# Optimization under Uncertainty Workshop

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# In order to break the ice ...

An incomplete statement of the « optimization under uncertainty » problematic.

# Uncertainties and optimization : example

Models and working conditions are partially unknown → uncertainties.
These uncertainties need to be taken into account during design.

Ex : a +/- 1mm dispersion in the manufacturing of the air admission line  can degrade the engine's performance (g CO2/km) by +20% (worst case).

# Optimization terminology (1)

Formulation of the deterministic optimization problem

$$\begin{cases} \min\limits_{x \in S} f(x) \\ g(x) \leqslant 0 \end{cases}$$

$x$ : optimization variables
$f$ : objective functions
$g$ : optimization constraints
$f, g$ : optimization (performance) criteria

# Optimization terminology (2) : the double (x,u) parameterization for uncertainties

$x$ is a vector of deterministic optimization (controlled) variables.
$x \in S$ , the search space.

Introduce $u \in U$ , a vector of uncertain parameters that affect the optimization criteria,

$$f(x)\,,\ g(x) \ \rightarrow\ f(x,u)\,,\ g(x,u)$$

*u* used to describe

- aleatoric uncertainties, e.g., noise in measurements … .
- systematic or epistemic uncertainties, e.g., lack of model knowledge.

# Steps in optimization under uncertainties

- **Uncertainty representation.**
- **Formulation of the optimization problem under uncertainty.**
- **Mathematical approximations, numerical solving.**

# Uncertainty representation

• As random variables. But which distribution ? Do we have sufficient data for inferring a distribution ?

• Using possibility or evidence theory (as a solution to lack of data). But uncertainty propagation is costly.

• Using a finite set of well-chosen values (against numerical difficulties and to fall back on deterministic methods). But how to choose these values ?

# Formulation of optimization with uncertainties
## (example using random variables)

Replace the noisy optimization criteria by statistical measures (mean, standard deviations, quantiles, probabilities).

Example :

$OK(x)$ is the random event "all constraints are satisfied" ,

$$OK(x) = \bigcap_i \{ g_i(x, U) \leqslant 0 \}$$

$\min_{x \in S} \; q_\alpha^c(x)$ (conditional $\alpha$-quantile)

such that $P(OK(x)) \geqslant 1 - \varepsilon$

where $P(f(x, U) \leqslant q_\alpha^c(x) \mid OK(x)) = \alpha$

$\varepsilon > 0$ , small

(an « ideal » formulation, aka chance constrained prob., that is never addressed because of the associated computational challenge)

# Formulation of optimization with uncertainties
(example using discrete deterministic variables)

$$U = \{u^1, u^2, \dots, u^n\}$$

$$\min_{x \in S} \left( \sup_{u \in U} f(x,u) \right)$$

such that $g(x,u) \leqslant 0 \quad , \quad \forall u \in U$

how to choose the $u^i$'s in such a worst case formulation (e.g., control the failure probability) ?

# Mathematical approximation, numerical solving

Often, when the optimization criteria are based on complex numerical simulation, the numerical cost of optimization under uncertainty is a bottle neck (curse of the double loop).

Typical approaches :

**Safety factors**

$$u = u^{\text{ref}}$$
$$\min_{x \in S} f\left(x, u^{\text{ref}}\right)$$
$$\text{such that} \quad g\left(x, u^{\text{ref}}\right) + S \leqslant 0 \quad , \quad S > 0$$

**FOSM, FORM, SORM**

$$\min_{x \in S} f\left(x\right)$$
$$\text{such that} \quad distance\left(u^{\text{ref}}, u^{\text{MPF}}\right) > safe\_dist$$
$$\text{where} \quad u^{\text{MPF}} = arg \max_{u \mid \hat{g}(x,u)=0} P\left(u\right)$$

Approaches based on **surrogates** of *f, g*, mean, variance, quantiles, probabilities, ...

# Now : 2 round tables, 2 questions

1) In order to know each other :
what have you done in the past related to optimization under uncertainty ?
( presentations by Younes Aoues and Victor Picheny and everyone )

2) Look ahead :
what are the next challenges and the main research directions in optimization under uncertainty ?