Reliability Importance Measures: From Local to Global

Emanuele Borgonovo

Dept. of Decision Sciences, Bocconi University

Workshop on Statistical methods

for safety and decommissioning

Avignon, November 21-22, 2022

Summary

- Overview of the literature
- Mathematical Foundations
- Structural Importance Measures
- Reliability Importance Measures with Aleatory Uncertainty
- Reliability Importance Measures with Epistemic Uncertainty

Motivation

- Reliability Importance Measures are a central tool in supporting engineering decision making
- They allow us to identify important components in a system under a variety of settings
- Over the years several Reliability Importance Measures developed for various tasks

Some Applications

- *Prioritization* (Birnbaum, 1969)
- *Redundancy Allocation in the Design Phase*
- *Graded Quality Assurance Programs* (NRC, 2002)
- *Maintenance Prioritization* (Nguyen, Do, & Grall, 2017)
- *Remaining Useful Life* (Do and Berenguer, 2022, Zhu et al, 2022)

Risk Analysis

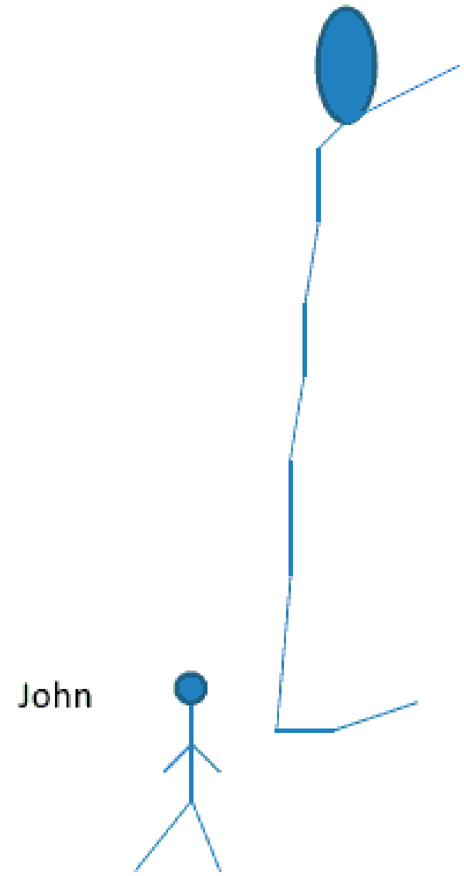


Fig. 1. Boulder example: risk associated with the dislodging of the boulder.

Definitions of Risk

- Several definitions of risk (e.g., Risk equals the expected loss; (2) Risk equals the expected disutility. (3) Risk is a measure of the probability and severity of adverse effects, etc.)
- Kaplan & Garrick 1981

Kaplan and Garrick Risk Triplets

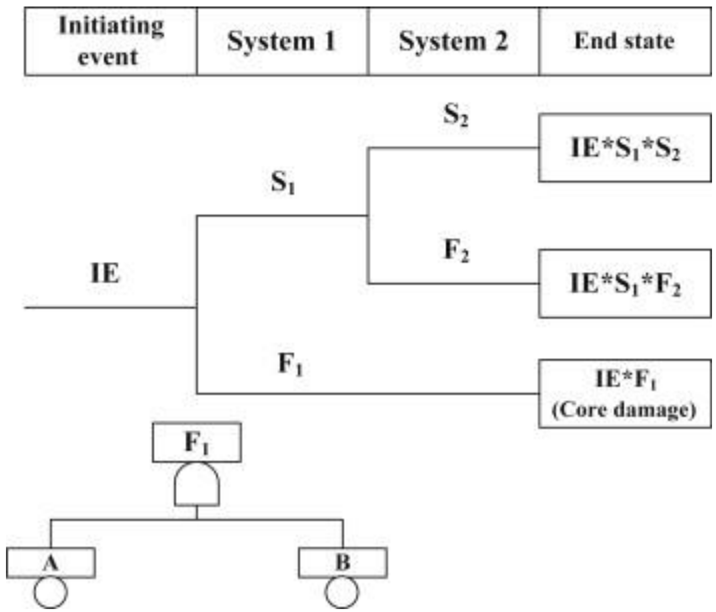
- **Risk** is a triplet of

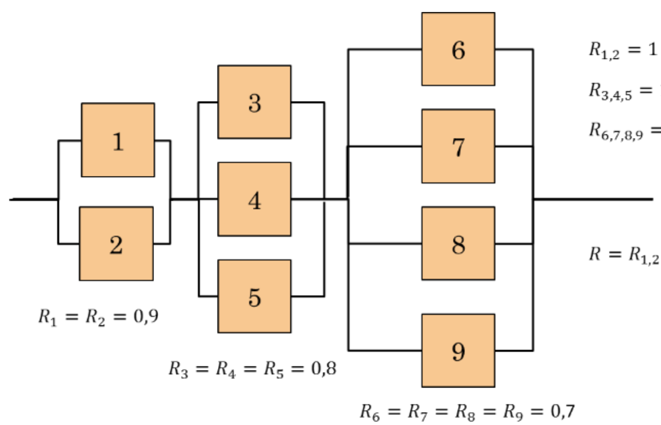
$$R = \{< S_n, l_n, x_n >: n = 1, \ldots, N\}$$

- Scenarios (S_n): What can happen?
- Likelihoods (l_n): How likely is it to happen?
- Consequences (x_n): What is the end result?
- **Hazard:** a set of doublets

$$H = \{< S_n, x_n >: n = 1, \ldots, N\}$$

Probabilistic Safety Assessment





$R_{1,2} = 1 - (1 - 0{,}8)^2 = 0{,}99$

$R_{3,4,5} = 1 - (1 - 0{,}8)^3 = 0{,}992$

$R_{6,7,8,9} = 1 - (1 - 0{,}7)^4 = 0{,}9919$

$R = R_{1,2} \cdot R_{3,4,5} \cdot R_{6,7,8,9} = 0{,}974$

Mathematical Foundations

- R. E. BARLOW, J. B. FUSSELL, AND N. D. SINGPURWALLA, EDS. "Reliability and Fault Tree Analysis," Society for Industrial and Applied Mathematics, Philadelphia, PA, 1977
- N. D. Singpurwalla, 1988: "FOUNDATIONAL ISSUES IN RELIABILITY AND RISK ANALYSIS", Siam Review, Vol. 30, No. 2, June 1988

The Boolean Background

- Structure function

$$\Psi = \Psi(\phi_1, \phi_2, ..., \phi_n) = \Psi(\phi)$$

- where

$$\phi_i = \begin{cases} 1 & \text{component i has failed} \\ 0 & \text{component i is working correctly} \end{cases}$$

- Coherent system: the structure function is increasing

$$\Psi(0) = 0, \Psi(1) = 1,$$

$$\Psi(\phi) \leq \Psi(\alpha) \text{ if } \phi \leq \alpha$$

Birnbaum Relevance

- Birnbaum (1969, p. 583-584):

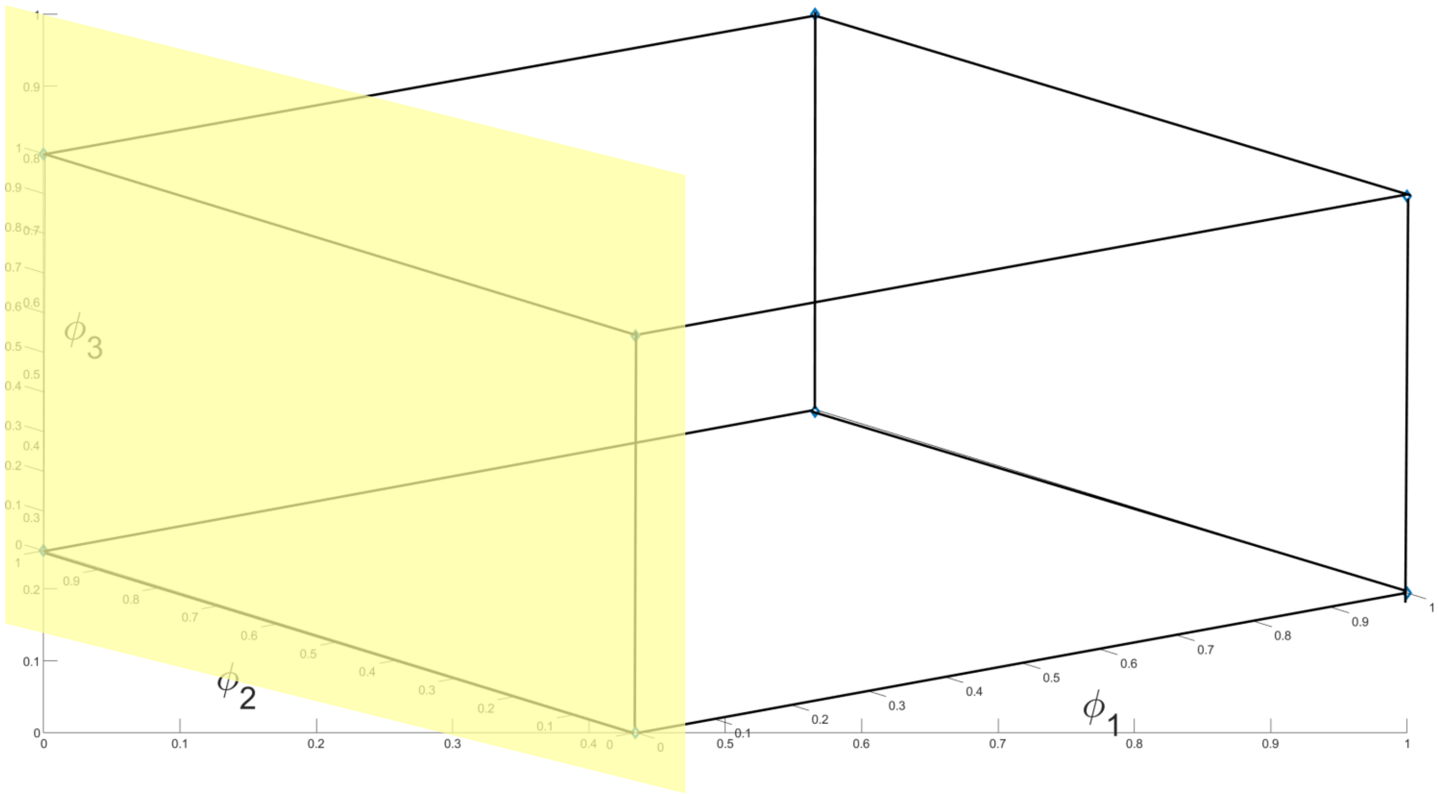
$$\delta_i = \boxed{\Psi(1_i, \phi_{-j})} - \boxed{\Psi(0_j, \phi_{-j})}$$

- Structural Importance

$$I_j^{\text{Str}} = \frac{\sum_x \delta_j(x)}{2^n}$$

- Further studied in several works, such as Meng (2000)

A visual representation



Bocconi

Aleatory Uncertainty and Conditioning

System Reliability

- The system reliability at time t is the probability that the system has not failed at time t:

$$R(t) = \Pr(\Psi = 0; t)$$

- Considering the failure probability

$$F(t) = \Pr(\Psi = 1; t)$$

- We have

$$R(t) = 1 - F(t)$$

Properties

- Let $p_i(t) = P(x_i$=1;t) be the (conditional) failure probability of component i
- Then R(t) is a multilinear function of $p_i(t)$ for both coherent and non coherent systems

$$R(t) = \sum_{k=1}^{n} \alpha_i p_i(k) + \sum_{i_1<i_2}^{n} \alpha_{i_1,i_2} p_{i_1}(t) p_{i_2}(t) + \ldots$$

with dependent and independent failures (B., 2010)

Conditioning and Criticality

- Component i is critical for the system if the system is in such a state that the change in state of the component causes the system to tail
- Coherent system: only changes from working to failed can cause the system to fail
- Non-coherent system: both ways
- To calculate criticality, we condition on the component being "up" or "down"

The Birnbaum Importance

- Re-consider the expression

$$\delta_i(t) = \Psi(1_i, x_{-j}; t) - \Psi(0_j, x_{-j}; t)$$

- $\delta_i(t) = 1$ the component i is critical at time t if the system is coherent
- Then, Birnbaum (1969) defines

$$B_i(t) = P[\delta_i(t) = 1] = P[\phi(1_i, x; t)] - P[\phi(0_i, x; t)]$$

- B_i(t) is the Birnbaum importance measure

Risk Achievement Worth

- Risk Achievement Worth (Vesely et al, 1983)

$$RAW_i(t) = \frac{\Pr(\Psi(1_i, \phi_{-i}) = 1; t)}{\Pr(\Psi = 1; t)} = \frac{F_{\Phi|X_i=1}(t)}{F_\Phi(t)}$$

conditional risk metric given that component i has failed

Risk Reduction Worth

- Risk Reduction Worth

$$RRW_i(t) = \frac{\Pr(\Psi = 1; t)}{\Pr(\Psi(0_i, x) = 1; t)} = \frac{F_\Phi(t)}{F_{\Phi|X_i=0}(t)}$$

- Conditional risk metric given that component i is always working
- Probabilistic Relationship

$$1 = p_i RAW_i + \frac{1 - p_i}{RRW_i}$$

Fussell-Vesely

- Let

$$Q(i) = \bigvee_{m:\varphi_i \in C_m} Q_m$$

be the union of the Min Cut Sets containing component i

- The Fussell-Vesely importance is defined as

$$FV_i = P(Q(i) = 1 | \Psi = 1)$$

- It can be shown that

$$FV_i \simeq \frac{P(\Psi) - P(\Psi_i^-)}{P(\Psi)}$$

Bocconi

A summary of Relationships

Table 1: Relationships between Risk Importance Measures. Proofs in Appendix A.

	RAW_i	RRW_i	FV_i	B_i
RAW_i	$-$	$(1-\frac{1-p_i}{RRW_i})\frac{1}{p_i}$	$1+\frac{(1-p_i)FV_i}{p_i}$	$1+\frac{B_i-\Delta\Psi_i^-}{P(\Psi)}$
RRW_i	$\frac{1-p_i}{1-RAW_i p_i}$	$-$	$\frac{1}{1-FV_i}$	$\frac{P(\Psi)}{P(\Psi_i^+)-B_i}$
FV_i	$\frac{p_i}{1-p_i}(RAW_i-1)$	$1-\frac{1}{RRW_i}$	$-$	$\frac{B_i}{P(\Psi)}p_i$
B_i	$\frac{P(\Psi)}{1-p_i}(RAW_i-1)$	$\frac{P(\Psi)}{p_i}(1-\frac{1}{RRW_i})$	$\frac{P(\Psi)}{p_i}FV_i$	$-$

Two Observations

- For coherent systems with iid failures

$$B_i(t) = \frac{\partial U(t)}{\partial p_i}$$

Thus, for a coherent system criticality and differentiation coincide

DETOUR ON IMPORTANCE MEASURES BASED ON DERIVATIVES

Literature Review

Table 2
Synthesis of the literature review on works concerning joint and differential reliability importance.

Work	Importance measure	System type	Interaction order
Birnbaum (1969)	B	Coherent	1
Hong and Lie (1993)	J^{II}	Coherent	2
Armstrong (1995)	J^{II}	Coherent	2
Borgonovo and Apostolakis (2001)	D	Coherent	1
Andrews and Beeson (2003)	B	Non-coherent	1
Zio and Podofillini (2006)	D^{II}	coherent	2
Lu and Jiang (2007)	J^{II}	Non-coherent	2
Gao et al. (2007)	J^{k}	Coherent	k
Do Van et al. (2008)	D^{k}	Coherent	k

The Criticality and Joint Importance Measures

- Criticality Importance Measure (Cheok et al, 1998)

$$C_i(t) = \frac{\partial U(t)}{\partial p_i} \frac{p_i(t)}{U(t)} = B_i(t) \frac{p_i(t)}{U(t)}$$

- Joint Reliability Importance (Hong & Li, 1993)

$$J_{i,m}^2(t) = \frac{\partial^2 U(t)}{\partial p_i \partial p_m}$$

How the Birnbaum importance of component I changes as the importance of component k changes

The Differential Importance Measure

$$D_i(t) = \frac{\frac{\partial U(t)}{\partial p_i} dp_i}{\sum_{j=1}^{n} \frac{\partial U(t)}{\partial p_j} dp_j}$$

- Fraction of the differential of the risk metric associated with a perturbation in the failure probability of component i

(Borgonovo and Apostolakis, 2001)

DIM Properties

- Additivity:

$$D_{i,j,\ldots,k}(t) = D_i(t) + D_j(t) + \ldots + D_k(t)$$

- Relationship to Birnbaum and Criticality
- Uniform perturbations in the p's Differential Importance coincides with Birnbaum
- Proportional Perturbations Differential importance coincides with Criticality

Considering Interactions

- Total Order Reliability Importance:

$$D_l^T := \frac{\Delta^T G_l}{\Delta G} = \frac{B_l \Delta x_l + \sum_{k=2}^{T} \sum_{\substack{i_1 < i_2, \ldots, < i_k \\ l \in i_1 < i_2, \ldots, < i_k}} J^k_{i_1, i_2, \ldots, i_k}(\mathbf{x}^0) \prod_{s=1}^{k} \Delta x_{i_s}}{\sum_{i=1}^{N} B_i(\mathbf{x}^0) \cdot \Delta x_i + \sum_{k=2}^{T} \sum_{i_1 < i_2, \ldots, < i_k} J^k_{i_1, i_2, \ldots, i_k}(\mathbf{x}^0) \prod_{s=1}^{k} \Delta x_{i_s}}$$

Includes the Birnbaum and all the joint reliability importance measures of all orders

Can be efficiently

- Borgonovo 2010, EJOR

Bocconi

END OF DETOUR

BIRNBAUM AND NON-COHERENT SYSTEMS

Two Extensions

- Andrews and Beeson (2003) extend the Birnbaum importance as

$$I_i^{AB}(t) = \frac{\partial U(t)}{\partial q_i} + \frac{\partial U(t)}{\partial p_i}$$

- q_i is the probability of component success
- Vaurio (2016)

$$I_i^{AB}(t) = \frac{\partial U(t)}{\partial q_i} - \frac{\partial U(t)}{\partial p_i}$$

Bocconi

Boolean Expression for Criticality

Definition 1. *We say that component/basic event i is*

(1) failure-critical *if the system is in such a state that* $\Psi_i^A = 1$, $\Psi_i^B = 0$, *and* $\Psi_i^C = 0$

(2) repair-critical *if the system is in such a state that* $\Psi_i^A = 0$, $\Psi_i^B = 1$, *and* $\Psi_i^C = 0$.

The above definition gives rise to the following Boolean expression (Aliee, B., Glass, Teich, 2017):

Definition 2. *The Boolean variable*

$$\Psi_i^{ABGT} := \left[\Psi(1_i, \varphi_{\sim i}) \wedge \overline{\Psi}(0_i, \varphi_{\sim i})\right] \vee \left[\overline{\Psi}(1_i, \varphi_{\sim i}) \wedge \Psi(0_i, \varphi_{\sim i})\right]$$

is called the criticality indicator *variable of component i.*

Bocconi

Birnbaum Importance For Coherent and Non Coherent Systems

$$B_i := \Pr[\Psi_i^{ABGT} = 1]$$

- The Birnbaum importance is then the probability that component i is critical for system failure when working or when failed

Main Implication

- For non coherent systems, derivatives can be negative
- Therefore the probability of a component being critical is no more equal to the partial derivative of the system with respect to the probability of component i

TIME INDEPENDENT IMPORTANCE MEASURES

Barlow-Proschan Importance

- For a coherent system, the B-P importance (Barlow and Proschan, 1975):

$$I_i^{BP} = \int_0^{\infty} \Pr[\Psi(1_i, \boldsymbol{\varphi}_{\sim i}) - \Psi(0_i, \boldsymbol{\varphi}_{\sim i})] dF_i(t)$$

- The probability that component i is critical, independently of time.

Other Time Independent Importance measures

- Lambert's Enabler importance measure (Lambert 1975)
- Natvig's importance measure (1979)
- A general definition by Xie (1987)

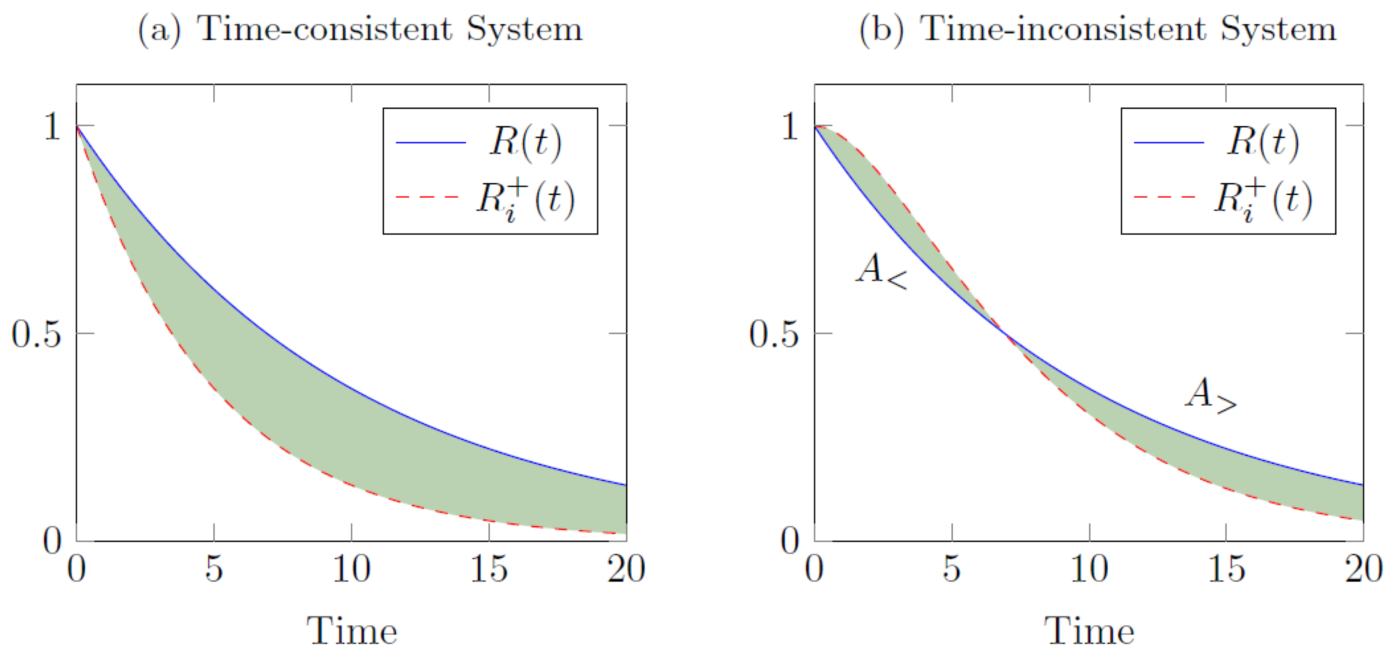
$$X_i = \int_0^{\infty} Y'(t) B_i(t) dF_i(t)$$

- Where Y(t) is a differentiable function.
- For instance, if $Y(t) = t^r, r \geq 0$
- For $r = 0$ we have the Barlow-Proshan, for $r = 1$, we have the importance measure of Bergman (1985).

THE NOTION OF TIME CONSISTENCY

Conditional Failure

- We expect that if a component is always failed, then the probability of the system working given that i has failed is always greater than the original probability



Time Consistency

- A system is time consistent for component failure if F$(t) \geq F_i(t|\phi_i = 1)$ for every t and for all components.
- Implication: if a system is not time consistent, there is one or more times after which if we do not perform repair, we are better off.

MTTF Importance

$$I_i^{BAGT} = \left| MTTF - MTTF_i \right| = \left| \mathrm{E}[T] - \mathrm{E}[T \mid \phi_i = 1] \right|$$

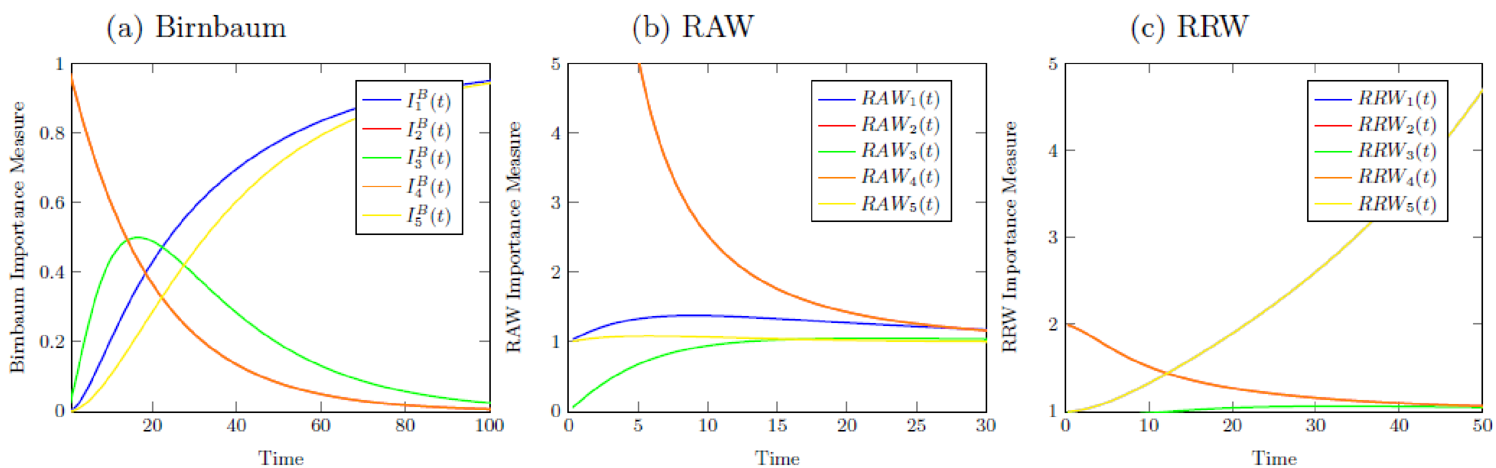
- Difference between MTTF and MTTF given that component i has failed
- The most important component is the one that creates the greatest shift in MTTF

A relevant result

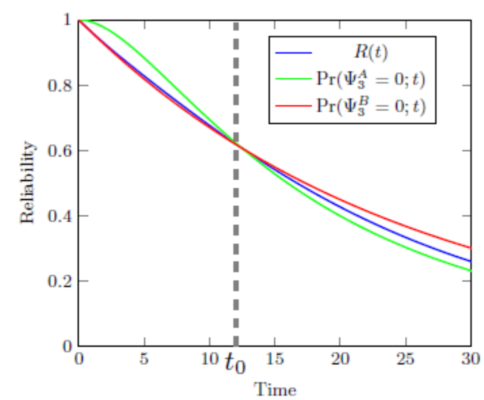
- A system is time consistent with respect to component i if and only if

$$I_i^{BAGT} = | MTTF - MTTF_i | = \int_0^{\infty} | F(t \mid \phi_i = 1) - F(t) | \, dt$$

Example of a non-time consistent system



$$\Psi = (\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3) \vee (\varphi_2 \wedge \overline{\varphi}_3)$$



Borgonovo, Aliee, Glass Teich (2016)

Extensions

- Importance Measures for Multistate Systems; works of Ramirez-Marquez, Coit, Natvig, Huseby.
- Importance Measures in Repairable and Non-Repairable Systems, Works of Natvig & Gasemyr, etc..
- Importance Measures for Thresholds linking value of information and importance measures (Borgonovo and Cillo, 2016)
- … several others

IMPORTANCE MEASURES AND EPISTEMIC UNCERTAINTY

How uncertainty affects IM Ranking

- Works has been done starting from Lambert (1975), Modarres and Aggarwal (1996) to account for the effect of epistemic uncertainty in importance measure ranking.
- A set of approaches is covered in Borgonovo (2008)
- Borgonovo and Smith (2015) introduce the epistemic risk achievement worth (ERAW)

Global Sensitivity Measures

- Epistemic Uncertainty can be addressed also using global sensitivity measures
- A variety of techniques, from variance-based (Homma and Saltelli 1996) to moment independent (Borgonovo 2007)

More General Settings

- Risk Metric Y
- Uncertainty in the paramters $X_1, X_2, \ldots$

$$Y = g(X_1, X_2, \ldots, X_n)$$

- $F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ probability distribution of X.
- Uncertainty propagates from X to Y (usually via Monte Carlo simulation).

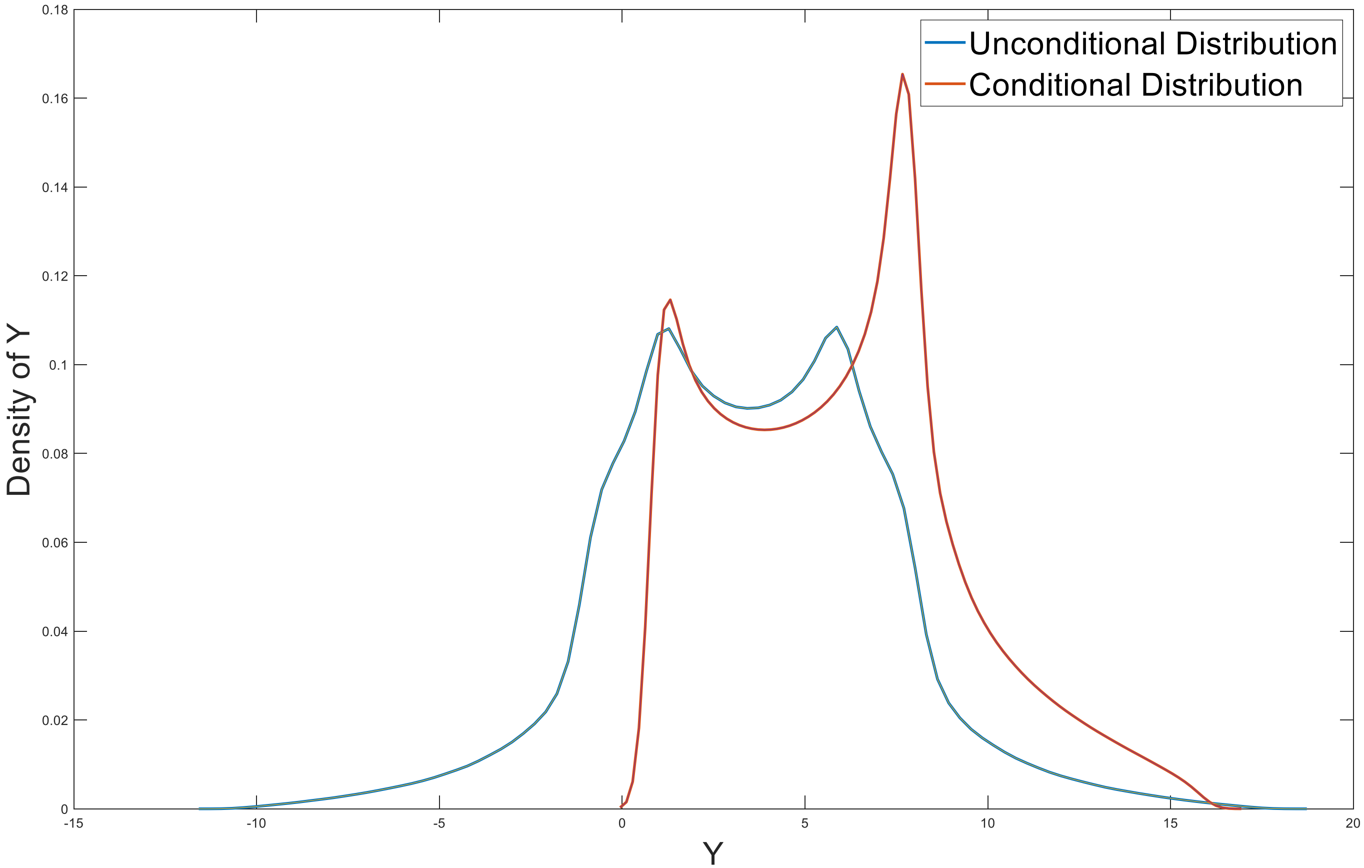
An Intuition

- Conditioning on the value of a parameter

- Marginal distribution: $F_Y(y)$

- Conditional Distribution: $F_{Y|X_i}(y)$

A Visualization



Bocconi

Università Commerciale Luigi Bocconi

Common Rationale

A General Framework:

$$\xi_X = E\left[d\left(F_Y, F_{Y|X_i}\right)\right]$$

where $d(\cdot,\cdot)$ is meant to accommodate several sensitivity measures.

Global Importance Measures

If we set $d(\cdot,\cdot)$ to be the L1 norm between densities,

$$d\left(F_Y, F_{Y|X_i}\right) = \int |f_{Y(y)} - f_{Y|X_i}(y, x_i)|dy$$

we find the delta importance measure (B. 2007):

$$\delta_i = \frac{1}{2} E_{X_i}[\int |f_{Y(y)} - f_{Y|X_i}(y, x_i)|dy$$

And Several Others

- Gamboa et al (2018), based on the Cramer von Mises Distance between Distributions
- Chatterjee (2020)
- Wiesel (2022), B. et al (2022) based on the Wasserstein distance (Optimal Transport)

Multiple Risk Metrics

- Loss of crew or loss of mission are two criteria simultaneously of interest
- Reliability and Cost are also two conflicting criteria
- Case a): they can be combined in a unique objective function
- Case b): we cannot combine them

Global Sensitivity Analysis for Multivariate Output

- Extending the framework to multivariate responses
- Several works by Da Veiga, Iooss, Lamboni, Marrel and others (no time to review all of them)

Recent Approaches based on Optimal-Transport-Theory

- Wasserstein Distance:

$$W_p(\nu,\nu') = \inf_{\pi\in\Pi(\nu,\nu')} \int |y-y'|^p \,\mathrm{d}\pi(y,y')$$

- Corresponding global sensitivity measure (B., Figalli, et al. 2022):

$$\xi^{\mathrm{Wp}}(Y,X) = \mathrm{E}\left[W_p(\mathrm{P}_Y,\mathrm{P}_{Y|X})\right]$$

Properties

- Zero-Independence
  - It is null if and only if Y is independent of X
- Max-Functionality
  - It is maximal if and only if learning Y removes uncertainty in X completely
- Monotonicity for information refinements
  - For the same X, if information is less refined then the value of the importance measure of X is smaller than if we have more refined information

And Value of Information

Let $L(y, a)$ a loss function where Y is a random variable and a is an alternative belonging to a set of alternatives A. The decision problem is to solve

$$\max_{a \in A} E[L(Y, a)]$$

Optimal choice: a^* such that

$$a^* = argmax_{a \in A} E[L(Y, a)]$$

Value of Information

The expected value of information about an uncertainty X is given by:

$$\varepsilon_X = E_X[d(x)]$$

Where

$$\mathrm{d}(x) = \max_{a \in \mathrm{A}} E_Y[L(Y,a)|X = x] - E[L(Y,a^*)]$$

which is information gain for getting to know X (conditioning on X=x).

Thank you for your attention!

References (1)

Barlow, R. E., & Proschan, F (1975). Importance of System Components and Fault Tree Events. *Stochastic Processes with Their Applications*, *3*, 153–173.

Birnbaum, L. W. (1969). On the importance of different elements in a multielement system. *Multivariate Analysis, New York Academic Press*, *2*, 1–15.

Borgonovo, E. (2007). A new uncertainty importance measure. *Reliability Engineering and System Safety*, *92*(6), 771–784.

Borgonovo, E. (2008). Epistemic uncertainty in the ranking and categorization of probabilistic safety assessment model elements: issues and findings. *Risk Analysis : An Official Publication of the Society for Risk Analysis*, *28*(4), 983–1001.

Borgonovo, E., & Smith, C. L. (2012). Composite multilinearity, epistemic uncertainty and risk achievement worth. *European Journal of Operational Research*, *222*(2), 301–311.

Borgonovo, E. (2010). The reliability importance of components and prime implicants in coherent and non-coherent systems including total-order interactions. *European Journal of Operational Research*, *204*(3), 485–495.

References (2)

Borgonovo, E., & Cillo, A. (2016). Deciding with Thresholds: Importance Measures and Value of Information. *Risk Analysis, to Appear*.

Borgonovo E., Figalli A., Plischke E. and Savarè G.: "Global Sensitivity Analysis with Optimal Transport", 2022, submitted.

Cheok, M. C., Parry, G. W., & Sherry, R. R. (1998). Use of importance measures in risk-informed regulatory applications. *Reliability Engineering & System Safety*, *60*(3), 213–226.

Homma, T., & Saltelli, A. (1996). Importance Measures in Global Sensitivity Analysis of Nonlinear Models. *Reliability Engineering & System Safety*, *52*(1), 1–17.

Kuo, W., & Zhu, X. (2012a). *Importance Measures in Reliability, Risk and Optimization*. Wiley and Sons, UK.

Kuo, W., & Zhu, X. (2012b). Some Recent Advances on Importance Measures in Reliability. *IEEE Transactions on Reliability*, *61*(2), 344–360.

Lambert, H. E. (1975). Measures of Importance of Events and Cut Sets in Fault Trees. In *Reliability and Fault Tree Analysis, Theoretical and Applied Aspects of System Reliability and Safety Assessment* (pp. 77–100). SIAM, Philadelphia.

Modarres, M., & Agarwal, M. (1996). Consideration of Probabilistic uncertainty in Risk-Based Importance Measures. In *Proceedings of PSA `96*. Park City, Utah: American Nuclear Society, La Grange Park, Illinois.

Natvig, B. (1979). A suggestion of a New Measure of Component Importance. *Stocastic Processes and Their Applications*, *9*, 319–330.

Natvig, B., Huseby, A.B., Reistadbakk, M.O. Measures of component importance in repairable multistate systems - A numerical study (2011) Reliability Engineering and System Safety, 96 (12), pp. 1680-1690

References (3)

Nguyen, K.-A., Do, P., & Grall, A. (2017). Joint predictive maintenance and inventory strategy for multi-component systems using Birnbaum's structural importance. *Reliability Engineering & System Safety*, *forthcomin*.

NRC. (2002). *Regulatory Guide 1174: An Approach for Using Probabilistic Risk Assessment in Risk-Informed Decisions on Plant-Specific Changes to the Licensing Basis*. Retrieved from https://www.nrc.gov/reading-rm/doc-collections/reg-guides/

Peng, H., Coit, D. W., & Feng, Q. (2012). Component reliability criticality or importance measures for systems with degrading components. *IEEE Transactions on Reliability*, *61*(1), 4–12.

Ramirez-Marquez, J. E., & Coit, D. W. (2005). Composite importance measures for multi-state systems with multi-state components. *IEEE Transactions on Reliability*, *54*(3), 517–529.

Shrestha, A., Xing, L., & Coit, D. W. (2009). Multi-state component importance analysis using multi-state multi-valued decision diagrams. In *Proceedings of 2009 8th International Conference on Reliability, Maintainability and Safety, ICRMS 2009* (pp. 99–103).

Singpurwalla, N. D. (1988). Foundtional Issues in Reliability and Risk Analysis. *SIAM Review*, *30*(2), 264–282. https://doi.org/10.1126/science.151.3712.859-b

Xie, M. (1987). Stochastic Processes and their Applications. *Stochastic Processes and Their Applications*, *25*, 273–280.

Vaurio, J. K. (2016). Importances of components and events in non-coherent systems and risk models. *Reliability Engineering and System Safety*, *147*, 117–122.