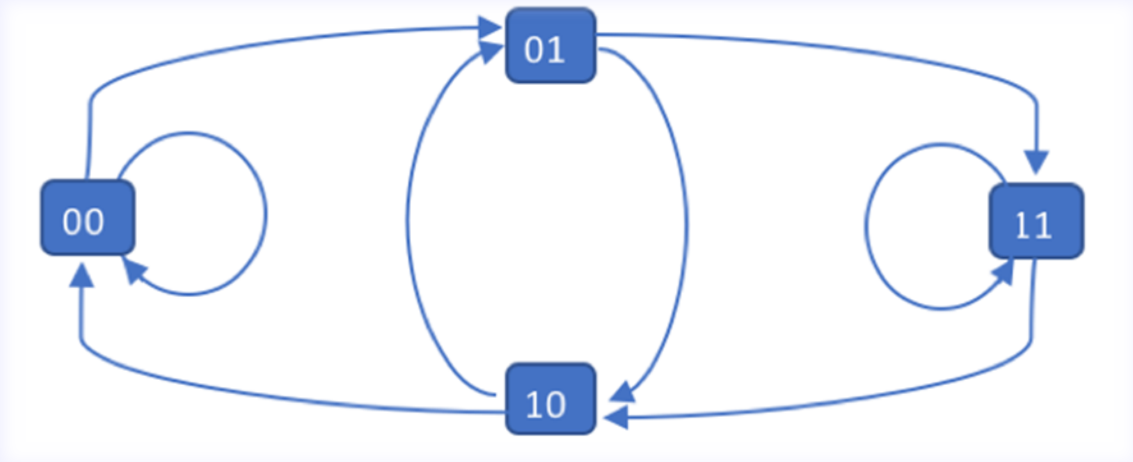
De Bruijn Graphs

- Directed graphs where nodes consist of all possible length m sequences (words) given a set of symbols
- m is the word length which controls how spread the correlation is (how many points the current point is dependent on)
- A probability is associated with each arc of the graph – gives the probability of transitioning from word to word

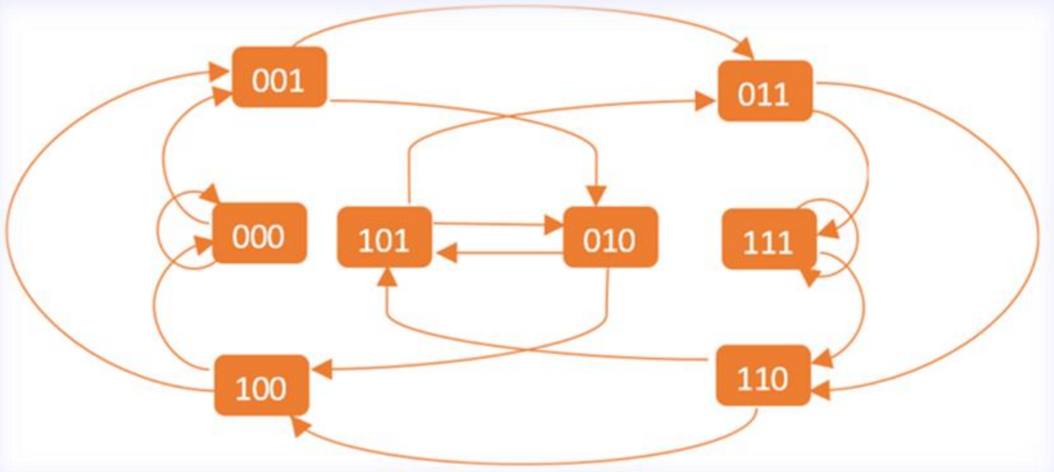
P_i^j – probability of transitioning from word i to word j

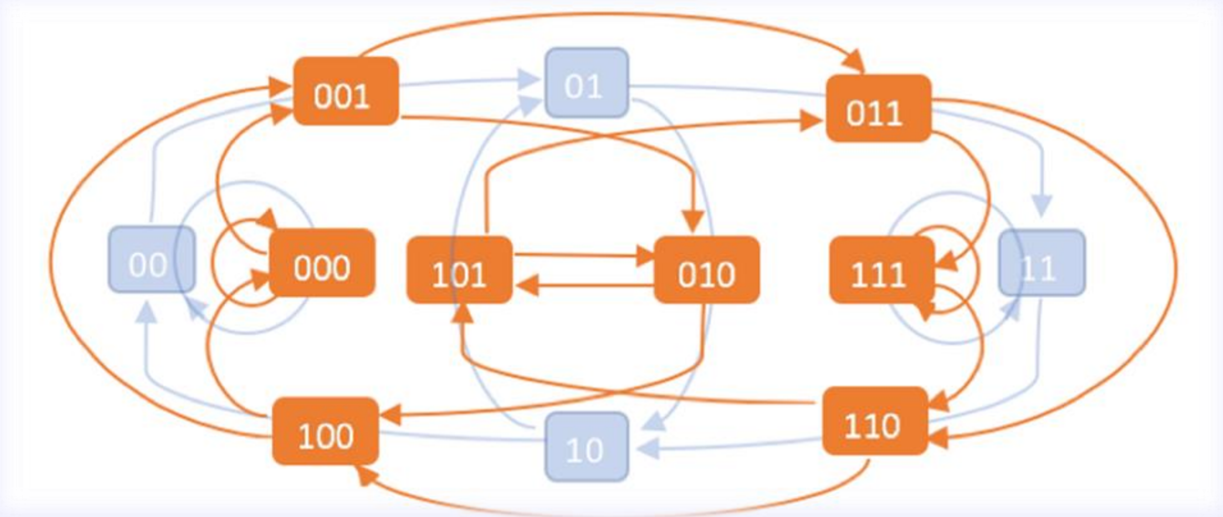
- Use symbols 0 and 1 to correspond to regions

Word Length m = 2



Word Length m = 3



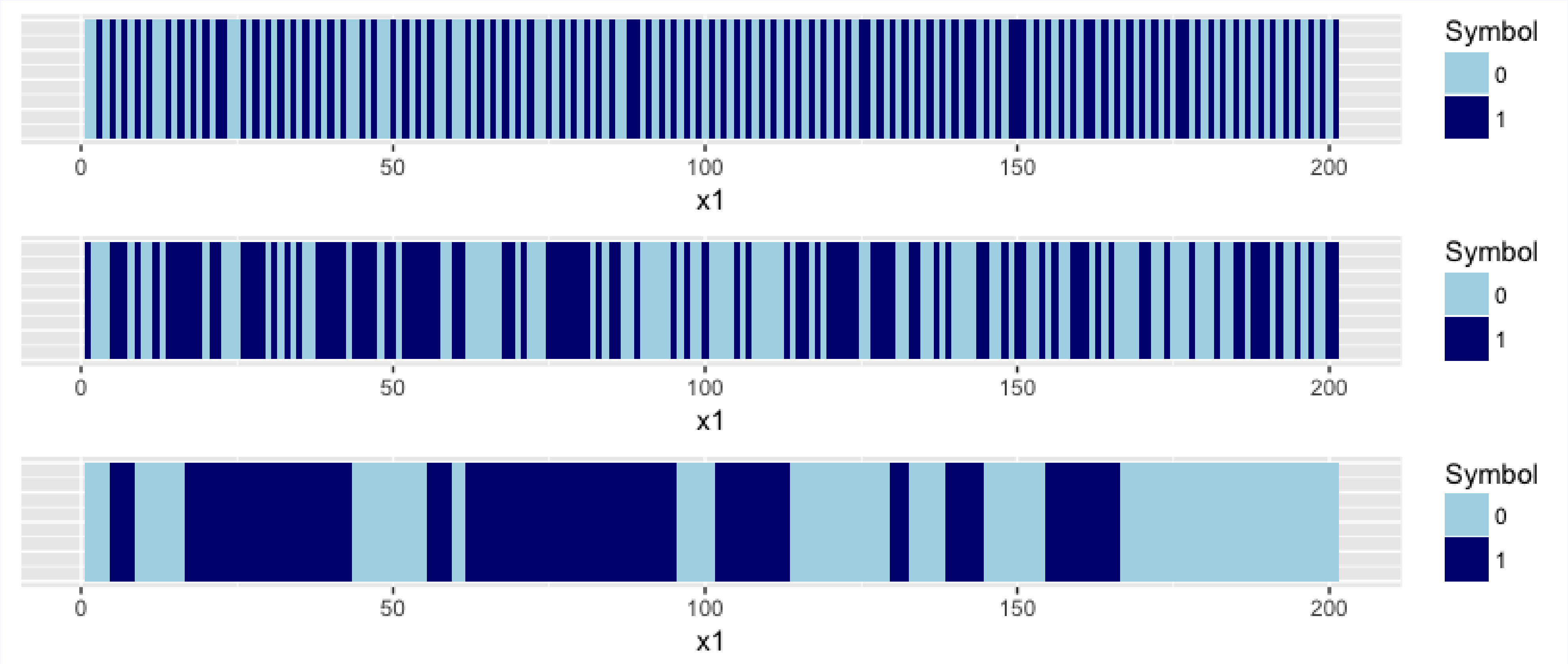


Markov Properties

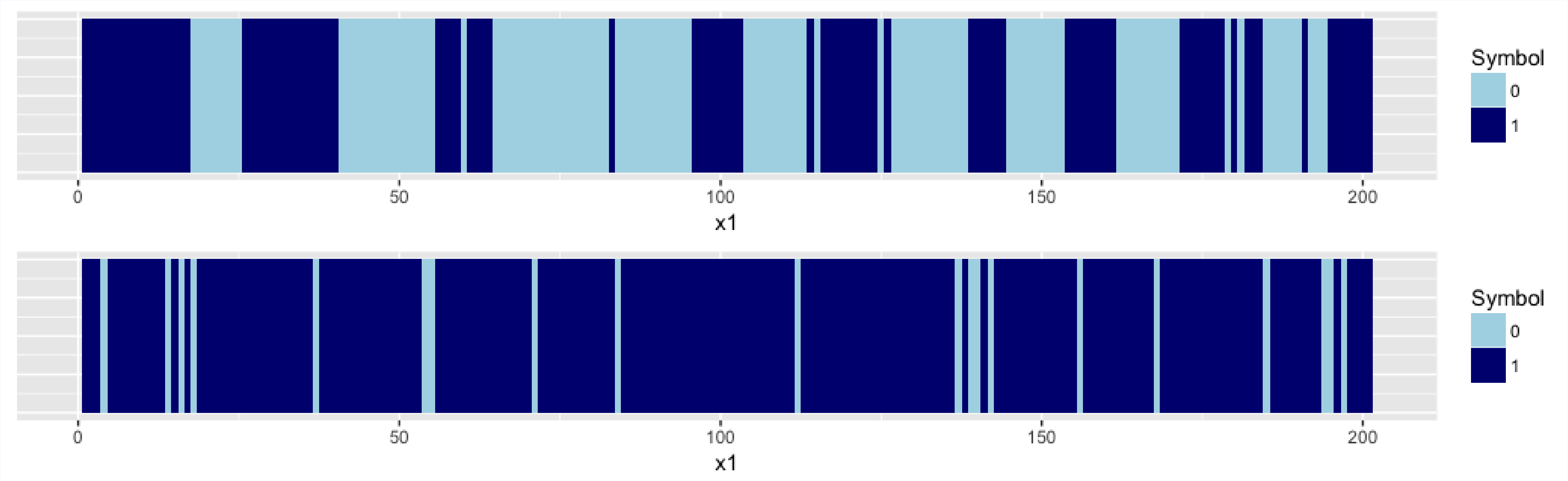
- Can write the transition probabilities in matrix form
- Then can use this to generate chains of 0s and 1s
- Can create stickiness in the chains by choosing specific transition probabilities
- Marginal probabilities stay the same over time but 0s and 1s are grouped together

$$T = \begin{pmatrix} 1 - p_{00}^{01} & p_{00}^{01} & 0 & 0 \\ 0 & 0 & 1 - p_{01}^{11} & p_{01}^{11} \\ 1 - p_{10}^{01} & p_{10}^{01} & 0 & 0 \\ 0 & 0 & 1 - p_{11}^{11} & p_{11}^{11} \end{pmatrix}$$

Examples


Symbol
0
1
0
50
100
150
200
x1
Symbol
0
1
0
50
100
150
200
x1
Symbol
0
1
0
50
100
150
200
x1

Examples



Run Length Distribution

Word Length
m = 2

$$P(\textbf{run length} = n) = \begin{cases} p_{01}^{10} & \text{for } \; n = 1 \\ p_{01}^{11}(p_{11}^{11})^{n-2}p_{11}^{10} & \text{for } \; n \geq 2, \end{cases}$$

$$E\left[\textbf{run length}\right] = p_{01}^{10} + \frac{p_{01}^{11}\left(1 - (p_{11}^{10})^2\right)}{p_{11}^{11}p_{11}^{10}}.$$

Run Length Distribution

Word Length
$m \geq 3$

$$P(\textbf{Run Length} = n)$$

$$= \begin{cases} \sum_{i=0}^{2^{m-2}-1} \pi(i)\, p_{4i+1}^{2^3(i \bmod 2^{m-3})+2} & \text{for } n = 1 \\ \sum_{i=0}^{2^{m-2}-1} \pi(i)\, p_{4i+1}^{2^3(i \bmod 2^{m-3})+3} & \text{for } n = 2 : m-1 \\ \quad \times \left[\prod_{j=1}^{n-1} p_{2^{j+2}(i \bmod 2^{m-j-2})+(2^{j+1}-1)}^{2^{j+3}(i \bmod 2^{m-j-3})+(2^{j+2}-1)-\mathbf{1}_{j=n-1}} \right] & \\ \sum_{i=0}^{2^{m-2}-1} \pi(i)\, p_{4i+1}^{2^3(i \bmod 2^{m-3})+3} & \\ \quad \times \left[\prod_{j=1}^{m-2} p_{2^{j+2}(i \bmod 2^{m-j-2})+(2^{j+1}-1)}^{2^{j+3}(i \bmod 2^{m-j-3})+(2^{j+2}-1)} \right] & \text{for } n \geq m. \\ \quad \times \left[\left(p_{2^m-1}^{2^m-1} \right)^{n-m} p_{2^m-1}^{2^m-2} \right] & \end{cases}$$

where,

$$\pi(i) = \sum_{j=0}^{2^m-1} \prod_{k=0}^{m-3} \left[p_{2^k(j \bmod 2^{m-k})+\sum_{s=1}^{k} 2^{k-s}[(\frac{1}{2^{m-s-2}}(i-(i \bmod 2^{m-s-2})))\bmod 2]}^{2^{k+1}(j \bmod 2^{m-k-1})+\sum_{s=1}^{k+1} 2^{k-s+1}[(\frac{1}{2^{m-s-2}}(i-(i \bmod 2^{m-s-2})))\bmod 2]} \right] \pi(j)$$

Transition Likelihood

p_i^j – transition probability for the word ij

n_i^j – number of words, ij, in the sequence

m = 2

$$\mathcal{L} = (p_{00}^{00})^{n_{00}^{00}} (p_{00}^{01})^{n_{00}^{01}} (p_{01}^{10})^{n_{01}^{10}} (p_{01}^{11})^{n_{01}^{11}} (p_{10}^{00})^{n_{10}^{00}} (p_{10}^{01})^{n_{10}^{01}} (p_{11}^{10})^{n_{11}^{10}} (p_{11}^{11})^{n_{11}^{11}}$$
$$= (1 - p_{00}^{01})^{n_{00}^{00}} (p_{00}^{01})^{n_{00}^{01}} (1 - p_{01}^{11})^{n_{01}^{10}} (p_{01}^{11})^{n_{01}^{11}} (1 - p_{10}^{01})^{n_{10}^{00}} (p_{10}^{01})^{n_{10}^{01}} (1 - p_{11}^{11})^{n_{11}^{10}} (p_{11}^{11})^{n_{11}^{11}}$$

m ≥ 3

$$\mathcal{L} = \prod_{i=0}^{2^{m+1}-1} \left(p_{\frac{1}{4}(-1)^{i+1}[2(-1)^{i+1}(i+1)-3(-1)^{i+1}-1]}^{i \bmod (2^{m+1}-1)}\right)^{n_{\frac{1}{4}(-1)^{i+1}[2(-1)^{i+1}(i+1)-3(-1)^{i+1}-1]}^{i \bmod (2^{m+1}-1)}}$$
$$= \prod_{i=0}^{2^m-1} \left(1 - p_i^{(2i+1) \bmod 2^m}\right)^{n_i^{((2i+1) \bmod 2^m)-1}} \left(p_i^{(2i+1) \bmod 2^m}\right)^{n_i^{(2i+1) \bmod 2^m}}$$

Conjugate Prior

$$P(p|seq) = \frac{P(seq|p)\,P(p)}{P(seq)}$$

Likelihood (m=2)

$$\mathcal{L} = (p_{00}^{00})^{n_{00}^{00}}\,(p_{00}^{01})^{n_{00}^{01}}\,(p_{01}^{10})^{n_{01}^{10}}\,(p_{01}^{11})^{n_{01}^{11}}\,(p_{10}^{00})^{n_{10}^{00}}\,(p_{10}^{01})^{n_{10}^{01}}\,(p_{11}^{10})^{n_{11}^{10}}\,(p_{11}^{11})^{n_{11}^{11}}$$

Posterior with beta prior

$$\begin{aligned}
P \propto\ & (p_{00}^{00})^{n_{00}^{00}}\,(p_{00}^{01})^{n_{00}^{01}}\,(p_{00}^{00})^{\beta_1-1}\,(p_{00}^{01})^{\alpha_1-1} \times \\
& (p_{01}^{10})^{n_{01}^{10}}\,(p_{01}^{11})^{n_{01}^{11}}\,(p_{01}^{10})^{\beta_2-1}\,(p_{01}^{11})^{\alpha_2-1} \times \\
& (p_{10}^{00})^{n_{10}^{00}}\,(p_{10}^{01})^{n_{10}^{01}}\,(p_{01}^{10})^{\beta_3-1}\,(p_{01}^{11})^{\alpha_3-1} \times \\
& (p_{11}^{10})^{n_{11}^{10}}\,(p_{11}^{11})^{n_{11}^{11}}\,(p_{01}^{10})^{\beta_4-1}\,(p_{01}^{11})^{\alpha_4-1}
\end{aligned}$$

pdf of beta distribution:

$$P(x;\alpha,\beta) \propto x^{\alpha-1}(1-x)^{\beta-1}$$

$$\begin{aligned}
= & (p_{00}^{00})^{n_{00}^{00}+\beta_1-1}\,(p_{00}^{01})^{n_{00}^{01}+\alpha_1-1}\,(p_{01}^{10})^{n_{01}^{10}+\beta_2-1}\,(p_{01}^{11})^{n_{01}^{11}+\alpha_2-1} \times \\
& (p_{10}^{00})^{n_{10}^{00}+\beta_3-1}\,(p_{10}^{01})^{n_{10}^{01}+\alpha_3-1}\,(p_{11}^{10})^{n_{11}^{10}+\beta_4-1}\,(p_{11}^{11})^{n_{11}^{11}+\alpha_4-1}
\end{aligned}$$

Inference for word length m

The posterior is now a product of beta densities. With the conjugate relationship, we can state the following:

$$P(seq) = \int P(seq|p)P(p)dp = \frac{\Gamma(n_{00}^{00}+\beta_1)\Gamma(n_{00}^{01}+\alpha_1)}{\Gamma(n_{00}^{00}+n_{00}^{01}+\beta_1+\alpha_1)} \times \frac{\Gamma(n_{01}^{10}+\beta_2)\Gamma(n_{01}^{11}+\alpha_2)}{\Gamma(n_{01}^{10}+n_{01}^{11}+\beta_2+\alpha_2)} \times \frac{\Gamma(n_{10}^{00}+\beta_3)\Gamma(n_{10}^{01}+\alpha_3)}{\Gamma(n_{10}^{00}+n_{10}^{01}+\beta_3+\alpha_3)} \times \frac{\Gamma(n_{11}^{10}+\beta_4)\Gamma(n_{11}^{11}+\alpha_4)}{\Gamma(n_{00}^{10}+n_{11}^{11}+\beta_4+\alpha_4)} \times$$

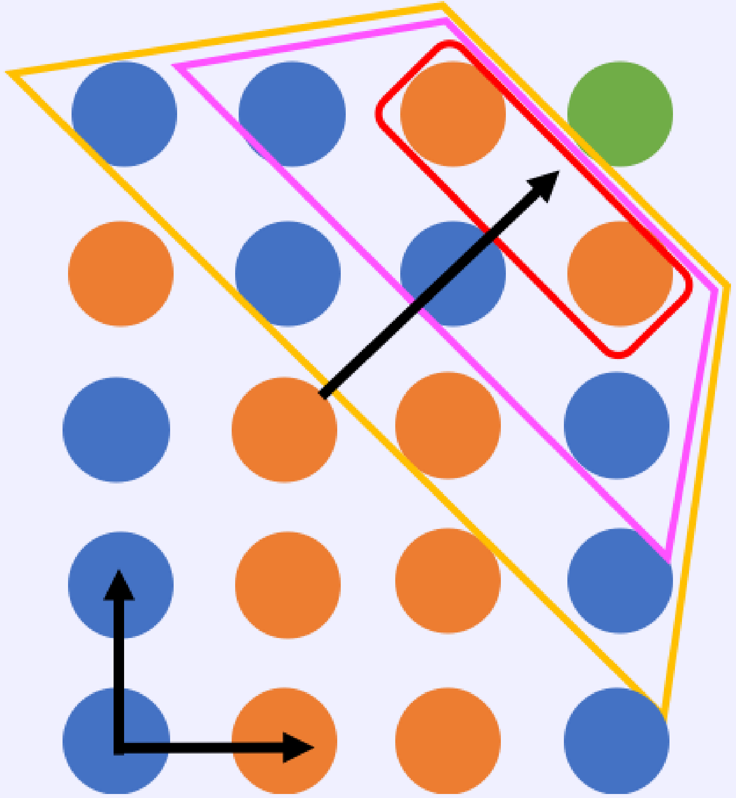
For m≥3, this becomes:

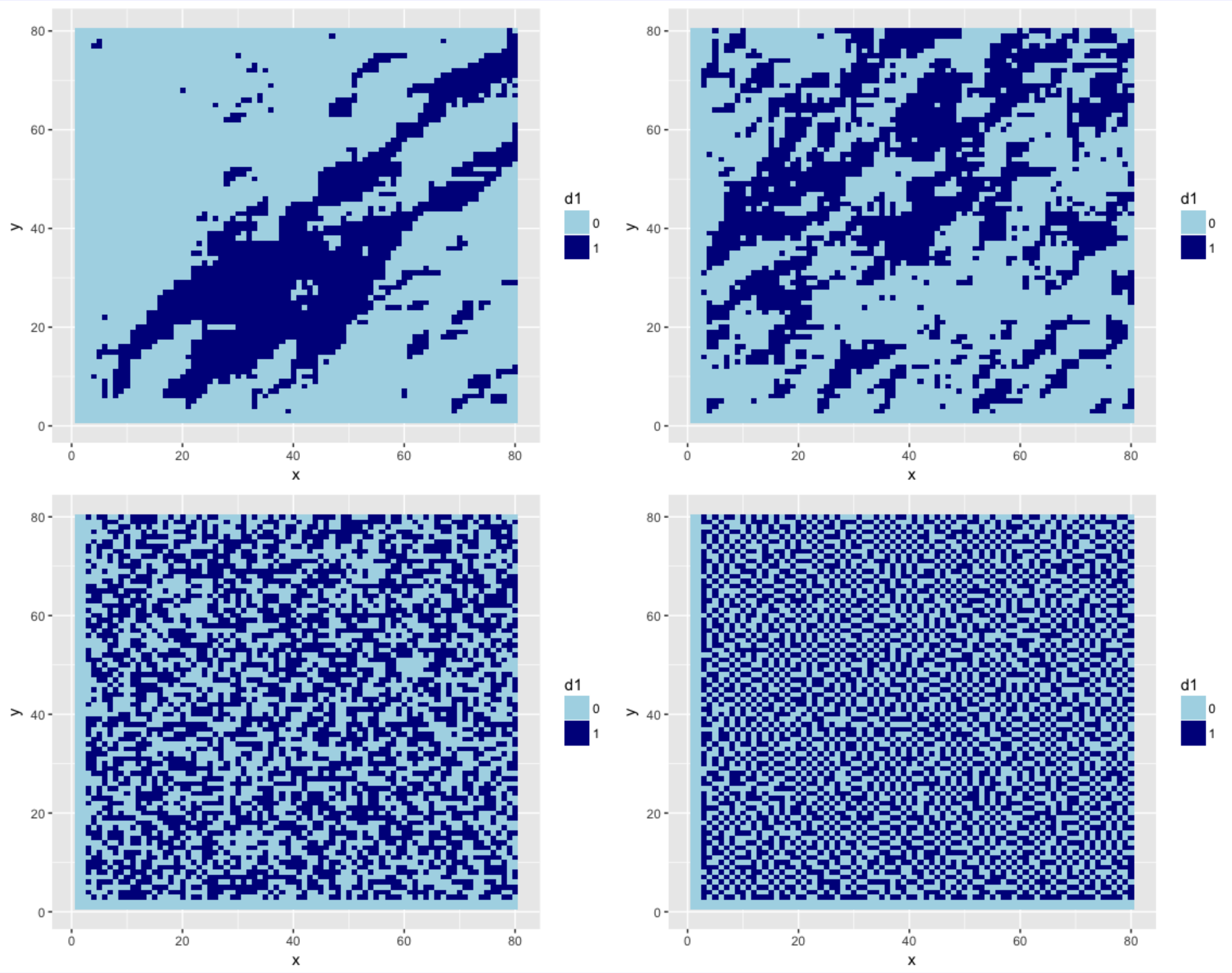
$$\int P(seq|p)P(p)dp = \prod_{i=0}^{2^m-1} \frac{\Gamma(n_i^{((2i+1)\ \mathrm{mod} 2^m)-1}+\beta_{i+1})\Gamma(n_i^{((2i+1)\ \mathrm{mod} 2^m)})+\alpha_{i+1})}{\Gamma(n_i^{((2i+1)\ \mathrm{mod} 2^m)-1}+n_i^{((2i+1)\ \mathrm{mod} 2^m)}+\beta_{i+1}+\alpha_{i+1})}$$

Bayes factors are calculated for each model with word lengths m= 1,…,10, so that the word length that best represents the given sequence is chosen

2d De Bruijn Graph

- Similar to the 1d version, but with a different word structure
- Words are formed by including all points that are a certain number of points away moving only upwards and to the right
- Can find the 1d equivalence for each 2d word so that we can apply the same theory
- Should be extendable to n dimensions




d1
0
1
x
y
0
20
40
60
80

Non-directional de Bruijn process

- Direction does not make logical sense in a spatial grid
- Attempt to remove the direction, but keep the de Bruijn structure
- Change the form of the word, but inference remains the same

Conclusion

- Create chains of 0's and 1's with correlation using de Bruijn graphs
- Developed a run length distribution and inference
- Working on the 2d version – with hope to eventually take out the directionality
- Apply the method to applications with classification problems