Reduced Basis methods: an introduction

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Non-intrusive RB

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Context and Motivations

Context : parametrized PDE

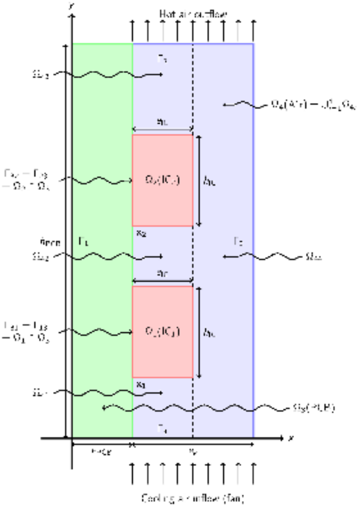
- input-parameter examples : geometric configuration, physical properties, boundary conditions, sources.
- output examples : mean temperature over a subdomain, flux on a boundary, etc.

Motivation : rapid and reliable evaluation of input-output relationships.

- Real-time context : parameter-estimation, control.
- Many-query context : sensitivity analysis, multi-model simulation.

OPUS Heat Transfer Benchmark

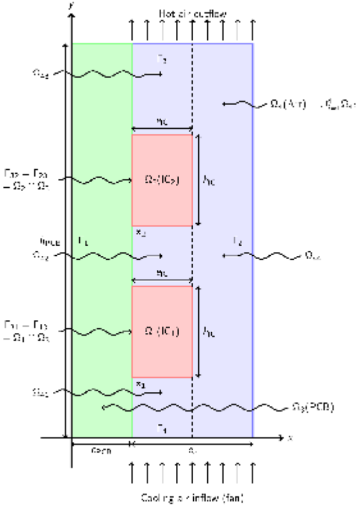
Thermal Testcase Description



Overview

- Heat-Transfer with conduction and convection possibly coupled with Navier-Stokes
- Simple but complex enough to contain all difficulties to test the certified reduced basis
  - non symmetric, non compliant
  - steady/unsteady
  - physical and geometrical parameters
  - coupled models
- Testcase can be easily complexified

Thermal Testcase Description



Heat transfer equation

$$\rho C_i\left(\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right) - \nabla\cdot(k_i \nabla T) = Q_i, \quad i = 1,2,3,4$$

Inputs

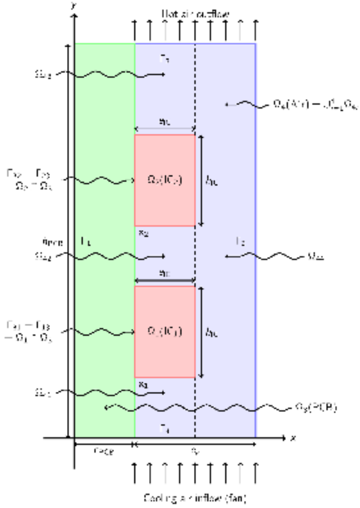
$$\mu = \{e_a; k_{Ic}; D; Q; r\}.$$

Outputs

$$s_1(\mu) = \frac{1}{e_{IC} h_{IC}} \int_{\Omega_2} T$$

$$s_2(\mu) = \frac{1}{e_a} \int_{\Omega_4 \cap \Gamma_3} T$$

Thermal Testcase Description



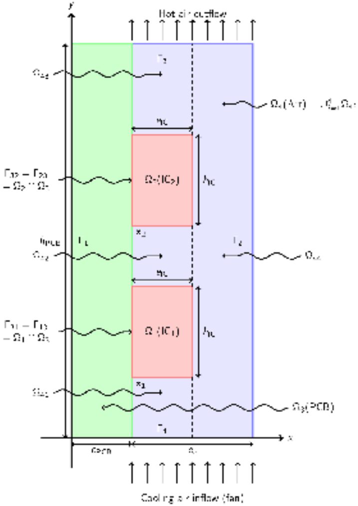
Fluid model

Poiseuille flow or Navier-Stokes flow

Boundary conditions

- on $\Gamma_3 \cap \Omega_3$, a zero flux
- on $\Gamma_3 \cap \Omega_4$, outflow
- on Γ_4, $(0 \leq x \leq e_{Pcb} + e_a, y = 0)$ temperature is set
- Γ_1 and Γ_2 periodic
- at interfaces between the ICs and PCB, thermal discontinuity (conductance)
- on other internal boundaries, the continuity of the heat flux and temperature

Thermal Testcase Description



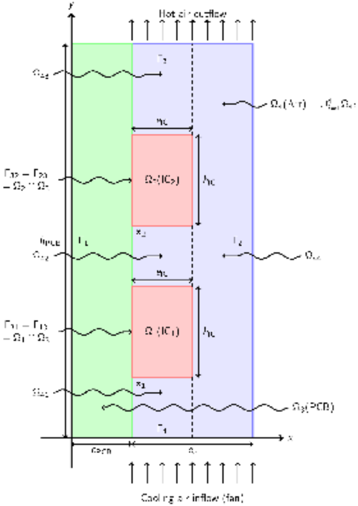
Finite element method

- $\mathbb{P}_k, k = 1, ..., 4$ Lagrange elements
- Weak treatment of Dirichlet conditions
- CIP Stabilisation
- Locally Discontinuous FEM functions

Validation

- Comparison between Comsol(EADS) and Feel++
- Extensive testing and comparisons
- Implementation validated, ref. config. max rel error $< 1\%$
- Diff. : mesh, stabilisation, Dirichlet

Thermal Testcase Description



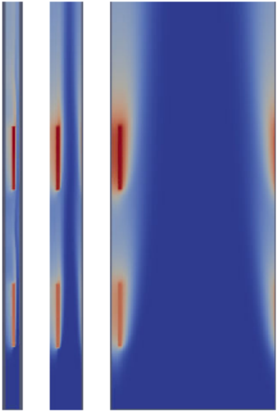


Figure : Temperature plot for $e_a \in \{2.5e-3, 8e-3, 5e-2\}$

HiFiMagnet project

High Field Magnet Modeling

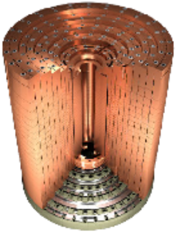
Laboratoire National des Champs Magnétiques Intenses

Large scale user facility in France

- High magnetic field : from 24 T
- Grenoble : continuous magnetic field (36 T)
- Toulouse : pulsed magnetic field (90 T)

Application domains

- Magnetoscience
- Solide state physic
- Chemistry
- Biochemistry



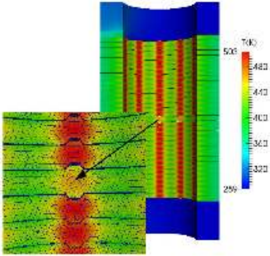
Magnetic Field

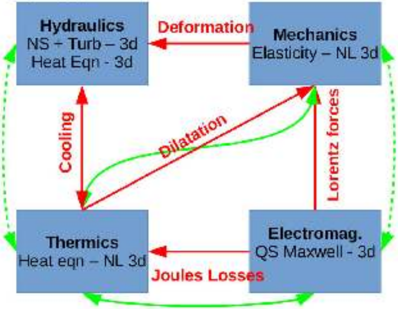
- Earth : $5.8 \cdot 10^{4}\,T$
- Supraconductors : $24\,T$
- Continuous field : $36\,T$
- Pulsed field : $90\,T$

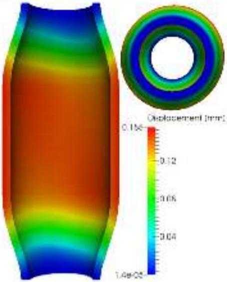
Access

- Call for Magnet Time : $2 \times$ per year
- $\approx$ 140 projects per year

High Field Magnet Modeling







Why use Reduced Basis Methods ?

Challenges

- Modeling : multi-physics non-linear models, complex geometries, genericity
- Account for uncertainties : uncertainty quantification, sensitivity analysis
- Optimization : shape of magnets, robustness of design

Objective 1 : Fast

- Complex geometries
  - Large number of dofs
- Uncertainty quantification
  - Large number of runs

Objective 2 : Reliable

- Field quality
- Design optimization
  - Certified bounds
  - Reach material limits

An open reduced basis framework

Objectives

- Provide an open framework for certified reduced basis methods
- Provide a rapid prototyping framework using the Feel++ language for the standard finite/spectral element methods
- Provide interfaces to various open "mathematical" programming environments such as Python/OpenTURNS(UQ) or Octave

Where to get it ?

- sources are available at `http://www.feelpp.org`
- available as Debian/Ubuntu packages

Feel++

Features

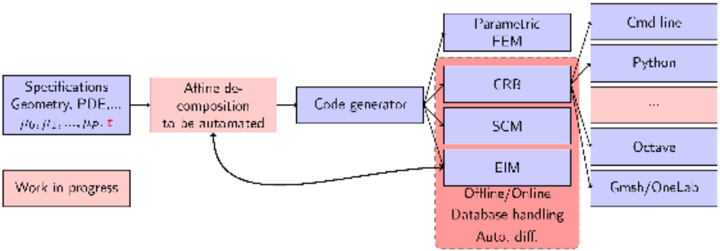
- Galerkin methods (fem,sem, cG, dG) in 1D, 2D and 3D on simplices and hypercubes
- Interfaces to PETSc/SLEPc
- Language embedded in C++ close to variational formulation language that shortens tremendously the "time to results"

```
// T_h = {elements}
auto mesh = loadMesh( new Mesh<Simplex<3> > );
// X_h = {v ∈ C^0(Ω) | v_K ∈ P_2(K), ∀K ∈ T_h}
auto Xh = Pch<2>(mesh);
auto u = Xh->element(), v = Xh->element();
auto a = form2( _test=Xh, _trial=Xh, );
// a(u,v) = ∫_Ω ∇u · ∇v
a=integrate(elements(mesh), gradt(u)*trans(grad(v)));
auto b=form2( _test=Xh, _trial=Xh );
// a(u,v) = ∫_Ω u v
b = integrate( elements(mesh), idt(u)*id(v) );
```

Feel++ Reduced Basis Features

Lower bound for coercivity/inf-sup	OK	90%
CRB Linear Elliptic case	OK	100% (coercive) 90% (non-coercive)
CRB Linear Parabolic case	OK	100%
Automatic differentiation	OK	90%
EIM (space/time)	OK	100%
CRB Nonlinear case	Ok	80%
CRB automated affine decomposition	Not OK	50%
CRB for multiphysics	OK	80%

Feel++ Framework : the User point of view



Wrappers : Python/OpenTURNS

Wrappers are automatically generated by the framework

Python Code (OpenTURNS wrapper for UQ)

```
# fem code
modelfem = NumericalMathFunction("modelfem")
# crb code
modelrb =  NumericalMathFunction("modelrb")
mu[0] = 10
mu[1] = 7e-3
outputfem = modelfem(mu)
outputrb = modelrb(mu)
```

Wrapper : Octave

Wrappers are automatically generated by the framework

Octave code

```
# kIC : thermal conductivity (default: 2)
inP(1) = 1.0e+1;
# D : fluid flow rate (default: 5e-3)
inP(2) = 7.0e-3;
# Q : heat flux (default: 1e6)
inP(3) = 1.0e+6;
# r : conductance (default: 100)
inP(4) = 1.0e+2;
# ea : length air flow channel (default: 4e-3)
inP(5) = 4.0e-3;
for i=1:N
  inP(1)= 0.2+(i-1)*(150-0.2)/N;  D=[D inP(1)];
  s= [s opuseadscrb( inP )];
end
```

Reduced Basis Methodology

Linear Compliant Elliptic Problems

Notations and Definitions

Notations

- $(\cdot)^{\mathcal{N}}$ finite element approximation
- $(\cdot)_N$ reduced basis approximation
- μ input parameter (physical, geometrical,...)
- $s(t;\mu) \approx s^{\mathcal{N}}(t;\mu) \approx s_N(t;\mu)$ output approximations
- $\mu \to s(t;\mu)$ input-output relationship

Definitions

- $\Omega \subset \mathbb{R}^d$ spatial domain
- μ P-uplet
- $\mathcal{D}^\mu \subset \mathbb{R}^P$ parameter space
- s output, ℓ, f functionals
- u field variable
- X function space $H^1_0(\Omega)^\nu \subset X \subset H^1(\Omega)^\nu$ ($\nu = 1$ for simplicity)
  $(\cdot,\cdot)_X$ scalar product and $\|\cdot\|_X$ norm associated to X

Problem Statement

The formal problem statement reads : Given $\mu \in \mathcal{D}^\mu$, evaluate

$$s(\mu) = \ell(u(\mu); \mu)$$

where $u(x; \mu) \in X$ satisfies

$$a(u(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X$$

Remark

We consider first the case of linear affine compliant elliptic problem and then complexify

Hypothesis : Reference Geometry

In these notes Ω is considered parameter independent

- To apply the reduced basis methodology exposed later, we need to setup a reference spatial domain Ω_{ref}
- We introduce an affine mapping $T(\cdot;\mu) : \Omega(\equiv \Omega_{\mathrm{ref}} = \Omega_o(\bar{\mu})) \rightarrow \Omega_o(\mu)$ such that

$$a(u, v; \mu) = a_o(u_o \circ \mathcal{T}_\mu, v_o \circ \mathcal{T}_\mu; \mu)$$

Hypothesis : Continuity, stability, compliance

We consider the following $\mu-$PDE

$a(\cdot,\cdot;\mu)$	bilinear symmetric continuous coercive $(\forall\mu\in\mathcal{D}^\mu)$
$f(\cdot;\mu),\ell(\cdot;\mu)$	linear bounded $(\forall\mu\in\mathcal{D}^\mu)$

and in particular, to start, the compliant case

- a symmetric
- $f(\cdot;\mu)=\ell(\cdot;\mu)\quad\forall\mu\in\mathcal{D}^\mu$

Hypothesis : Affine dependence in the parameter

We require for the RB methodology

$$a(u, v; \mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\ a^q(u, v),$$

where for $q = 1, \ldots, Q_a$

$$\begin{aligned} \Theta_a^q : \quad & \mathcal{D}^\mu \to \mathbb{R} \qquad \mu - \text{dependent functions} \\ a^q : \quad & X \times X \to \mathbb{R} \quad \mu - \text{independent bilinear forms} \end{aligned}$$

Remark

- *similar decomposition is required for $\ell(v; \mu)$ and $f(v; \mu)$, and denote Q_ℓ and Q_f the corresponding number of terms*
- *applicable to a large class of problems including geometric variations*
- *can be relaxed (see non affine/non linear problems)*

Inner Products and Norms

We next define the

- energy inner product and associated norm (parameter dependent)

$$(((w,v)))_\mu = a(w,v;\mu) \qquad \forall u,v \in X$$
$$|||v|||_\mu = \sqrt{a(v,v;\mu)} \qquad \forall v \in X$$

- X-inner product and associated norm (parameter independent)

$$(w,v)_X = (((w,v)))_{\bar\mu} \; (\equiv a(w,v;\bar\mu)) \qquad \forall u,v \in X$$
$$||v||_X = |||v|||_{\bar\mu} \; (\equiv \sqrt{a(v,v;\bar\mu)}) \qquad \forall v \in X$$

Coercivity and Continuity Constants

Recall that

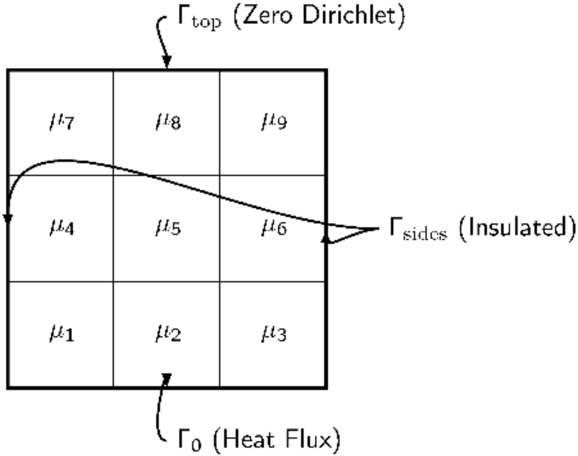
- coercivity constant

$$(0 <)\alpha(\mu) \equiv \inf_{v \in X} \frac{a(v, v; \mu)}{||v||_X^2}$$

- Continuity constant

$$\gamma(\mu) \equiv \sup_{w \in X} \sup_{v \in X} \frac{a(w, v; \mu)}{\|w\|_X \|v\|_X} (< \infty)$$

Example Thermal Block : Heat Transfer



Example Thermal Block : Problem statement

Given $\mu \in (\mu_1, ...\mu_P) \in \mathcal{D}^\mu \equiv [\mu^{\mathsf{min}}, \mu^{\mathsf{max}}]^P$, evaluate (recall that $\ell = f$)

$$s(\mu) = f(u(\mu))$$

where $u(\mu) \in X \equiv \{v \in H^1(\Omega), v|_{\Gamma_{\mathbf{top}}} = 0\}$ satisfies

$$a(u(\mu), v; \mu) = f(v; \mu) \quad \forall v \in X$$

we have $P = 8$ and given $1 < \mu_r < \infty$ we set

$$\mu^{\min} = 1/\sqrt{\mu_r}, \quad \mu^{\max} = \sqrt{\mu_r}$$

such that $\mu^{\max}/\mu^{\min} = \mu_r$.

Example Thermal Block

Recall we are in the compliant case $\ell = f$, we have

$$f(v) = \int_{\Gamma_0} v \quad \forall v \in X$$

and

$$a(u, v; \mu) = \sum_{i=1}^{P} \mu_i \int_{\Omega_i} \nabla u \cdot \nabla v + 1 \int_{\Omega_{P+1}} \nabla u \cdot \nabla v \quad \forall u,\ v\ \in X$$

where $\Omega = \cup_{i=1}^{P+1} \Omega_i$.

Example Thermal Block

The inner product is defined as follows

$$(u,v)_X = \sum_{i=1}^{P} \bar{\mu}_i \int_{\Omega_i} \nabla u \cdot \nabla v + 1 \int_{\Omega_{P+1}} \nabla u \cdot \nabla v$$

where $\bar{\mu}_i$ is a reference parameter. We have readily that a is

- symmetric
- parametrically coercive

$$0 < \frac{1}{\sqrt{\mu_r}} \leq \min(\mu_1/\bar{\mu}_1, \dots, \mu_P/\bar{\mu}_P, 1) \leq \alpha(\mu)$$

- and continuous

$$\gamma(\mu) \leq \max(\mu_1/\bar{\mu}_1, \dots, \mu_P/\bar{\mu}_P, 1) \leq \sqrt{\mu_r} < \infty$$

and the linear form f is bounded.

Example Thermal Block : Affine decomposition

We obtain the affine decomposition

$$a(u, v; \mu) = \sum_{q=1}^{P+1} \Theta^q(\mu) a^q(u, v)$$

with

$$\Theta^1(\mu) = \mu_1 \qquad a^1(u, v) = \int_{\Omega_1} \nabla u \cdot \nabla v$$

$$\vdots$$

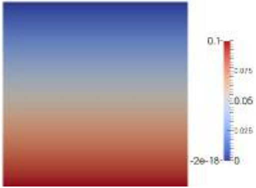
$$\Theta^P(\mu) = \mu_P \qquad a^P(u, v) = \int_{\Omega_P} \nabla u \cdot \nabla v$$

$$\Theta^{P+1}(\mu) = 1 \qquad a^{P+1}(u, v) = \int_{\Omega_{P+1}} \nabla u \cdot \nabla v$$

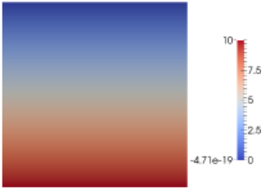
Example Thermal Block

- Homogeneous parameters

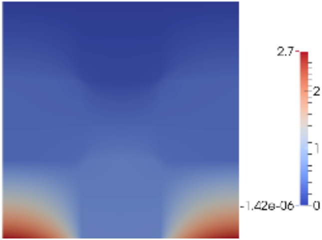
- **Maximum parameters values.**



- **Minimum parameters values.**



- Heterogeneous parameters



"Truth" FEM Approximation

Let $\mu \in \mathcal{D}^{\mu}$, evaluate

$$s^{\mathcal{N}}(\mu) = \ell(u^{\mathcal{N}}(\mu)) \ ,$$

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathcal{N}} \ .$$

Here $X^{\mathcal{N}} \subset X$ is a Truth finite element approximation of dimension $\boxed{\mathcal{N} \gg 1}$ equiped with an inner product $(\cdot,\cdot)_X$ and induced norm $||\cdot||_X$. Denote also X' and associated norm

$$\ell \in X', \qquad ||\ell||_{X'} \equiv \sup_{v \in X} \frac{\ell(v)}{||v||_X}$$

.

Purpose

- Equate $u(\mu)$ and $u_{\mathcal{N}}(\mu)$ in the sense that

$$||u(\mu) - u_{\mathcal{N}}(\mu)||_X \leq \mathrm{tol} \quad \forall \mu \in \mathcal{D}^{\mu}$$

- Build the reduced basis approximation using the FEM approximation
- Measure the error associated with the reduced basis approximation relative to the FEM approximation

Reduced Basis Objectives

For any given accuracy ϵ, evaluate — Accuracy

$$\mu \in \mathcal{D}^\mu \rightarrow s_N(\mu) (\approx s^{\mathcal{N}}(\mu)) \text{ and } \Delta_N^s(\mu)$$

that provably achieves the desired accuracy — Reliability

$$|s^{\mathcal{N}}(\mu) - s_N(\mu)| \leq \Delta_N^s(\mu) \leq \epsilon$$

for a very low cost t_{comp} — Efficiency

Independent of $\mathcal{N}$ as $\mathcal{N} \rightarrow \infty$

where t_{comp} is the time to perform the input-output relationship

$$\mu \rightarrow (s_N(\mu), \Delta_N^s(\mu))$$

Reduced Basis Objective : Rapid Convergence

Build a rapidly convergent approximation of

$$s_N(\mu) \in \mathbb{R} \text{ and } u_N(\mu) \in X^N \subset X^{\mathcal{N}} \subset X$$

such that for all μ, we have

$$s_N(\mu) \to s^{\mathcal{N}}(\mu) \text{ and } u_N(\mu) \to u^{\mathcal{N}}(\mu)$$

rapidly as $N = \dim X_N \to \infty (= 10 - 200)$ (and independently of $\mathcal{N}$)

Reduced Basis Objective : Reliability and Sharpness

Provide a posteriori error bound $\Delta_N(\mu)$ and $\Delta_N^s(\mu)$:

$$1 \leq \frac{\Delta_N(\mu)}{\|u^{\mathcal{N}}(\mu) - u_N(\mu)\|_X} \leq \ E$$

and

$$1 \leq \frac{\Delta_N^s(\mu)}{|s^{\mathcal{N}}(\mu) - s_N(\mu)|} \leq \ E$$

for all $N = 1 \ldots N_{\max}$ and $\mu \in \mathcal{D}^\mu$.

Reduced Basis Objective : Efficiency

Develop a two stage strategy : Offline/Online

Offline : very expensive pre-processing, we have typically that for a given $\mu \in \mathcal{D}^\mu$

$$t_{\text{comp}}^{\text{offline}} >> t_{\text{comp}}^{\mu \rightarrow s^{\mathcal{N}}(\mu)}$$

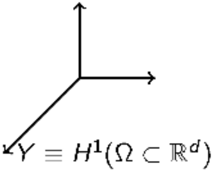
Online : very rapid convergent certified reduced basis input-output relationship

$$t_{\text{comp}}^{\text{online}} \text{ independent of } \mathcal{N}$$

Remark

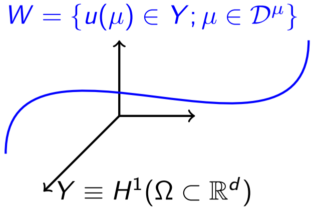
$\mathcal{N}$ may/should be chosen conservatively

Approximation opportunities : Low-Dimension Manifold



To approximate $u(\mu)$, and thus $s(\mu)$, we need not represent all functions in Y

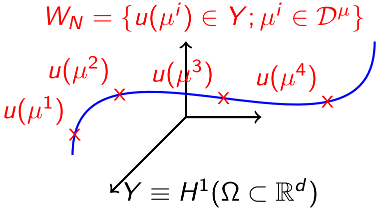
Approximation opportunities : Low-Dimension Manifold



To approximate $u(\mu)$, and thus $s(\mu)$, we need only approximate functions in low-dimensional manifold

$$W = \{u(\mu) \in Y; \mu \in \mathcal{D}^\mu\}$$

Approximation opportunities : Low-Dimension Manifold



To approximate $u(\mu)$, and thus $s(\mu)$, we construct the approximation space

$$W_N = \{u(\mu^i) \in Y; (\mu^i)_{i=1..N} \in \mathcal{D}^\mu\}$$

Spaces

Parameter Samples :

$$\text{Sample}: \quad S_N = \{\mu_1 \in \mathcal{D}^\mu, \ldots, \mu_N \in \mathcal{D}^\mu\} \quad 1 \leq N \leq N_{\max},$$

with

$$S_1 \subset S_2 \ldots S_{N_{\max}-1} \subset S_{N_{\max}} \subset \mathcal{D}^\mu$$

Lagrangian Hierarchical Space

$$W_N = \mathrm{span}\,\{\zeta_n \equiv \underbrace{u(\mu^n)}_{u^{\mathcal{N}}(\mu^n)}, n = 1, \ldots, N\}.$$

with

$$W_1 \subset W_2 \ldots \subset W_{N_{\max}} \subset X^{\mathcal{N}} \subset X$$

Sampling strategies ?

- Equidistributed points in $\mathcal{D}^\mu$ (curse of dimensionality)
- Log-random distributed points in $\mathcal{D}^\mu$
- See later for more efficient, adaptive strategies

Spaces : Remarks

Remark

We could include μ-derivatives (evaluation of the derivatives at given parameters) into W_N to obtain Hermite-like spaces (not covered here).

Remark

The basis functions $\zeta_n = u(\mu^n), n = 1...N_{\max}$ of W_N are orthonormalized via a Gram-Schmidt process. We then denote $X_N =$ Gram-Schmidt(W_N)

Formulation (Linear Compliant Case) : a Galerkin method

Galerkin Projection

Given $\mu \in \mathcal{D}^\mu$ evaluate

$$s_N(\mu) = f(u_N(\mu); \mu) \tag{1}$$

where $u_N(\mu) \in X_N$ satisfies

$$a(u_N(\mu), v; \mu) = f(v; \mu), \ \forall\, v \in X_N \ .$$

Formulation (Linear Compliant Case) : Optimality

For any $\mu \in \mathcal{D}^\mu$, we have the following optimality results (thanks to Galerkin)

$$|||u(\mu) - u_N(\mu)|||_\mu = \inf_{v_N \in X_N} |||u(\mu) - v_N(\mu)|||_\mu,$$

$$||u(\mu) - u_N(\mu)||_X \leq \sqrt{\frac{\gamma(\mu)}{\alpha(\mu)}} \inf_{v_N \in X_N} ||u(\mu) - v_N(\mu)||_X,$$

and

$$\begin{aligned} s(\mu) - s_N(\mu) &= |||u(\mu) - u_N(\mu)|||_\mu^2, \\ &= \inf_{v_N \in X_N} |||u(\mu) - v_N(\mu)|||_\mu^2, \end{aligned}$$

and finally

$$0 \leq s(\mu) - s_N(\mu) \leq \gamma(\mu) \inf_{v_N \in X_N} ||u(\mu) - v_N(\mu)||_X^2$$

Formulation (Linear Compliant Case) : offline-online decomposition

Expand our RB approximations :

$$u_N(\mu) \;=\; \sum_{j=1}^{N} \; u_{N\,j}(\mu)\;\zeta_j \tag{2}$$

Express $s_N(\mu)$

$$s_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\;\left\{\sum_{q=1}^{Q_f}\;\Theta_f^q(\mu)\;f^q(\zeta_j)\right\} \tag{3}$$

where $u_{N\,i}(\mu), 1 \leq i \leq N$ satisfies

$$\sum_{j=1}^{N}\left\{\sum_{q=1}^{Q_a}\;\Theta_a^q(\mu)\;a^q(\zeta_i,\zeta_j)\right\} u_{N\,j}(\mu) = \;\; \textstyle\sum_{q=1}^{Q_f}\;\Theta_f^q(\mu)\;f^q(\zeta_i),$$

$$1 \leq i \leq N \tag{4}$$

$$\tag{5}$$

Formulation (Linear Compliant Case) : matrix form

Solve

$$\underline{A}_N(\mu)\, \underline{u}_N(\mu) = \underline{F}_N$$

where

$$(A_N)_{i\,j}(\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\; a^q(\zeta_i, \zeta_j),$$

$$F_{N\,i} = \sum_{q=1}^{Q_f} \Theta_f^q(\mu) f^q(\zeta_i)\ .$$

$$1 \leq i, j \leq N, \quad 1 \leq i \leq N$$

Formulation (Linear Compliant Case) : complexity analysis

Offline : independent of μ

- Solve : N FEM system depending on $\mathcal{N}$
- Form and store : $f^q(\zeta_i)$
- Form and store : $a^q(\zeta_i, \zeta_j)$

Online : independent of $\mathcal{N}$

- Given a new $\mu \in \mathcal{D}^\mu$
- Form and solve $A_N(\mu)$: $O(QN^2)$ and $O(N^3)$
- Compute $s_N(\mu)$

Online : $N << \mathcal{N}$
Online we realize often orders of magnitude computational economies relative to FEM in the context of many μ-queries

Formulation (Linear Compliant Case) : Condition number

Proposition

Thanks to the orthonormalization of the basis function, we have that the condition number of $A_N(\mu)$ is bounded by the ratio $\gamma(\mu)/\alpha(\mu)$.

Démonstration.

- Write the Rayleigh Quotient

$$\frac{v_N^T A_N(\mu) v_N}{v_N^T v_N}, \quad \forall v_N \in \mathbb{R}^N$$

- Express

$$v_N = \sum_{n=1}^{N} v_{N_n} \zeta^n$$

- Use coercivity, continuity and orthonormality.

□

Questions

- What is the accuracy of $u_N(\mu)$ and $s_N(\mu)$? Online

$$\|u(\mu) - u_N(\mu)\|_X \leq \epsilon_{\mathrm{tol}}, \quad \forall \mu \in \mathcal{D}^\mu$$
$$|s(\mu) - s_N(\mu)\|_X \leq \epsilon^s_{\mathrm{tol}}, \quad \forall \mu \in \mathcal{D}^\mu$$

- What is the best value for N ? Offline/Online
  - N too large $\Rightarrow$ computational inefficiency
  - N too small $\Rightarrow$ unacceptable error
- How should we build S_N ? is there an optimal construction ? Offline
  - Good approximation of the manifold $\mathcal{M}$ through the RB space, but
  - need for well conditioned RB matrices

A Posteriori Error Estimation : Requirements

We shall develop the following error bounds $\Delta_N(\mu)$ and $\Delta_N^s(\mu)$ with the following properties

- rigorous $1 \leq N \leq N_{\max}$

$$\|u(\mu) - u_N(\mu)\|_X \leq \Delta_N(\mu), \quad \forall \mu \in \mathcal{D}^\mu$$

$$|s(\mu) - s_N(\mu)| \leq \Delta_N^s(\mu), \quad \forall \mu \in \mathcal{D}^\mu$$

- reasonably sharp $1 \leq N \leq N_{\max}$

$$\frac{\Delta_N(\mu)}{\|u(\mu) - u_N(\mu)\|_X} \leq C, \frac{\Delta_N^s(\mu)}{|s(\mu) - s_N(\mu)|} \leq C,$$

$$C \approx 1$$

- efficient Online cost depend only on Q and N and not $\mathcal{N}$

$u_N(\mu)$: Error equation and residual dual norm

Given our RB approximation $u_N(\mu)$, we have

$$e(\mu) \equiv u(\mu) - u_N(\mu)$$

that satisfies

$$a(e(\mu), v; \mu) \;=\; r(u_N(\mu), v; \mu), \forall v \in X$$

where $r(u_N(\mu), v; \mu) = f(v) - a(u_N(\mu), v; \mu)$ is the residual. We have then from coercivity and the definitions above that

$$||e(\mu)||_X \;\leq\; \frac{||r(u_N(\mu), v; \mu)||_{X'}}{\alpha(\mu)} \;=\; \frac{\varepsilon_N(\mu)}{\alpha(\mu)}$$

A Posteriori error estimation : Dual norm of the residual

Proposition

Given $\mu \in \mathcal{D}^\mu$, the dual norm of $r(u_N(\mu), \cdot; \mu)$ is defined as follows

$$\begin{aligned} ||r(u_N(\mu), \cdot; \mu)||_{X'} &\equiv \sup_{v \in X} \frac{r(u_N(\mu), v; \mu)}{||v||_X} \\ &= ||\hat{e}(\mu)||_X \end{aligned}$$

where $\hat{e}(\mu) \in X$ satisfies

$$(\hat{e}(\mu), v)_X = r(u_N(\mu), v; \mu)$$

The error residual equation can then be rewritten

$$a(e(\mu), v; \mu) = (\hat{e}(\mu), v)_X, \quad \forall v \in X$$

$u_N(\mu)$: Definitions of energy error bounds and effectivity

Given $\alpha_{\mathrm{LB}}(\mu)$ a nonnegative lower bound of $\alpha(\mu)$:

$$\alpha(\mu) \geq \alpha_{\mathrm{LB}}(\mu) \geq \epsilon_\alpha \alpha(\mu),\ \epsilon_\alpha \ \in]0,1[,\ \forall \mu \in \mathcal{D}^\mu \tag{6}$$

Denote $\varepsilon_N(\mu) = \|\hat{e}(\mu)\|_X = \|r(u_N(\mu), v; \mu\|_{X'}$

Definition : Energy error bound

$$\Delta_N(\mu) \ \equiv \ \frac{\varepsilon_N(\mu)}{\sqrt{\alpha_{\mathrm{LB}}(\mu)}} \tag{7}$$

Definition : Effectivity

$$\eta_N(\mu) \ \equiv \ \frac{\Delta_N(\mu)}{|||e(\mu)|||_\mu} \tag{8}$$

$u_N(\mu)$: Rigorous sharp error bounds

One can prove that

$$1 \;\leq\; \eta_N(\mu) \;\leq \sqrt{\frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}}, \quad 1 \leq N \leq N_{\max}, \quad \forall \mu \;\in\; \mathcal{D}^{\mu} \qquad (9)$$

Remarks

- Rigorous : Left inequality ensures rigorous upper bound measured in $||\cdot||_X$, i.e. $||e(\mu)||_X \leq \Delta_N(\mu),\ \forall \mu \in \mathcal{D}^{\mu}$
- Sharp : Right inequality states that $\Delta_N(\mu)$ overestimates the "true" error by at most $\gamma(\mu)/\alpha_{\mathrm{LB}}(\mu)$

$s_N(\mu)$: error bounds

It follows from (100) and (102)

$$|s(\mu) - s_N(\mu)| \;\leq\; \Delta_N^s(\mu), \quad \mu \;\in\; \mathcal{D}^\mu \tag{10}$$

where

$$\Delta_N^s(\mu) = \frac{\Delta_N(\mu)^2}{\alpha_{\mathrm{LB}}(\mu)} \tag{11}$$

Rapid convergence of the error in the output

Note that the error in the output vanishes quadratically

Offline-Online decomposition

Denote $\hat{e}(\mu) \in X$

$$||\hat{e}(\mu)||_X = \varepsilon_N(\mu) = ||r(u_N(\mu), \cdot; \mu)||_X$$

such that

$$(\hat{e}(\mu), v)_X = -r(u_N(\mu), v; \mu), \quad \forall v \in X$$

And recall that

$$-r(u_N(\mu), v; \mu) = f(v) - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\ u_{N\,n}(\mu)\ a^q(\zeta_n, v), \quad \forall v \ \in \ X$$

Offline-Online decomposition

- It follows next that $\hat{e}(\mu) \in X$ satisfies

$$(\hat{e}(\mu), v)_X \;=\; f(v) - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\; u_{N\,n}(\mu)\; a^q(\zeta_n, v), \quad \forall v \;\in\; X$$

- Observe then that the rhs is the *sum* of products of parameter dependent functions and parameter independent linear functionals, thus invoking linear superposition

$$\hat{e}(\mu) \;=\; \mathcal{C} - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\; u_{N\,n}(\mu)\; \mathcal{L}_n^q$$

where
  - $\mathcal{C} \in X$ satisfies
  $$(\mathcal{C}, v) = f(v), \forall v \in X$$
  - $\mathcal{L} \in X$ satisfies
  $$(\mathcal{L}_n^q, v)_X = -a^q(\zeta_n, v), \forall v \in X,\; 1 \le n \le N, 1 \le q \le Q$$

  which are parameter independent problems

Offline-Online decomposition : Error bounds

From (20) we get

$$||\hat{e}(\mu)||_X^2 = (\mathcal{C},\mathcal{C})_X + \sum_{q=1}^{Q}\sum_{n=1}^{N} \Theta^q(\mu)\, u_{N\,n}(\mu) \Big\{ 2(\mathcal{C},\mathcal{L}_n^q)_X + \sum_{q'=1}^{Q'}\sum_{n'=1}^{N'} \Theta^{q'}(\mu)\, u_{N\,n'}(\mu)\, (\mathcal{L}_n^q,\mathcal{L}_{n'}^{q'})_X \Big\} \quad (12)$$

Remark

In (12), $||\hat{e}(\mu)||_X^2$ is the sum of products of

- parameter dependent (simple/known) functions and
- parameter independent inner-product,

the offline-online for the error bounds is now clear.

Offline-Online decomposition : steps and complexity

Offline :

- Solve for $\mathcal{C}$ and $\mathcal{L}_n^q,\ 1 \leq n \leq N,\ 1 \leq q \leq Q$
- Form and save $(\mathcal{C}, \mathcal{C})_X$, $(\mathcal{C}, \mathcal{L}_n^q)_X$ and $(\mathcal{L}_n^q, \mathcal{L}_{n'}^{q'})_X$, $1 \leq n, n' \leq N,\ 1 \leq q, q' \leq Q$

Online

- Given a new $\mu \in \mathcal{D}^\mu$
- Evaluate the sum $||\hat{e}(\mu)||_X^2$ (12) in terms of $\Theta^q(\mu)$ and $u_{N\,n}(\mu)$
- Complexity in $O(Q^2 N^2)$ independent of $\mathcal{N}$

Offline-Online Scenarii

Offline

Given a tolerance τ, build S_N and W_N s.t.

$$\forall \; \mu \; \in \mathcal{P} \equiv \mathcal{D}^\mu \; , \; \Delta_N(\mu) < \tau$$

Online

Given μ and a tolerance τ, find N^* and thus $s_{N^*}(\mu)$ s.t.

$$N^* = \arg\max_N \; (\Delta_N(\mu) < \tau)$$

or given μ and a max execution time T, find N^* and thus $s_{N^*}(\mu)$ s.t.

$$N^* = \arg\min_N \; (\Delta_N(\mu) \text{ and execution time } < T)$$

S_N and W_N Generation Strategies

Offline Generation

- Given a tolerance ϵ, set $N = 0$ and $S_0 = \emptyset$
- While $\Delta_N^{\max} > \epsilon$
- $N = N + 1$
- If N == 1 ; then Pick ((log-)randomly) $\mu_1 \in \mathcal{D}^\mu$
- Build $S_N := \{\mu_N\} \cup S_{N-1}$
- Build $W_N := \{\xi = u(\mu_N)\} \cup W_{N-1}$
- Compute $\Delta_N^{\max} := \max_{\mu \in \mathcal{D}^\mu} \Delta_N(\mu)$
- $\mu^{N+1} := \arg\max_{\mu \in \mathcal{D}^\mu} \Delta_N(\mu)$
- End While

Condition number

recall that the ζ_n are orthonormalized, this ensures that the condition number will stay bounded by $\gamma(\mu)/\alpha(\mu)$

Online Algorithm I

μ adaptive online

- Given $\mu \in \mathcal{D}^\mu$, compute $(s_{N^*}(\mu), \Delta_{N^*}(\mu))$ such that $\Delta_{N^*}(\mu) < \tau$.
- $N = 2$
- While $\Delta_N(\mu) > \tau$
- Compute $(s_N(\mu), \Delta_N(\mu))$ using (S_N, W_N)
- $N = N * 2$
  use the (very) fast convergence properties of RB
- End While

Online Algorithm II

Offline

- While $i <= \mathrm{Imax} >> 1$
- Pick log-randomly $\mu \in \mathcal{D}^\mu$
- Store in table $\mathcal{T}, \Delta_N(\mu)$ if worst case for $N = 1, \ldots, N^{\max}$
- $i = i + 1$; End While

Online Algorithm II – μ adaptive online – worst case

- Given $\mu \in \mathcal{D}^\mu$, compute $(s_{N^*}(\mu), \Delta_{N^*}(\mu))$ such that $\Delta_{N^*}(\mu) < \tau$.
- $N^* := \mathrm{argmax}_{\mathcal{T}}\, \Delta_N(\mu) \ < \ \tau$
- Use W_{N^*} to compute $(s_{N^*}(\mu), \Delta_{N^*}(\mu))$

Online Algorithm II

Offline

- While $i <= \mathrm{Imax} >> 1$
- Pick log-randomly $\mu \in \mathcal{D}^\mu$
- Store in table $\mathcal{T}, \Delta_N(\mu)$ if worst case for $N = 1, ..., N^{\max}$
- $i = i + 1$; End While

Online Algorithm II – μ adaptive online – worst case

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Online Algorithm II

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Online Algorithm II – μ adaptive online – worst case

- Given $\mu \in \mathcal{D}^\mu$, compute $(s_{N^*}(\mu), \Delta_{N^*}(\mu))$ such that $\Delta_{N^*}(\mu) < \tau$.
- $N^* := \mathrm{argmax}_{\mathcal{T}}\, \Delta_N(\mu) \; < \; \tau$
- Use W_{N^*} to compute $(s_{N^*}(\mu), \Delta_{N^*}(\mu))$

Lower bound for coercivity constant

We require a lower bound $\alpha_{\mathrm{LB}}(\mu)$ for $\alpha(\mu) = \alpha_c(\mu), \ \forall \mu \in \mathcal{D}^\mu$
Two strategies are available :

- "Min *Theta*" approach if *a* is parametrically coercive (*i.e.* the coercivity constant depends solely on μ)
- and more generally the Successive Constraint Method(SCM) which can also be applied in case of "Inf-Sup" stable problems (Stokes, Helmholtz,...)

"Min *Theta*" approach : Lower bound for $\alpha(\mu)$

- $\Theta^q(\mu) > 0, \; \forall \mu \in \mathcal{D}^\mu$ and
- $a^q(v, v) \geq 0, \; \forall v \in X, \; 1 \leq q \leq Q$

We define

$$\Theta_a^{\min,\bar{\mu}}(\mu) = \min_{q=1\dots Q} \frac{\Theta^q(\mu)}{\Theta^q(\bar{\mu})} \leq \alpha(\mu)$$

for $\bar{\mu} \in \mathcal{D}^\mu$ which was used to define the X-inner product and induced norm

$$(u, v)_X = a(u, v; \bar{\mu}), \quad \forall u, v \in X$$
$$\|v\|_X = \sqrt{(u, v)_X}, \quad \forall v \in X$$

"Min *Theta*" approach : Upper bound for $\gamma(\mu)$

Similarly we develop an upper bound $\gamma_{\mathrm{UB}}(\mu)$ for $\gamma(\mu)$. We define

$$\infty > \Theta_a^{\max,\bar{\mu}}(\mu) = \max_{q=1\dots Q} \frac{\Theta^q(\mu)}{\Theta^q(\bar{\mu})} \geq \gamma(\mu)$$

Remark

$\gamma_{\mathrm{UB}}(\mu)$ *is actually not required in practice but relevant in the theory.*

"Min *Theta*" approach : Summary

if a is parametrically coercive we then choose

- the coercivity constant lower bound to be

$$\alpha_{\mathrm{LB}}(\mu) \equiv \Theta_a^{\min,\bar{\mu}}(\mu)$$

- and the continuity constant upper bound to be (a symmetric)

$$\gamma_{\mathrm{UB}}(\mu) \equiv \Theta_a^{\max,\bar{\mu}}(\mu)$$

Remark

- *Online cost to evaluate* $\alpha_{\mathrm{LB}}(\mu)$ *:* $O(Q_a)$
- *Choice of inner product important* $(u, v)_X = a(u, v; \bar{\mu})$ *(see multiple inner products approach)*
- *Extends to non-symmetric problems by considering the symmetric part*

$$a_s(u, v; \mu) = \frac{1}{2}(a(u, v; \mu) + a(v, u; \mu))$$

Stability estimates

We wish to compute $\alpha_{\rm LB} : \mathcal{D} \rightarrow \mathbb{R}$ such that

$$0 < \alpha_{\rm LB}(\mu) \leq \alpha^{\mathcal{N}}(\mu), \quad \mu \in \mathcal{D} \tag{13}$$

and it computation is rapid $O(1)$ where

$$\alpha^{\mathcal{N}}(\mu) = \inf_{w \in X^{\mathcal{N}}} \frac{a(w, w; \mu)}{\|w\|_X^2} \tag{14}$$

Computation of $\alpha^{\mathcal{N}}(\mu)$

$\alpha^{\mathcal{N}}(\mu)$ is the minimum eigenvalue of the following generalized eigenvalue problem

$$a(w, v; \mu) = \lambda(\mu)\ m(w, v; \mu), \quad (Aw = \lambda Bw) \tag{15}$$

where $m(\cdot, \cdot)$ is the bi-linear form associated with $\|\cdot\|_X$ and B is the associated matrix.

Reformulation

The problem as a minimization one

First recall

$$a(w, v; \mu) = \sum_{q=1}^{Q_a} \theta_q(\mu)\ a_q(w, v)$$

Hence we have

$$\alpha^{\mathcal{N}}(\mu) = \inf_{w \in X^{\mathcal{N}}} \sum_{q=1}^{Q} {}_a\ \theta_q(\mu) \frac{a_q(w, w)}{\|w\|_X^2}$$

and we note

$$\mathcal{J}^{\mathrm{obj}}(w; \mu) = \sum_{q=1}^{Q_a} \theta_q(\mu) \frac{a_q(w, w)}{\|w\|_X^2}$$

Reformulation

We have the following optimisation problem

$$\alpha^{\mathcal{N}}(\mu) = \inf_{y \in \mathcal{Y}} \mathcal{J}^{\mathrm{obj}}(\mu; y)$$

where

$$\mathcal{J}^{\mathrm{obj}}(\mu; y) \equiv \sum_{q=1}^{Q_a} \theta_q(\mu) y_q$$

and

$$\mathcal{Y} = \left\{ y \in \mathbb{R}^{Q_a} | \; \exists w \in X^{\mathcal{N}} \text{ s.t. } y_q = \frac{a_q(w,w)}{\|w\|^2_{X^{\mathcal{N}}}}, 1 \leq q \leq Q_a \right\}$$

We now need to characterize $\mathcal{Y}$, to do this we construct two sets $\mathcal{Y}_{\mathrm{LB}}$ and $\mathcal{Y}_{\mathrm{UB}}$ such that $\mathcal{Y}_{\mathrm{UB}} \subset \mathcal{Y} \subset \mathcal{Y}_{\mathrm{LB}}$ over which finding $\alpha^{\mathcal{N}}(\mu)$ is feasible.

First we set the design space for the minimisation problem (69). We introduce

$$\mathcal{B} = \prod_{q=1}^{Q_a} \left[\inf_{w \in X^{\mathcal{N}}} \frac{a_q(w,w)}{\|w\|_X^2}; \sup_{w \in X^{\mathcal{N}}} \frac{a_q(w,w)}{\|w\|_X^2}\right]$$

$$\Xi = \left\{\mu_i \in \mathcal{D}; i = 1, ..., J\right\}$$

and

$$C_K = \left\{\mu_i \in \Xi; i = 1, ..., K\right\} \subset \Xi$$

Ξ is constructed using a $\frac{1}{2^p}$ division of $\mathcal{D}$: in 1D, $0, 1; \frac{1}{2}; \frac{1}{4}, \frac{3}{4}; \ldots$. C_K will be constructed using a greedy algorithm.

Finally we shall denote $P_M(\mu; E)$ the set of M points closest to μ in the set E. We shall need this type of set to construct the lower bounds.

Lower bounds : $\mathcal{Y}_{\rm LB}$

Given $M_\alpha, M_+ \in \mathbb{N}$ we are now ready to define $\mathcal{Y}_{\rm LB}$

$$\mathcal{Y}_{\rm LB}(\mu; C_K) = \Big\{ y \in \mathbb{R}^{Q_a} \mid y \in \mathcal{B},$$

$$\sum_{q=1}^{Q_a} \theta_q(\mu') y_q \geq \alpha^{\mathcal{N}}(\mu'),\ \forall \mu' \in P_{M_\alpha}(\mu; C_K)$$

$$\sum_{q=1}^{Q_a} \theta_q(\mu') y_q \geq \alpha_{\rm LB}(\mu'; C_{K-1}),\ \forall \mu' \in P_{M_+}(\mu; \Xi \backslash C_K) \Big\}$$

We now set

$$\alpha_{\rm LB}(\mu; C_K) = \inf_{y \in \mathcal{Y}_{\rm LB(\mu; C_K)}} \mathcal{J}^{\rm obj}(\mu; y)$$

Computing $\alpha_{\rm LB}(\mu; C_K)$ is in fact a linear program with Q_a design variables, y_q, and $2Q_a + M_\alpha + M_+$ constraints online. It requires the construction of C_K offline.

Let

$$\mathcal{Y}_{\mathrm{UB}}(C_K) = \left\{ y^*(\mu_k), 1 \leq k \leq K \right\}$$

with

$$y^*(\mu) = \mathrm{argmin}_{y \in \mathcal{Y}} \ \mathcal{J}^{\mathrm{obj}}(\mu; y)$$

We set

$$\alpha_{\mathrm{UB}}(\mu; C_K) = \inf_{y \in \mathcal{Y}_{\mathrm{UB}}(C_K)} \ \mathcal{J}^{\mathrm{obj}}(\mu; y)$$

$\mathcal{Y}_{\mathrm{UB}}$ requires K eigensolves to compute the eigenmode η_k associated with $w_k, k = 1, ..., K$ and $KQ\mathcal{N}$ inner products to compute the $y^*_q(w_k) = \frac{a_q(\eta_k, \eta_k; \mu)}{\|\eta_k\|^2_{X^{\mathcal{N}}}}, k = 1, ..., K$ offline . Then computing $\alpha_{\mathrm{UB}}(\mu; C_K)$ is a simple enumeration online.

$[C_{K_{\max}}] = \mathrm{Greedy}(\Xi, \epsilon)$

Given Ξ and $\epsilon \in [0;1]$

- While $\max_{\mu\in\Xi} \frac{\alpha_{\mathrm{UB}}(\mu;C_K)-\alpha_{\mathrm{LB}}(\mu;C_K)}{\alpha_{\mathrm{UB}}(\mu;C_K)} > \epsilon$
  - $\mu_{K+1} = \mathrm{argmax}_{\mu\in\Xi} \frac{\alpha_{\mathrm{UB}}(\mu;C_K)-\alpha_{\mathrm{LB}}(\mu;C_K)}{\alpha_{\mathrm{UB}}(\mu;C_K)}$
  - $C_{K+1} = C_K \cup \{\mu_{K+1}\}$
  - $K \leftarrow K+1$
- set $K_{\max} = K$

Offline-Online

Offline

- $2Q_a + M_\alpha + M_+$ eigensolves $\alpha^{\mathcal{N}}(\mu), y^*(\mu_k)$
- $n_\Xi K_{\max} LP(Q, M_\alpha, M_+)$ to build $C_{K_{\max}}$
- $K_{\max} Q$ inner products over $X^{\mathcal{N}} \Rightarrow \mathcal{Y}_{\mathrm{UB}}$

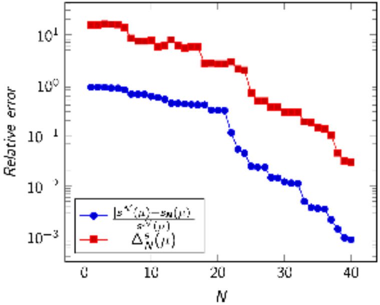
$[\alpha_{\mathrm{LB}}(\mu)] = \mathrm{Online}(\mu, C_{K_{\max}}, M_\alpha, M_+)$

Given $\mu \in \mathcal{D}$

- sort over $C_{K_{\max}} \Rightarrow P_{M_\alpha}(\mu; C_{K_{\max}})$ and $P_{M_+}(\mu; \Xi \backslash C_{K_{\max}})$
- $(M_\alpha + M_+ + 2)Q_a$ evaluation of $\theta_q(\mu')$
- M_α lookups to get $\mu' \to \alpha^{\mathcal{N}}(\mu')$
- $LP(Q_a, M_\alpha, M+)$ to get $\alpha_{\mathrm{LB}}(\mu)$

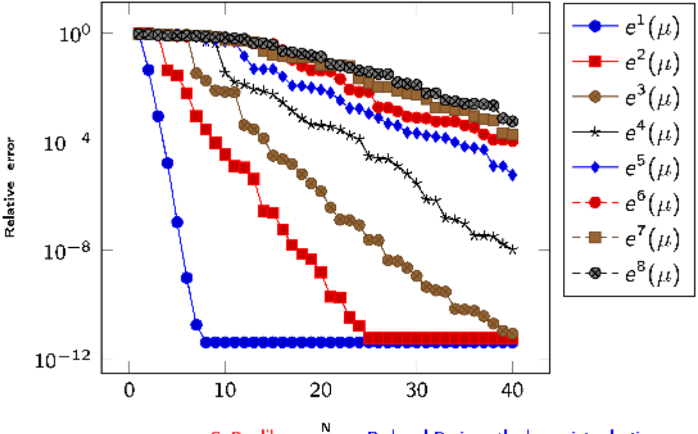
Example Thermal Block

- Configuration :
  - 47 600 dofs ;
  - Preconditionner : LU – Solver : MUMPS
  - Ξ : parameter sampling of dimension 1 000.
- Plot $\max_{\mu\in\Xi} \frac{|s^{\mathcal{N}}(\mu)-s_N(\mu)|}{s^{\mathcal{N}}(\mu)}$



Example Thermal Block

- More parameters there are, more rich the problem is ;
- Notations :
  - $e^i(\mu)$ is the relative error on the output when i parameters vary.



Non Compliant Output and/or Non-Symmetric Elliptic Problems

“Truth” FEM Approximation

Let $\mu \in \mathcal{D}^\mu$, evaluate

$$s^{\mathcal{N}}(\mu) = \ell(u^{\mathcal{N}}(\mu)) \ ,$$

where $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}} \subset X$ satisfies

$$a(u^{\mathcal{N}}(\mu), v; \mu) = f(v), \quad \forall\, v \in X^{\mathcal{N}} \ .$$

and we suppose that

- $a(\cdot,\cdot;\mu)$ is bilinear, $f(\cdot;\mu$ and $\ell(\cdot;\mu)$ are linear
- f and ℓ are bounded
- $\ell \neq f$ (non-compliance) and/or a is non-symmetric

"Truth" FEM Approximation : Hypothesis

We assume that $a : X^{\mathcal{N}} \times X^{\mathcal{N}} \to \mathbb{R}$ is

- coercive

$$(0 <)\alpha(\mu) \equiv \inf_{v \in X} \frac{a(v, v; \mu)}{||v||_X^2}$$

- Continuous

$$\gamma(\mu) \equiv \sup_{w \in X} \sup_{v \in X} \frac{a(w, v; \mu)}{\|w\|_X \|v\|_X} (< \infty)$$

- and enjoys affine parametric dependence

$$a(u, v; \mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu)\ a^q(u, v), \quad \forall u, v \in X$$

"Truth" FEM Approximation : Inner products and Norms

We next define the

- energy inner product and associated norm (parameter dependent)

$$(((w,v)))_\mu = a_s(w,v;\mu) \qquad \forall u,v \in X$$
$$|||v|||_\mu = \sqrt{a_S(v,v;\mu)} \qquad \forall v \in X$$

- X-inner product and associated norm (parameter independent)

$$(w,v)_X = (((w,v)))_{\bar\mu} \; (\equiv a_S(w,v;\bar\mu)) \qquad \forall u,v \in X$$
$$||v||_X = |||v|||_{\bar\mu} \; (\equiv \sqrt{a_S(v,v;\bar\mu)}) \qquad \forall v \in X$$

where a_s denotes the symmetric part of a.

Spaces

Parameter Samples :

$$\text{Sample}: \quad S_N = \{\mu_1 \in \mathcal{D}^\mu, \ldots, \mu_N \in \mathcal{D}^\mu\} \quad 1 \leq N \leq N_{\max},$$

with

$$S_1 \subset S_2 \ldots S_{N_{\max}} \subset \mathcal{D}^\mu$$

Lagrangian Hierarchical Space

$$W_N = \mathrm{span}\,\{\zeta_n \equiv \underbrace{u(\mu^n)}_{u^{\mathcal{N}}(\mu^n)}, n = 1, \ldots, N\}.$$

with

$$W_1 \subset W_2 \ldots W_{N_{\max}} \subset X^{\mathcal{N}} \subset X$$

Formulation : Galerkin method

Galerkin Projection

Given $\mu \in \mathcal{D}^\mu$ evaluate

$$s_N(\mu) = \ell(u_N(\mu); \mu)$$

where $u_N(\mu) \in X_N$ satisfies

$$a(u_N(\mu), v; \mu) = f(v; \mu), \ \forall\, v \in X_N \ .$$

- RB Space $X_N = \mathrm{GramSchmidt}(W_N)$
- Well posed problem (there exists a unique solution : coercivity, continuity, linear independence)

Formulation : a priori convergence

For any $\mu \in \mathcal{D}^\mu$, we have the following optimality(thanks to Galerkin) results

$$||u(\mu) - u_N(\mu)||_X \leq \left(1 + \frac{\gamma(\mu)}{\alpha(\mu)}\right) \inf_{v_N \in X_N} ||u(\mu) - v_N(\mu)||_X,$$

and

$$|s(\mu) - s_N(\mu)| \leq C||u(\mu) - v_N(\mu)||_X$$

Remember that

- symmetry : $\left(1 + \frac{\gamma(\mu)}{\alpha(\mu)}\right) \Rightarrow \sqrt{\frac{\gamma(\mu)}{\alpha(\mu)}}$;
- compliance : "quadratic convergence" in the output (not the case anymore)

Formulation : a posteriori error estimation

We wish to develop rigorous, sharp and efficient online a posteriori error estimation $\Delta_N(\mu)$, $\Delta_N^s(\mu)$ such that $\forall \mu \in \mathcal{D}^\mu$

$$\|u(\mu) - u_N(\mu)\|_X \leq \Delta_N(\mu)$$
$$|s(\mu) - s_N(\mu)\| \leq \Delta_N^s(\mu)$$

Coercivity Lower Bound	OK
Error Bounds	OK (using a_s)

However two issues remain.

Formulation : coercivity lower bound

For a non-symmetric we introduce

$$a_s(u, v; \mu) = \sum_{q=1}^{Q_{a_s}} \Theta^q_{a_s}(\mu)\ a^q_s(u, v), \quad \forall u, v \in X$$

where

$$a_s(u, v; \mu) = \frac{1}{2}(a(u, v; \mu) + a(v, u; \mu))$$

We then apply either

- the "min Θ" approach if a_s is parametrically coercive

$$\alpha_{\mathrm{LB}}(\mu) \equiv \Theta^{\min,\bar{\mu}}_{a_s} = \min_{q \in \{1 \dots Q_{a_s}} \frac{\Theta^q_{a_s}(\mu)}{\Theta^q_{a_s}(\bar{\mu})}$$

- or the SCM (a_s)

Formulation : a posteriori error bounds

Given our RB approximation $u_N(\mu)$, we have

$$e(\mu) \equiv u(\mu) - u_N(\mu)$$

that satisfies

$$a(e(\mu), v; \mu) \ = \ r(u_N(\mu), v; \mu), \forall v \in X$$

where $r(u_N(\mu), v; \mu) = f(v) - a(u_N(\mu), v; \mu)$ is the residual. We have then from coercivity and the definitions above that

$$||e(\mu)||_X \ \leq \ \frac{||r(u_N(\mu), v; \mu)||_{X'}}{\alpha(\mu)} \ = \ \frac{\varepsilon_N(\mu)}{\alpha(\mu)}$$

A Posteriori error estimation : Dual norm of the residual

Proposition

Given $\mu \in \mathcal{D}^\mu$, the dual norm of $r(u_N(\mu), \cdot; \mu)$ is defined as follows

$$\begin{aligned} ||r(u_N(\mu), \cdot; \mu)||_{X'} &\equiv \sup_{v \in X} \frac{r(u_N(\mu), v; \mu)}{||v||_X} \\ &= ||\hat{e}(\mu)||_X \end{aligned}$$

where $\hat{e}(\mu) \in X$ satisfies

$$(\hat{e}(\mu), v)_X = r(u_N(\mu), v; \mu)$$

The error residual equation can then be rewritten

$$a(e(\mu), v; \mu) = (\hat{e}(\mu), v)_X, \quad \forall v \in X$$

A Posteriori error estimation : Dual norm of the residual

Then we can define

Definition : Energy error bound

$$\Delta_N(\mu) \equiv \frac{\|\hat{e}(\mu)\|_X}{\alpha_{\mathrm{LB}}(\mu)}$$

Definition : Effectivity

$$\eta_N(\mu) \equiv \frac{\Delta_N(\mu)}{\|e(\mu)\|_X}$$

Proposition

for $N = 1 \ldots N_{\max}$, the effectivity $\eta_N(\mu)$ verifies

$$1 \le \eta_N(\mu) \le \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}, \quad \forall \mu \in \mathcal{D}^{\mu}.$$

A Posteriori error estimation : Output error bound

Then we can define

Definition : Output error bound

$$\Delta_N^s(\mu) \equiv \|\ell(\cdot,\mu)\|_{X'} \Delta_N(\mu)$$

Definition : Output Effectivity

$$\eta_N^s(\mu) \equiv \frac{\Delta_N^s(\mu)}{|s(\mu) - s_N(\mu)|}$$

Proposition

for $N = 1 \ldots N_{\max}$, the error $|s(\mu) - s_N(\mu)|$ verifies

$$|s(\mu) - s_N(\mu)| \leq \Delta_N^s(\mu), \quad \forall \mu \in \mathcal{D}^\mu$$

A Posteriori error estimation : Remarks and Motivations for a Primal/Dual formulation

- Very similar to the compliant case : need only $\|\ell(\cdot,\mu)\|_{X'}$ find $\hat{e}_\ell \in X$ (Riesz representation) such that

$$(\hat{e}_\ell, v)_X = \ell(v,\mu), \quad \forall v \in X$$

  and apply offline-online decomposition similarly to other terms
- Rigorous error bounds
- Best approach if many outputs (little overhead), however in case of few outputs a primal-dual formulation is preferable
  1. Loss of quadratic convergence
  2. Effectivities possibly unbounded,

$$\eta_N^s(\mu) \geq \frac{\|\ell(\cdot,\mu)\|_{X'}}{\gamma(\mu)\|u(\mu)-u_N(\mu)\|_X}$$

  from output error bound (taking $\ell = f$) and energy error bound.

Formulation (Linear Case)

$$\text{Sample}: \quad S_N = \{\mu_1 \in \mathcal{D}^\mu, \ldots, \mu_N \in \mathcal{D}^\mu\} \;.$$

$$\text{Sample}: \quad S^{\mathrm{du}}_{N^{\mathrm{du}}} = \{\mu^{\mathrm{du}} \in \mathcal{D}^\mu, \ldots, \mu^{\mathrm{du}}_{N^{\mathrm{du}}} \in \mathcal{D}^\mu\} \;.$$

$$\text{Space}: \quad W_N = \mathrm{span}\,\{\zeta_n \equiv \underbrace{u(\mu^n)}_{u^{\mathcal{N}}(\mu^n)}, n = 1, \ldots, N\}.$$

$$\text{Space}: \quad W^{\mathrm{du}}_{N^{\mathrm{du}}} = \mathrm{span}\,\{\zeta^{\mathrm{du}}_n \equiv \underbrace{\Psi(\mu^{\mathrm{du}}_n)}_{\Psi^{\mathcal{N}}(\mu^{\mathrm{du}}_n)}, n = 1, \ldots, N^{\mathrm{du}}\}.$$

Sampling strategies ?

- Equidistributed points in $\mathcal{D}^\mu$ (curse of dimensionality)
- Log-random distributed points in $\mathcal{D}^\mu$
- See later for more efficient, adaptive strategies

Formulation (Linear Case) : a Galerkin method

Galerkin Projection

Given $\mu \in \mathcal{D}^\mu$ evaluate

$$s_N(\mu) = \ell(u_N(\mu)) - r(u_N(\mu), \Psi_{N^{\mathrm{du}}}(\mu); \mu) \;;$$

where $u_N(\mu) \in X_N$ and $\Psi_{N^{\mathrm{du}}}(\mu) \in X^{\mathrm{du}}_{N^{\mathrm{du}}}$ satisfy

$$a(u_N(\mu), v; \mu) = f(v), \; \forall\, v \in X_N \;.$$

and

$$a(v, \Psi_{N^{\mathrm{du}}}(\mu) \;; \mu) = -\ell(v; \mu), \; \forall\, v \in X^{\mathrm{du}}_{N^{\mathrm{du}}} \;.$$

- Note that RB Space $X_N = \mathrm{GramSchmidt}(W_N), X^{\mathrm{du}}_N = \mathrm{GramSchmidt}(W^{\mathrm{du}}_N)$
- In general $N \neq N^{\mathrm{du}}$ (primal and dual are different problems)

Formulation (Linear Case) : back to the compliant case

Recall that in compliance

- a is symmetric
- $\ell = f$

such that $\Psi(\mu) = -u(\mu)$.
We may take $N^{\rm du} = N$, $S_N^{\rm du} = S_N$ and $X_N^{\rm du} = X_N$ and get

$$\Psi_N(\mu) = -u_N(\mu)$$

Compliant case

- The dual problem is never formed/solved
- We simply identify $\Psi_N(\mu) = -u_N(\mu)$
- We get a 50% cost reduction

Formulation : A priory convergence

Proposition

For any $\mu \in \mathcal{D}^\mu$, we have

$$|s(\mu) - s_N(\mu)| \leq C \left(\inf_{v_N \in X_N} ||u(\mu) - v_N(\mu)||_X \right) \times \left(\inf_{v_N^{\mathrm{du}} \in X_N^{\mathrm{du}}} ||\psi(\mu) - \psi_N^{\mathrm{du}}(\mu)||_X \right)$$

- Recovery of quadratic convergence for the output !
- Alternative : build RB space comprising both primal and dual basis functions (output dual correction not needed however more costly and conditioning issues)

Formulation (Linear Case) : offline-online decomposition

Expand our RB approximations :

$$u_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\,\zeta_j$$

$$\Psi_{N^{\rm du}}(\mu) = \sum_{j=1}^{N^{\rm du}} \Psi_{N\,j}(\mu)\,\zeta_j^{\rm du}$$

Express $s_N(\mu)$

$$s_N(\mu) = \sum_{j=1}^{N} u_{N\,j}(\mu)\,\ell(\zeta_j) - \sum_{j=1}^{N^{\rm du}} \Psi_{N\,j}(\mu)\,f(\zeta_j^{\rm du}) + \sum_{j=1}^{N}\sum_{j'=1}^{N^{\rm du}}\sum_{q=1}^{Q} u_{N\,j}(\mu)\,\Psi_{N\,j'}(\mu)\,\Theta^q(\mu)\,a^q(\zeta_j,\zeta_{j'}^{\rm du})$$

Formulation (Linear Case) : offline-online decomposition

$u_{N\,i}(\mu), 1 \leq i \leq N$ and $\Psi_{N\,i}(\mu), 1 \leq i \leq N^{\mathrm{du}}$ satisfy

$$\sum_{j=1}^{N} \left\{ \sum_{q=1}^{Q} \Theta^q(\mu)\; a^q(\zeta_i, \zeta_j) \right\} u_{N\,j}(\mu) = f(\zeta_i), \quad 1 \leq i \leq N$$

$$\sum_{j=1}^{N^{\mathrm{du}}} \left\{ \sum_{q=1}^{Q} \Theta^q(\mu)\; a^q(\zeta_i^{\mathrm{du}}, \zeta_j^{\mathrm{du}}) \right\} \Psi_{N^{\mathrm{du}}\,j}(\mu) = -\ell(\zeta_i^{\mathrm{du}}), \quad 1 \leq i \leq N^{\mathrm{du}}$$

Formulation (Linear Case) : matrix form

Solve

$$\underline{A}_N(\mu)\,\underline{u}_N(\mu) = \underline{F}_N$$

and

$$\underline{A}^{\mathrm{du}}_{N^{\mathrm{du}}}(\mu)\,\underline{\Psi}_{N^{\mathrm{du}}}(\mu) = -\underline{L}_N$$

where

$$(A_N)_{i\,j}(\mu) = \sum_{q=1}^{Q}\ \Theta^q(\mu)\ a^q(\zeta_i,\zeta_j),\quad F_{N\,i} = f(\zeta_i)\ . \qquad 1 \le i,j \le N \qquad 1 \le i \le N$$

and

$$(A^{\mathrm{du}}_{N^{\mathrm{du}}})_{i\,j}(\mu) = \sum_{q=1}^{Q}\ \Theta^q(\mu)\ a^q(\zeta^{\mathrm{du}}_i,\zeta^{\mathrm{du}}_j),\quad L_{N\,i} = \ell(\zeta^{\mathrm{du}}_i)\ . \qquad 1 \le i,j \le N^{\mathrm{du}} \qquad 1 \le i \le N^{\mathrm{du}}$$

Formulation (Linear Case) : complexity analysis

Offline : independent of μ

- Solve : $N + N^{\mathrm{du}}$ FEM system depending on $\mathcal{N}$
- Form and store : $f(\zeta_i)$, $\ell(\zeta_i)$, $f(\zeta_i^{\mathrm{du}})$, $\ell(\zeta_i^{\mathrm{du}})$
- Form and store : $a^q(\zeta_i, \zeta_j)$, $a^q(\zeta_i^{\mathrm{du}}, \zeta_j^{\mathrm{du}})$, $a^q(\zeta_i, \zeta_j^{\mathrm{du}})$

Online : independent of $\mathcal{N}$

- Given a new $\mu \in \mathcal{D}^\mu$
- Form and solve $A_N(\mu)$: $O(QN^2)$ and $O(N^3)$
- Form and solve ${A_{N^{\mathrm{du}}}}^{\mathrm{du}}(\mu)$: $O(Q{N^{\mathrm{du}}}^2)$ and $O({N^{\mathrm{du}}}^3)$
- Compute $s_N(\mu)$

Online : $N, N^{\mathrm{du}} << \mathcal{N}$

Online we realize often orders of magnitude computational economies relative to FEM in the context of many μ-queries

$u_N(\mu)$: Error equation and residual dual norm

Given our RB approximation $u_N(\mu)$, we have

$$e(\mu) \equiv u(\mu) - u_N(\mu)$$

that satisfies

$$a(e(\mu), v; \mu) \;=\; r(u_N(\mu), v; \mu), \forall v \in X$$

where $r(u_N(\mu), v; \mu) = f(v) - a(u_N(\mu), v; \mu)$ in the linear case is the residual. We have then

$$||e(\mu)||_X \;\leq\; \frac{||r(u_N(\mu), v; \mu)||_{X'}}{\alpha(\mu)} \;=\; \frac{\varepsilon_N(\mu)}{\alpha(\mu)}$$

$u_N(\mu)$: Defintions of energy error bounds and effectivity

Given $\alpha_{\mathrm{LB}}(\mu)$ a nonnegative lower bound of $\alpha(\mu)$:

$$\alpha(\mu) \geq \alpha_{\mathrm{LB}}(\mu) \geq \epsilon_\alpha \alpha(\mu),\ \epsilon_\alpha \ \in]0,1[,\ \forall \mu \in \mathcal{D}^\mu$$

Definition : Energy error bound

$$\Delta_N(\mu) \ \equiv \ \frac{\varepsilon_N(\mu)}{\alpha_{\mathrm{LB}}(\mu)}$$

Definition : Effectivity

$$\eta_N(\mu) \ \equiv \ \frac{\Delta_N(\mu)}{||e(\mu)||_X}$$

$u_N(\mu)$: Rigorous sharp error bounds

One can prove that

$$1 \;\leq\; \eta_N(\mu) \;\leq \frac{\gamma(\mu)}{\alpha_{\mathrm{LB}}(\mu)}, \quad 1 \leq N \leq N_{\max}, \quad \forall \mu \;\in\; \mathcal{D}^\mu$$

Remarks

- Rigorous : Left inequality ensures rigorous upper bound measured in $||\cdot||_X$, i.e. $||e(\mu)||_X \leq \Delta_N(\mu),\ \forall \mu \in \mathcal{D}^\mu$
- Sharp : Right inequality states that $\Delta_N(\mu)$ overestimates the "true" error by at most $\gamma(\mu)/\alpha_{\mathrm{LB}}(\mu)$

$\Psi_N(\mu)$: error bounds

We have a similar result for the dual problem

$$||\Psi(\mu) - \Psi_{N^{\mathrm{du}}}||_X \;\leq\; \Delta_N^{\mathrm{du}}(\mu), \quad 1 \;\leq\; N^{\mathrm{du}} \;\leq\; N_{\mathrm{max}}^{\mathrm{du}}, \quad \forall \mu \;\in\; \mathcal{D}^{\mu}$$

where

$$\Delta_N^{\mathrm{du}}(\mu) \;\equiv\; \frac{\varepsilon_N^{\mathrm{du}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)} \;\equiv\; \frac{||-\ell(\cdot) - a(\cdot, \Psi_{N^{\mathrm{du}}}(\mu); \mu)||_{X'}}{\alpha_{\mathrm{LB}}(\mu)}$$

$\varepsilon_N^{\mathrm{du}}(\mu)$ is the dual norm of the residual.

$s_N(\mu)$: error bounds

From primal and dual energy error bounds we have

$$|s(\mu) - s_N(\mu)| \;\leq\; \Delta^s_N(\mu), \quad \mu \;\in\; \mathcal{D}^\mu$$

where

$$\Delta^s_N(\mu) \equiv \varepsilon_N(\mu)\Delta^{\mathrm{du}}_N(\mu)(\mu) \;=\; \alpha_{\mathrm{LB}}(\mu)\Delta_N(\mu)\Delta^{\mathrm{du}}_N(\mu)$$

Rapid convergence of the error in the output

Note that the error in the output vanishes as the product of the error in the primal and dual error

Back to compliance : a symmetric and $\ell = f$

We obtain

$$\Delta^s_N(\mu) \equiv \frac{\varepsilon^2_N(\mu)}{\alpha_{\mathrm{LB}}(\mu)}, \quad \forall \mu \in \mathcal{D}^\mu$$

Offline-Online decomposition (Primal problem)

«««< HEADDual problem : similar treatmentDenote $\hat{e}(\mu) \in Y$

$$||\hat{e}(\mu)||_Y = \varepsilon_N(\mu) = ||g(u_N(\mu), \cdot; \mu)||_Y \tag{16}$$

such that

$$(\hat{e}(\mu), v)_Y = -g(u_N(\mu), v; \mu), \quad \forall v \in Y \tag{17}$$

Recall that

$$-g(u_N(\mu), v; \mu) = f(v) - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu)\, u_{N\,n}(\mu)\, a^q(\zeta_n, v), \quad \forall v \in X \tag{18}$$

Offline-Online decomposition (Primal problem)

- It follows next that $\hat{e}(\mu) \in Y$ satisfies

$$(\hat{e}(\mu), v)_Y \; = \; f(v) - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu) \; u_{N\,n}(\mu) \; a^q(\zeta_n, v), \quad \forall v \; \in \; X \tag{19}$$

- Observe then that the rhs is the *sum* of products of parameter dependent functions and parameter independent linear functionals, thus invoking linear superposition

$$\hat{e}(\mu) \; = \; \mathcal{C} - \sum_{q=1}^{Q} \sum_{n=1}^{N} \Theta^q(\mu) \; u_{N\,n}(\mu) \; \mathcal{L}_n^q \tag{20}$$

  - $\mathcal{C} \in Y$ satisfies

$$(\mathcal{C}, v) = f(v), \forall v \in Y \tag{21}$$

  - $\mathcal{L} \in Y$ satisfies

$$(\mathcal{L}_n^q, v)_Y = -a^q(\zeta_n, v), \forall v \in Y, \; 1 \leq n \leq N, 1 \leq q \leq Q \tag{22}$$

(22) are parameter independent Poisson problems

Offline-Online decomposition : Error bounds

From (20) we get

$$||\hat{e}(\mu)||_Y^2 = (\mathcal{C},\mathcal{C})_Y + \sum_{q=1}^{Q}\sum_{n=1}^{N} \Theta^q(\mu)\, u_{N\,n}(\mu) \Big\{ 2(\mathcal{C},\mathcal{L}_n^q)_Y + \sum_{q'=1}^{Q'}\sum_{n'=1}^{N'} \Theta^{q'}(\mu)\, u_{N\,n'}(\mu)\, (\mathcal{L}_n^q,\mathcal{L}_{n'}^{q'})_Y \Big\} \quad (23)$$

Remark

In (23), $||\hat{e}(\mu)||_Y^2$ is the sum of products of

- parameter dependent (simple/known) functions and
- parameter independent inner-product,

the offline-online for the error bounds is now clear.

Offline-Online decomposition : steps and complexity

Offline :

- Solve for $\mathcal{C}$ and $\mathcal{L}_n^q,\ 1 \leq n \leq N,\ 1 \leq q \leq Q$
- Form and save $(\mathcal{C},\mathcal{C})_Y$, $(\mathcal{C},\mathcal{L}_n^q)_Y$ and $(\mathcal{L}_n^q,\mathcal{L}_{n'}^{q'})_Y$, $1 \leq n, n' \leq N,\ 1 \leq q, q' \leq Q$

Online

- Given a new $\mu \in \mathcal{D}^\mu$
- Evaluate the sum (23) in terms of $\Theta^q(\mu)$ and $u_{N\,n}(\mu)$
- Complexity in $O(Q^2N^2)$ independent of $\mathcal{N}$

The linear symmetric coercive case

- We require a lower bound $\beta_{\mathrm{LB}}(\mu)$ for $\beta(\mu) = \alpha_c(\mu), \ \forall \mu \in \mathcal{D}^\mu$
- If
- Primal-Dual problem : similar treatment as in Primal-only formulation
- New ingredient introduced is the correction term for the output $r(u_N(\mu), \Psi_{N^{\mathrm{du}}}(\mu); \mu) = f(\psi_N^{\mathrm{du}}(\mu); \mu) - a(u_N(\mu), \psi_N^{\mathrm{du}}(\mu); \mu)$ which requires cross terms between primal and dual problems
  - $f^q(\zeta_n^{\mathrm{du}}), \quad 1 \leq n \leq N^{\mathrm{du}}, 1 \leq q \leq Q_f$
  - $a^q(\zeta_n, \zeta_n^{\mathrm{du}}), \quad 1 \leq n \leq N, \ 1 \leq n \leq N^{\mathrm{du}}, \ 1 \leq q \leq Q_a$

Non-Affine and/or Non-Linear Problems

Notations

NonLinear μ-parametrized PDE

- Set of parameters : $\mu \in \mathcal{D}^\mu \subset \mathbb{R}^p$,
- Solution of the nonlinear μ-PDE : $u(\mu) \in X \equiv H^1(\Omega \subset \mathbb{R}^d)$,
- PDE weak formulation : We look for $u(\mu) \in X$ such that

$$g(u(\mu), v; \mu) = 0, \ \forall\, v \in X \ .$$

Truth approximation

"Truth" FEM approximation

- $X^{\mathcal{N}} \subset X$: finite element approximation of dimension $\mathcal{N} >> 1$.
- $u^{\mathcal{N}}(\mu) \in X^{\mathcal{N}}$ is solution of $g(u^{\mathcal{N}}(\mu), v; \mu) = 0, \ \forall\, v \in X^{\mathcal{N}}$.
- Solution strategies such as Newton or Picard iterations, e.g. given ${}^0u^{\mathcal{N}}$, build the nonlinear iterates ${}^1u^{\mathcal{N}}, \dots, {}^ku^{\mathcal{N}}, \dots$

$$j(\delta^k u^{\mathcal{N}}(\mu), v; {}^k u^{\mathcal{N}}(\mu), \mu) = -g({}^k u^{\mathcal{N}}(\mu), v; \mu)$$

  where

$$\delta^k u^{\mathcal{N}}(\mu) = {}^{k+1}u^{\mathcal{N}}(\mu) - {}^k u^{\mathcal{N}}(\mu).$$

  We recover the linear case.
- Equate $u(\mu)$ and $u^{\mathcal{N}}(\mu)$, i.e.

$$\|u(\mu) - u^{\mathcal{N}}(\mu)\|_X \leq \text{tol}, \ \forall\, \mu \in \mathcal{D}^\mu \ .$$

Recovering Affine decomposition (Nonlinear-case)

We wish to build a reduced basis approximation $u_N(\mu)$, to apply the methodology recall that we require affine dependence.

Generalized affine decomposition

Suppose that we can build $g_{AD}\left(u(\mu), v; \mu\right)$ such that

$$g\left(u(\mu), v; \mu\right) \approx g_{AD}\left(u(\mu), v; \mu\right) = \sum_{q=1}^{Q_g} \sum_{m=1}^{M_q^g} \beta_g^{qm}(\mu; u(\mu))\; g^{qm}(v)\ ,$$

and similarly for

$$j\left(u(\mu), v; \mu; w(\mu)\right) \approx j_{AD}\left(u(\mu), v; \mu; w(\mu)\right) = \sum_{q=1}^{Q_j} \sum_{m=1}^{M_q^j} \beta_j^{qm}(\mu; w(\mu))\; j^{qm}(u(\mu), v)\ , \qquad (24)$$

Generalized affine decomposition

Finite element or reduced basis approximations

The Newton algorithm now reads offline($\mathcal{N}$) and online (N) of the reduced basis methodology

$$j_{AD}\left(\delta^k u^{\mathcal{N}|N}(\mu), v; \mu; {}^k u^{\mathcal{N}|N}(\mu)\right) = -g_{AD}\left({}^k u^{\mathcal{N}|N}(\mu), v; \mu\right) \;, \qquad (25)$$

for the increment $\delta^k u^{\mathcal{N}|N}(\mu)$ defined by

$$\delta^k u^{\mathcal{N}|N}(\mu) = \left({}^{k+1} u^{\mathcal{N}|N}(\mu) - {}^k u^{\mathcal{N}|N}(\mu)\right) \;. \qquad (26)$$

Empirical Interpolation Method : Objective

We are given a parametrized function g that depend on

- space $x \in \Omega$ and a parameter $\mu \in \mathbb{R}^P$,

$$\sigma(x;\mu)$$

- and possibly a field $u(\mu)$ solution of a μ-PDE

$$\sigma(u(\mu);x;\mu)$$

EIM : build $\sigma_M \approx \sigma$

We wish to build the expansion

$$\sigma_M(u(\mu);x;\mu) = \sum_{m=1}^{M} \beta_m(\mu)\ q_m(x) \approx \sigma(u(\mu);x;\mu)$$

Empirical Interpolation Method : Objective

One we have σ_M we can for example

- compute integrals with varying μ

$$\int_\Omega \sigma(x;\mu) \approx \int_\Omega \sigma_M(x;\mu) = \sum_{m=1}^{M} \beta_m(\mu) \int_\Omega q_m(x)$$

  The last term $\int_\Omega \beta(\mu))\ q_m(x)$ (independent of μ) can be precomputed and hence it provides a very efficient method

- recover affine decomposition for bilinear/linear forms

$$\int_\Omega \sigma(x;\mu)\nabla u \cdot \nabla v \approx \int_\Omega \sigma_M(x;\mu)\nabla u \cdot \nabla v = \sum_{m=1}^{M} \beta_m(\mu) \int_\Omega q_m(x)\nabla u \cdot \nabla v$$

Empirical Interpolation Method : Examples

- Consider the following function with one parameter μ and 2D space dimension

$$\sigma(x;\mu) = \exp(-((x-0.5)^2 + (y-0.5)^2)/(2\mu^2))$$

- Consider the following function with the triplet parameter $\mu = (\mu_1, \mu_2, \mu_3)$

$$\sigma(u(\mu); x; \mu) = \frac{\mu_1}{1 + \mu_2(u(\mu) - \mu_3)}$$

where $u(\mu) \in X$ is solution of the μ-PDE

$$a(u(\mu), v; \mu) = f(v;\mu) \forall v \in X$$

Empirical Interpolation Method [Barrault et al., 2004]

Ingredients

- Training set $\Xi^{\mu}_{train} \subset \mathcal{D}^{\mu}$
- Offline step
  - Sample $S_M = \{\mu_1 \in \Xi^{\mu}_{train}, ..., \mu_M \in \Xi^{\mu}_{train}\}$, Interpolation points $t_1, \dots, t_M \in \Omega$
  - Approximation space $W_M = span\{q_1(x), \dots, q_M(x)\}$
    - Residual $r_m(x) = \sigma(u^{\mathcal{N}}(\mu_m), x, \mu_m) - \sigma_m(u^{\mathcal{N}}(\mu_m), x, \mu_m)$
    - $q_{m+1}(x) = \frac{r_m(x)}{r_m(t_m)}$ (matrix $(B_{i,j}) = q_j(t_i)$ lower triangular)
- Online step : Compute approximation coefficients $\beta_m(u^{\mathcal{N}|N}(\mu); \mu)$

$$\sigma_M(u^{\mathcal{N}|N}(\mu); t_i; \mu) = \sum_{m=1}^{M} \beta_m(u^{\mathcal{N}|N}(\mu); \mu)\ q_m(t_i) = \sigma(u^{\mathcal{N}|N}(\mu); t_i; \mu)$$

$$\forall i = 1, \dots, M$$

A language embedded in C++ : EIM expansion

Let u be the solution of $g(u, v; \mu) = 0 \ \forall v \in X^{\mathbb{N}}$ and $\sigma(u)$ the non linear expression involving a field.

```
parameterspace_ptrtype D; parameter_type mu; //μ ∈ D^μ
auto Pset = Dmu->sampling();
int eim_sampling_size = 1000;
Pset->randomize(eim_sampling_size);

//expression we want EIM expansion
auto sigma = ref(mu(0))/(1+ref(mu(2))*(idv(u)-u0));

//call Feel++ function eim
auto eim_sigma = eim( _model=solve( g(u,v;μ;x) = 0 ),
       _element=u,      // u_N(μ)
       _parameter=mu,  // μ
       _expr=sigma,     // σ(u)
       _space=X_N,
       _name="eim_sigma",
       _sampling=Pset );

//then we can have access to β coefficients
// of EIM expansion Σ_{m=1}^{M} β(μ, u_N(μ)) q_m(x)
std::vector<double> beta_sigma = eim_sigma->beta( mu );
```

Non-affine Non-Linear decomposition : Wrap-up

- Build EIM approximations of non-linear terms using the initial finite element approximation
  - Generate databases of $\mathcal{N}$ independent terms and $\mathcal{N}$ dependent terms
  - Optimisation opportunities in EIM right hand side online step evaluation
    $$\sigma(u^{\mathcal{N}|N}(\mu); t_i; \mu)$$
    by storing the (FEM or RB) basis functions associated to $u^{\mathcal{N}|N}(\mu)$ at the t_i.
- Build the generalized affine decomposition of the non-linear problem (Newton or Picard iterations)
- Compute the RB approximations (and associated reduced quantities) using the FEM approximation of the generalized affine problem
- Store all the $\mathcal{N}$-independent terms in database (we get rid of the finite element space dimension) and depend solely N, Q and the complexity of the generalized affine expansion.

Affine decomposition in Feel++

given Q_g , Q_j and Q_ℓ the EIM determines $(M_q^g)_{q=1,..,Q_g}$, $(M_q^j)_{q=1,..,Q_j}$ and $(M_q^\ell)_{q=1,..,Q_\ell}$ such that we can write :

$$g_{AD}\left({}^k u(\mu), v; \mu\right) = \sum_{q=1}^{Q_g} \sum_{m=1}^{M_q^g} \beta_g^{qm}(\mu; {}^k u(\mu))\ g^{qm}(v)\ ,$$

$$j_{AD}\left(u(\mu), v; \mu; {}^k u(\mu)\right) = \sum_{q=1}^{Q_j} \sum_{m=1}^{M_q^j} \beta_j^{qm}(\mu; {}^k u(\mu))\ j^{qm}(u(\mu), v)\ ,$$

and

$$\ell_{AD}(v;\mu) = \sum_{q=1}^{Q_\ell} \sum_{m=1}^{M_q^\ell} \beta_\ell^{qm}(\mu)\ \ell^{qm}(v)\ .$$

The standard affine decomposition is in fact a special case of the generalized one.

Linear Parabolic Problems

"Truth" FEM Approximation

Let $I = (0, T_f)$, with T_f is the final time. Let $\mu \in \mathcal{D}^\mu$ and $t \in I$, evaluate

$$s^{\mathcal{N}}(t;\mu) = \ell(t; u^{\mathcal{N}}(\mu)) \ ,$$

where $u^{\mathcal{N}}(t;\mu) \in X^{\mathcal{N}} \subset X$ satisfies

$$m\left(\frac{\partial u^{\mathcal{N}}(\mu)}{\partial t}, v; \mu\right) + a(u^{\mathcal{N}}(\mu), v; \mu) = f(t; v), \quad \forall\, v \in X^{\mathcal{N}} \quad \forall\, t \in I \ .$$

- $a(\cdot,\cdot;\mu)$ and $m(\cdot,\cdot;\mu)$ are bilinear, $f(\cdot;\mu)$ and $\ell(\cdot;\mu)$ are linear
- f and ℓ are bounded

"Truth" FEM Approximation : Inner products and Norms

Let Δt the time step defined by $\Delta t = \frac{T_f}{K}$.
$\forall w, v \in X$ and $1 \leq k \leq K$ we next define the

- energy inner product and associated norm (parameter dependent)

$$\left(\left(\left(w(t^k), v(t^k)\right)\right)\right)_\mu = m(w(t^k), v(t^k); \mu) + \sum_{k'=1}^{k} a_S(w(t^{k'}), v(t^{k'}); \mu)\Delta t$$

$$\left|\left|\left|v(t^k)\right|\right|\right|_\mu = \sqrt{m(v(t^k), v(t^k); \mu) + a_S(v(t^{k'}), v(t^{k'}); \mu)\Delta t}$$

- X-inner product and associated norm (parameter independent)

$$\left(w(t^k), v(t^k)\right)_X = \left(\left(\left(w(t^k), v(t^k)\right)\right)\right)_{\bar{\mu}} \qquad \forall w, v \in X, 1 \leq k \leq K$$

$$\left|\left|v(t^k)\right|\right|_X = \left|\left|\left|v(t^k)\right|\right|\right|_{\bar{\mu}} \qquad \forall v \in X, 1 \leq k \leq K$$

where a_s denotes the symmetric part of a.

Spaces

Parameter Samples :

$$\text{Sample}: \quad S_N = \{\mu_1 \in \mathcal{D}^\mu, \ldots, \mu_N \in \mathcal{D}^\mu\} \quad 1 \leq N \leq N_{\max},$$

with

$$S_1 \subset S_2 \ldots S_{N_{\max}} \subset \mathcal{D}^\mu$$

Let $Snap(\mu) = \left\{ u^{\mathcal{N}}(t^k, \mu), 1 \leq k \leq K \right\}$ a snapshot set with $\mu \in S_N$

$$W_N = \mathrm{span}\left\{ \zeta_n \equiv \arg \inf_{v \in Snap(\mu)} \left(\frac{1}{K} \sum_{k=1}^{K} \left|\left| u^{\mathcal{N}}(t^k, \mu) - v \right|\right|_X^2 \right), n = 1, \ldots, N \right\}.$$

with

$$W_1 \subset W_2 \ldots W_{N_{\max}} \subset X^{\mathcal{N}} \subset X$$

A posteriori error estimation

Given $\mu \in \mathcal{D}^\mu$ we define

$$\varepsilon_N(t^k;\mu) = \left|\left| r(u_N(t^k;\mu), v;\mu) \right|\right|_{X'} \ , \ 1 \le k \le K.$$

Let $e(t^k,\mu) = \left|\left| u^{\mathcal{N}}(t^k,\mu) - u_N(t^k,\mu) \right|\right|_X$, we can write

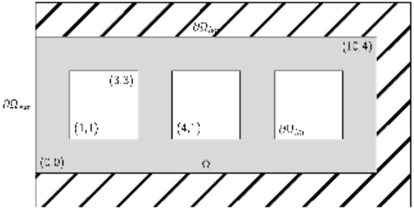
$$e(t^k,\mu) \le \Delta_N(t^k,\mu) \equiv \sqrt{\frac{\Delta t}{\alpha_{LB}(\mu)} \sum_{k'=1}^{k} \varepsilon_N(t^{k'};\mu)^2} \ , \ 1 \le k \le K.$$

$\forall \mu \in \mathcal{D}^\mu$ and $1 \le k \le K$ the output error bound is given by

$$\left| s^{\mathcal{N}}(t^k,\mu) - s_N(t^k,\mu) \right| \le \Delta_N^s(t^k,\mu) \equiv \Delta_N(t^k,\mu) \ \Delta_N^{du}(t^k,\mu).$$

Example Heat Shield : Problem statement

$$\begin{cases} -\Delta u + \frac{\partial u}{\partial t} = 0 & \text{in } \Omega, \\ \text{boundaries conditions} & \text{on } \partial\Omega. \end{cases}$$



Boundaries conditions

$\partial\Omega_{ext}$: heat transfert with $T_{air} = 1$

- $-\nabla u \cdot \mathbf{n} = Biot_{ext}(u - T_{air})$;

$\partial\Omega_{int}$: heat transfert, $T_{air} = 0$

- $-\nabla u \cdot \mathbf{n} = Biot_{int}(u - T_{air})$;

$\partial\Omega_{iso}$: Insulated.

2 parameters

- $Biot_{ext}$ and $Biot_{int}$.

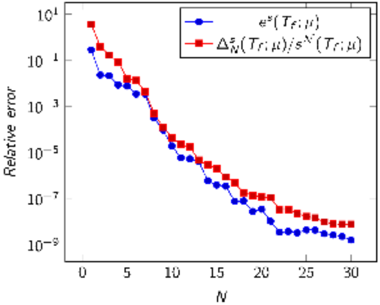
Example Heat Shield

- final time : 20 s ;
- time step : 0.2 s ;
- Minimum parameters values.

- Maximum parameters values.

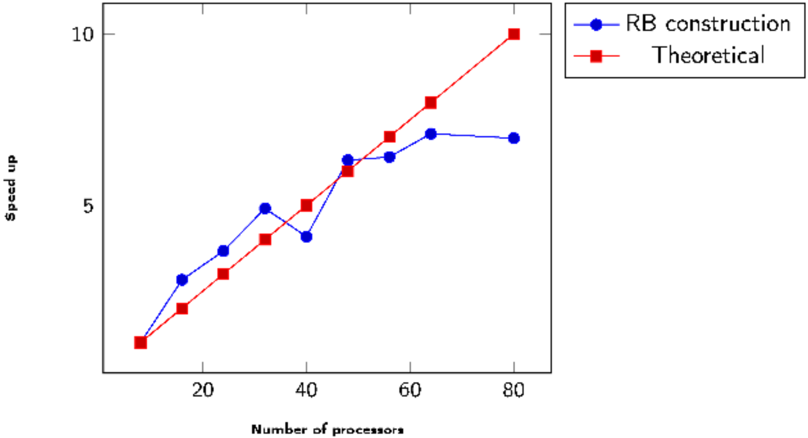
Example Heat Shield

- Configuration :
  - 33 000 dofs ;
  - Preconditionner : LU – Solver : MUMPS
  - Ξ : parameter sampling of dimension 430.
- Let $e^s(T_f;\mu) = \frac{|s^{\mathcal{N}}(T_f;\mu) - s_N(T_f;\mu)|}{s^{\mathcal{N}}(T_f;\mu)}$, plot $\max_{\mu\in\Xi} e^s(T_f;\mu)$



Example Heat Shield : Scalability

- Study the time to build the first reduced basis
- Configuration : 292 000 dofs for FEM approximation.



HiFiMagnet project

High Field Magnet Modeling

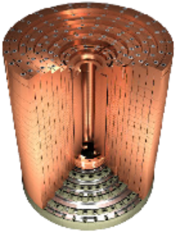
Laboratoire National des Champs Magnétiques Intenses

Large scale user facility in France

- High magnetic field : from 24 T
- Grenoble : continuous magnetic field (36 T)
- Toulouse : pulsed magnetic field (90 T)

Application domains

- Magnetoscience
- Solide state physic
- Chemistry
- Biochemistry



Magnetic Field

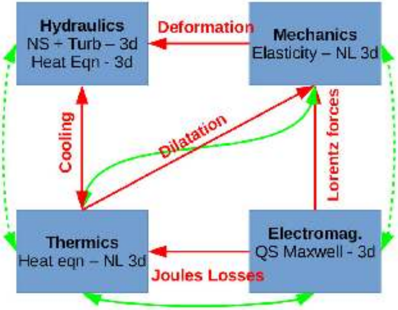
- Earth : $5.8 \cdot 10^{-4} T$
- Supraconductors : $24 T$
- Continuous field : $36 T$
- Pulsed field : $90 T$

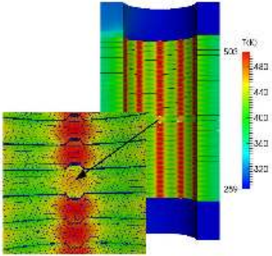
Access

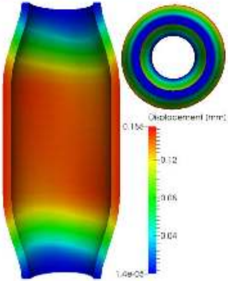
- Call for Magnet Time : $2 \times$ per year
- $\approx$ 140 projects per year

HiFiMagnet

High Field Magnet Modeling







Why use Reduced Basis Methods ?

Challenges

- Modeling : multi-physics non-linear models, complex geometries, genericity
- Account for uncertainties : uncertainty quantification, sensitivity analysis
- Optimization : shape of magnets, robustness of design

Objective 1 : Fast

- Complex geometries
  - Large number of dofs
- Uncertainty quantification
  - Large number of runs

Objective 2 : Reliable

- Field quality
- Design optimization
  - Certified bounds
  - Reach material limits

Electro-thermal model

- V : electrical potential
- T : temperature

$$\begin{cases} -\nabla \cdot (\sigma(T)\nabla V) = 0 \text{ on } \Omega \\ -\nabla \cdot (k(T)\nabla T) = \sigma(T)\nabla V \cdot \nabla V \text{ on } \Omega \end{cases}$$

Boundary conditions

- Applied potential
  - $V = 0$ on *Bottom*
  - $V = V_{Top}$ on *Top*
- Water / Glue electrically isolant
  - $-\sigma(T)\nabla V \cdot \mathbf{n} = 0$
- No thermic exchange with air
  - $-k(T)\nabla T \cdot \mathbf{n} = 0$
- Thermic exchange (h) with cooling water (T_w)
  - $-k(T)\nabla T \cdot \mathbf{n} = h(T - T_w)$

Non-linearity

Electrical conductivity

$$\sigma(T) = \frac{\sigma_0}{1 + \alpha(T - T_0)}$$

- $\sigma_0 = \sigma(ref = 20^\circ C)$
- $T_0 = 20^\circ C$
- α = temperature coeff.

Thermal conductivity

$$k(T) = LT\sigma(T)$$

- L = Lorentz number

Electro-thermal model : Inputs/Outputs

Parameters

Material properties

- Electrical conductivity (σ_0)
- Temperature coeff (α)
- Lorentz number (L)

Operating conditions

- Applied potential (V_D)
- Heat transfert coeff. (h)
- Water temperature (T_w)

$$\mu = (\sigma_0, \alpha, L, V_{Top}, h, T_w)$$

Outputs

$$s(\mu) = \ell(V(\mu), T(\mu))$$

Possibilities for ℓ :

- Mean temperature in the domain
- Magnetic field on specific point (Biot & Savart's law)
- Power of the magnet

Variationnal formulation

$$\begin{cases} \nabla\cdot(\sigma(T)\nabla V) = 0 \text{ on } \Omega_V \\ \nabla\cdot(k(T)\nabla T) = \sigma(T)\nabla V\cdot\nabla V \text{ on } \Omega_T \end{cases}$$

$$\begin{cases} V = V_D \text{ on } D_V \\ -\sigma(T)\nabla V\cdot\mathbf{n} = V_N \text{ on } N_V \end{cases} \qquad \begin{cases} -\sigma(T)\nabla T\cdot\mathbf{n} = 0 \text{ on } N_T \\ -\sigma(T)\nabla T\cdot\mathbf{n} = T_{R1}T + T_{R2} \text{ on } R_T \end{cases}$$

Electrical Potential

Find $V \in X \subset H_{\mathbf{1}}(\Omega)$ such that $\forall\ \phi_{\boldsymbol{V}} \in X$:

$$\int_\Omega \sigma(T)\nabla V\cdot\nabla\phi_{\boldsymbol{V}} \quad - \int_{\boldsymbol{D_V}} \sigma(T)(\nabla V\cdot\mathbf{n})\phi_{\boldsymbol{V}} + \int_{\boldsymbol{D_V}} \sigma(T)(\frac{\gamma}{h_{\boldsymbol{s}}} V\phi_{\boldsymbol{V}} - (\nabla\phi_{\boldsymbol{V}}\cdot\mathbf{n})V)$$
$$- \int_{\boldsymbol{D_V}} \sigma(T)(\frac{\gamma}{h_{\boldsymbol{s}}} V_{\boldsymbol{D}}\phi_{\boldsymbol{V}} - (\nabla\phi_{\boldsymbol{V}}\cdot\mathbf{n})V_{\boldsymbol{D}}) = 0$$

Temperature

Find $T \in X \subset H_{\mathbf{1}}(\Omega)$ such that $\forall\ \phi_{\boldsymbol{T}} \in X$:

$$\int_\Omega k(T)\nabla T\cdot\nabla\phi_{\boldsymbol{T}} + \int_{\boldsymbol{R_T}} T_{\boldsymbol{R1}} T\phi_{\boldsymbol{T}} = \int_\Omega \sigma(T)\nabla V\cdot\nabla V\phi_{\boldsymbol{T}} - \int_{\boldsymbol{R_T}} T_{\boldsymbol{R2}}\phi_{\boldsymbol{T}}$$

Non-affine parameter dependance

Considering first term of electrical potential formulation :

$$a_v = \int_\Omega \sigma(T) \nabla V \cdot \nabla \phi_V = \int_\Omega \frac{\sigma_0}{1 + \alpha(T - T_0)} \nabla V \cdot \nabla \phi_V$$

As σ_0 and α are input parameters :

$$\nexists\ a_q^v, \theta_q^v \ \mid\ a_v(V, T; \mu) = \sum_q \Theta_q^v(\mu) a_q^v(V, T, \phi_V, \phi_T)$$

EIM : Empirical Interpolation Method

Build an affine approximation a_v^{aff} of a_v such that :

$$a_v^{aff} = \sum_q \Theta_q^v(\mu) a_q^v(V, T, \phi_V, \phi_T)$$

exact on a set of interpolation points $\{t_i\}$: $a_v^{aff}(t_i) = a_v(t_i)\ \forall i$.

Electrical potential

$$\int_\Omega \sigma(T)\nabla V \cdot \nabla \phi_V \quad - \int_{D_V} \sigma(T)\left((\nabla V \cdot \mathbf{n})\phi_V + \frac{\gamma}{h_s} V\phi_V - (\nabla\phi_V \cdot \mathbf{n})V\right)$$

$$- \qquad \int_{D_V} \sigma(T) V_D(\frac{\gamma}{h_s}\phi_V - (\nabla\phi_V \cdot \mathbf{n})) = 0$$

$$\sigma(T) = \frac{\sigma_0}{1+\alpha(T-T_0)} \longrightarrow \quad \sigma^{aff}(T) = \sum_{m_\sigma=0}^{M_\sigma} \beta_{m_\sigma}(\mu) q_{m_\sigma}(T)$$

Temperature

$$\int_\Omega k(T)\nabla T \cdot \nabla\phi_T + \int_{R_T} T_{R1} T\phi_T = \int_\Omega \sigma(T)\nabla V \cdot \nabla V \phi_T - \int_{R_T} T_{R2}\phi_T$$

$$k(T) = \sigma(T)LT \longrightarrow \quad k^{aff}(T) = \sum_{m_k=0}^{M_k} \beta_{m_k}(\mu) q_{m_k}(T)$$

$$J(V,T) = \sigma(T)\nabla V \cdot \nabla V \longrightarrow \quad J^{aff}(V,T) = \sum_{m_J=0}^{M_J} \beta_{m_J}(\mu) q_{m_J}(V,T)$$

Coupled formulation

Find $(V,T) \in X \times X \subset [H_1(\Omega)]^2$ such that $\forall\ (\phi_V, \phi_T) \in X \times X$:

$$a((V,T),(\phi_V,\phi_T);\mu) = f((\phi_V,\phi_T);\mu) \quad \forall(\phi_V,\phi_T) \in X \times X$$

Affine decomposition

$$\sum_{q_a=1}^{Q_a} \Theta_a^q(\mu) a^q((V,T),(\phi_V,\phi_T)) = \sum_{q_f=1}^{Q_f} \Theta_f^q(\mu) f^q((\phi_V,\phi_T))$$

$$\sum_{m_\sigma=1}^{M_\sigma} \beta_{m_\sigma}(\mu) \left[\int_\Omega q_{m_\sigma}(T) \nabla V \cdot \nabla \phi_V \right.$$

$$\left. - \int_{D_V} q_{m_\sigma}(T) \left((\nabla V \cdot \mathbf{n})\phi_V + \frac{\gamma}{h_s} V \phi_V - (\nabla \phi_V \cdot \mathbf{n}) V \right) \right]$$

$$- \sum_{m_\sigma=1}^{M_\sigma} \beta_{m_\sigma}(\mu) V_D \int_{D_V} q_{m_\sigma}(T) \left(\frac{\gamma}{h_s} \phi_V - (\nabla \phi_V \cdot \mathbf{n}) \right)$$

$$+ \sum_{m_k=1}^{M_k} \beta_{m_k}(\mu) \int_\Omega q_{m_k}(T) \nabla T \cdot \nabla \phi_T + T_{R_1} \int_{R_T} T \phi_T$$

$$= \sum_{m_J=1}^{M_J} \beta_{m_J}(\mu) \int_\Omega q_{m_J}(V,T) \phi_T - T_{R_2} \int_{R_T} \phi_T$$

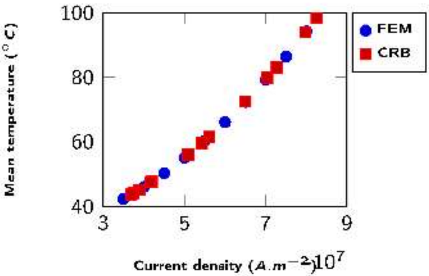
Reduced Electro-thermal model

From small towards large simulations

Parametric study on Bitter magnet

Maximum current density without thermal damages ?

$\Rightarrow$ Parametric study with $j \in [30; 90].10^6 A.m^{-2}$ (other μ_i fixed)



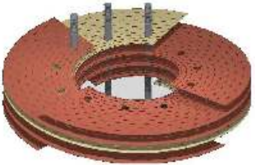
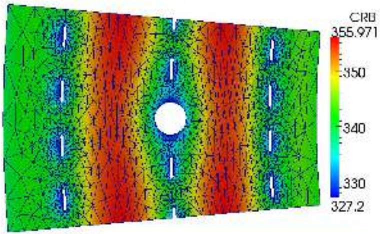


Figure : Bitter magnet



40 °C to 60 °C : + 1 Tesla

Performances

Simulation characteristics

- Number of dofs : $\approx 10^6$
  - 15 processors
  - Number of dofs / proc : ≈ 67000
- Number of inputs : 6
  - σ_0, α, L, U, h, T_w
- Non-Linear model (20 Picard iter.)
- Number od Reduced Basis : 10
- Error FEM / RB :

- FEM time $\approx 20min$
- RB online time $\approx 16sec$

Gain factor : $72\times$

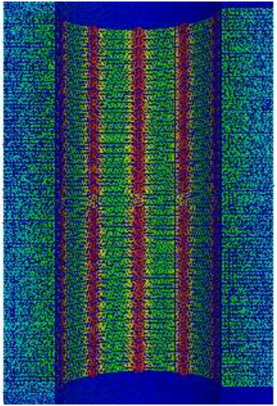


Figure : Helix mesh

Error FEM/CRB

Error estimation is not yet available for non-linear problems

- No error estimations : parameters are chosen randomly

Parameter chosen :

- $\sigma_0 = 50.013 \times 10^6 \ (S.m^{-1})$
- $\alpha = 3.3635 \times 10^{-3} (K^{-1})$
- $L = 2.5065 \times 10^{-8}$
- $U = 71.018(V)$
- $h = 82152(W.m^{-2}.K^{-1})$
- $T_w = 307.21(K)$

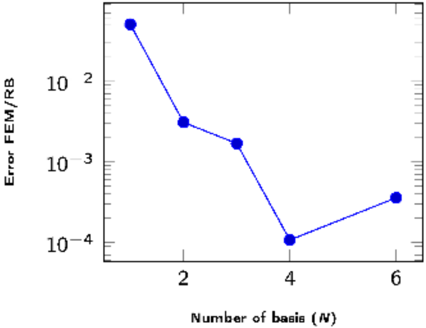
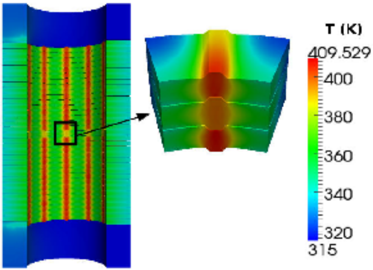


Figure : Error FEM/RB depending on reduced basis space size (N)

Sensitivity Analysis : CRB on helix sector

Sobol indices : $S_i = \dfrac{V(E[Y \mid X_i])}{V(Y)}$

σ_0	:	**0.000068**
α	:	**0.00045**
L	:	**0.0092**
U	:	**0.12**
h	:	**0.24**
T_w	:	**0.62**



Temperature range

Mean of outputs :
$T = 367.8K \approx 95C$
Standard deviation : 6, 2

$\Rightarrow$ Range for T :
$[361.6; 374](K) = [88.6; 101](C)$

Quantiles

Determine a threshold $q(\gamma)$ such that $P(Y < q(\gamma)) > \gamma$

99.0 % : 380 K = 107 C
80.0 % : 377.5 K = 104.5 C

Perspectives

Ongoing work on electro-thermal model

- Continue investigations with large simulations
  - Increase number of basis
  - Analyse convergence (EIM, RB)
  - ...
- Work on error estimation for such a model
  - Error estimation for EIM approximation
  - Dealing with non-linearity

Towards full reduced model

- Add Linear Elasticity model
- Add Magnetostatic model

References

Barrault, M., Maday, Y., Nguyen, N. C., and Patera, A. T. (2004).
An empirical interpolation method : application to efficient reduced-basis discretization of partial differential equations.
Comptes Rendus Mathematique, 339(9) :667–672.