Challenges of the Emissions Markets

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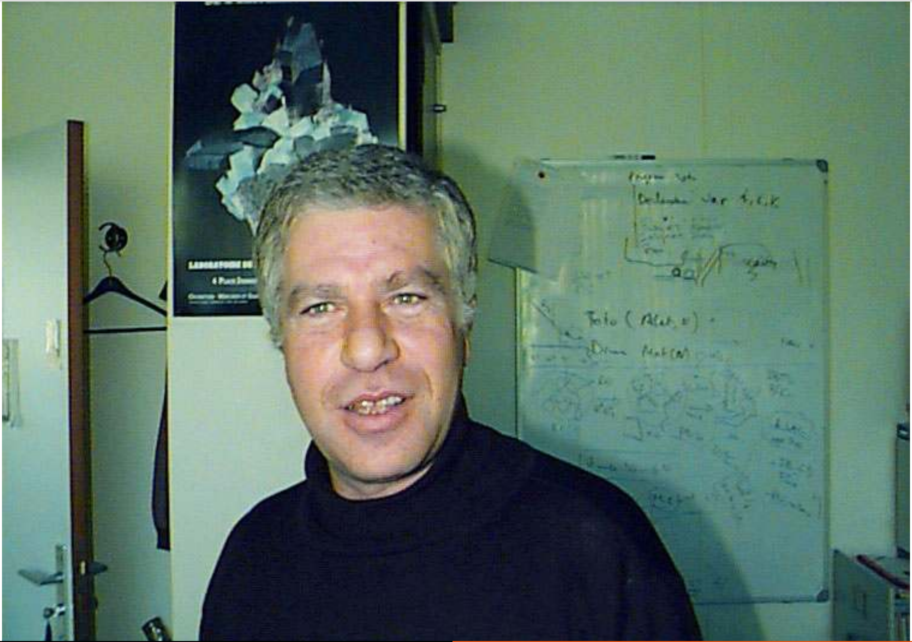
Villard de Lans, March 25, 2011

Quantification of cumulated physical fatigue at the workplace



Cap-and-Trade Schemes for Emission Control

- **Cap & Trade Schemes for CO_2 Emissions**
  - Kyoto Protocol
  - Mandatory Carbon Markets **(EU ETS, RGGI since 01/01/09)**
  - Lessons learned from the EU Experience
- **What Can we Learn from Mathematical (Equilibrium) Models**
  - **Joint Price Formation** for Goods and Emission Allowances
  - New **Designs** and Alternative Schemes
  - **Horizon** / **Time Scale Mismatch**
    - Long Term Emission Targets (2020, 2030, 2050)
    - Short Term Regulations (e.g. Kyoto 2008 – 2012)
- **Immaturity of the Markets**
  - **Option** Data

Goal of the Study

- Putting a Price on
  - CO_2 by **internalizing** its Social Cost
  - Goods whose Productions lead to **Emissions**
- Regulatory Economic Instruments
  - Carbon **TAX**
  - Permits Allocation & Trading (**Cap-and-Trade**)
- Calibrate the Different Schemes for
  - **MEANINGFUL** & **FAIR** comparisons

Equilibrium Analysis

- **Dynamic Stochastic General Equilibrium**
- **Inelastic** Demand
  - Electricity Production for the purpose of **illustration**
  - **Same** results in **multi-good** Markets
- **Random** Factors
  - Demands for goods $\{D_t^k\}_{t\geq 0}$
  - **Costs** of Production $\{C_t^{i,j,k}\}_{t\geq 0}$
    - Spot Price of Coal
    - Spot Price of Natural Gas

Case Study: **Japan**

TOKYO unveiled a **Carbon Scheme**

Japanese Electricity Market:

- Eastern & Western Regions (1GW Interconnection)
- Electricity Production: Nuclear, **Coal, Natural Gas**, Oil
  - Coal is **expensive**
  - Visible Impact of Regulation (**fuel switch**)
- **Regulation** Gory Details
  - **Cap** (Emission Target) 300 Mega-ton CO_2 = 20% w.r.t. 2012 BAU
  - Calibration for Fair Comparisons: **Meet Cap** 95% **of time**
    - Penalty 100 USD
    - Tax Level 40 USD
  - Numerical Solution of a **Stochastic Control** Problem (**HJB**) in 4-D

Comparisons

Economic Statics to be Compared

- **Actual Emissions**
- **Reduction (Abatment) Costs**
- **Social Costs**
- **Windfall Profits**

Controls to be Varied

- **Penalty**
- **Tax**
- **Allocation Mechanisms**
  - Free Initial Allocation
  - Auctions
  - Dynamic Proportional Allocation
  - Hybrid Allocation Schemes
  - . . . .

Description of the Economy

- **Finite set** $\mathcal{I}$ of **risk neutral firms**
- **Producing a finite set** $\mathcal{K}$ of **goods**
- Firm $i \in \mathcal{I}$ can use **technology** $j \in \mathcal{J}^{i,k}$ to produce good $k \in \mathcal{K}$
- **Discrete time** $\{0, 1, \cdots, T\}$
- **No Discounting** Work with T-Forward Prices
- **Inelastic Demand**

$$\{D^k(t);\ t = 0, 1, \cdots, T-1,\ k \in \mathcal{K}\}.$$

- . . . .

Regulator Input (EU ETS)

At inception of program (i.e. time $t = 0$)

- **INITIAL DISTRIBUTION** of θ_0 **allowance certificates**

$$\theta_0 = \sum_{i \in \mathcal{I}} \theta_0^i, \qquad \theta_0^i \text{ to firm } i \in \mathcal{I}.$$

- Set **PENALTY** π for emission unit **NOT** offset by allowance certificate at end of **compliance period**

Extensions postponed for later discussions.

- **Risk aversion** and agent preferences (existence theory easy)
- **Elastic** demand (e.g. smart meters for electricity)
- **Investments in new technologies** (wind, solar, CCS,...)
- ....

Goal of Equilibrium Analysis

Find **stochastic processes**

- **Price of one allowance**

$$A = \{A_t\}_{t \geq 0}$$

- **Prices of goods**

$$S = \{S_t^k\}_{k \in K,\, t \geq 0}$$

satisfying the usual conditions for the existence of a

competitive equilibrium

(to be spelled out below) and study the fine properties of these processes.

Individual Firm Problem

During each time period $[t, t+1)$

- Firm $i \in \mathcal{I}$ **produces** $\xi_t^{i,j,k}$ of good $k \in \mathcal{K}$ with technology $j \in \mathcal{J}^{i,k}$
- Firm $i \in \mathcal{I}$ **holds** a position θ_t^i in emission credits

$$\begin{aligned} L^{A,S,i}(\theta^i, \xi^i) := & \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}^{i,k}} \sum_{t=0}^{T-1} (S_t^k - C_t^{i,j,k}) \xi_t^{i,j,k} \\ & + \theta_0^i A_0 + \sum_{t=0}^{T-1} \theta_{t+1}^i (A_{t+1} - A_t) - \theta_{T+1}^i A_T \\ & - \pi(\Gamma^i + \Pi^i(\xi^i) - \theta_{T+1}^i)^+ \end{aligned}$$

where

$$\Gamma^i \text{ random}, \qquad \Pi^i(\xi^i) := \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}^{i,k}} \sum_{t=0}^{T-1} e^{i,j,k} \xi_t^{i,j,k}$$

Random Inputs

- Γ^i uncontrolled emissions
- $C_t^{i,j,k}$ costs of productions (e.g. fuel prices)

Individual Firm Problem (cont.)

Problem for (risk neutral) firm $i \in I$

$$\max_{(\theta^i,\xi^i)} \mathbb{E}\{L^{A,S,i}(\theta^i,\xi^i)\}$$

Choose

- Production strategy ξ^i
- Trading strategy θ^i

in order to

- Maximize its own **expected P&L**
- Satisfy the demand

Equilibrium Definition for Emissions Market

The processes $A^* = \{A_t^*\}_{t=0,1,\cdots,T}$ and $S^* = \{S_t^*\}_{t=0,1,\cdots,T}$ form an equilibrium if for each agent $i \in \mathcal{I}$ there exist strategies $\theta^{*i} = \{\theta_t^{*i}\}_{t=0,1,\cdots,T}$ (**trading**) and $\xi^{*i} = \{\xi_t^{*i}\}_{t=0,1,\cdots,T}$ (**production**)

- **(i) All financial positions are in constant net supply**

$$\sum_{i \in I} \theta_t^{*i} = \sum_{i \in I} \theta_0^i, \qquad \forall\ t = 0, \dots, T+1$$

- **(ii) Supply meets Demand**

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}^{i,k}} \xi_t^{*i,j,k} = D_t^k, \qquad \forall\ k \in \mathcal{K},\ \ t = 0, \dots, T-1$$

- **(iii) Each agent $i \in I$ is satisfied by its own strategy**

$$\mathbb{E}[L^{A^*,S^*,i}(\theta^{*i}, \xi^{*i})] \geq \mathbb{E}[L^{A^*,S^*,i}(\theta^i, \xi^i)] \qquad \text{for all } (\theta^i, \xi^i)$$

Business As Usual (i.e. $\pi = 0$)

The corresponding prices of the goods are

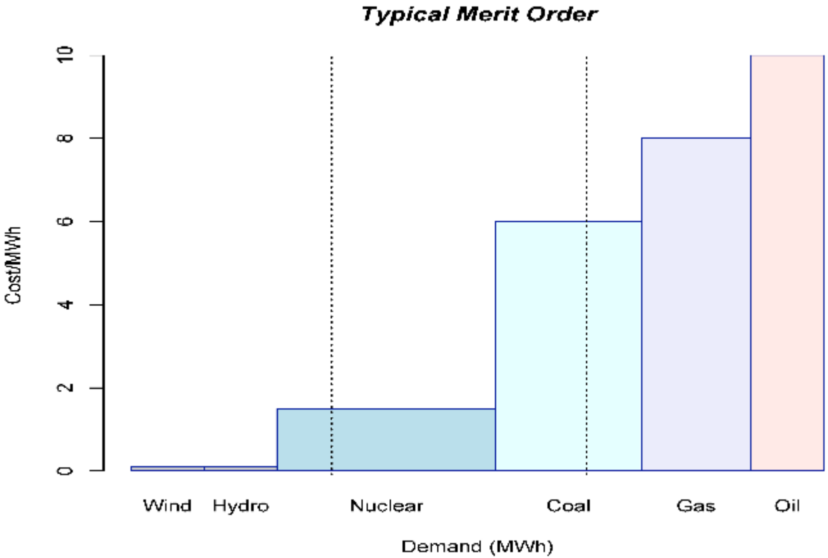
$$S_t^{*k} = \max_{i \in \mathcal{I},\, j \in \mathcal{J}^{i,k}} C_t^{i,j,k} \mathbf{1}_{\{\xi_t^{*i,j,k} > 0\}},$$

Classical **MERIT ORDER**

- At each time t and for each good k
- Production technologies ranked by increasing production costs $C_t^{i,j,k}$
- Demand D_t^k met by producing from the cheapest technology first
- Equilibrium spot price is the marginal cost of production of the most expansive production technoligy used to meet demand

Business As Usual
(typical scenario in Deregulated **electricity markets**)

Example of a Classical Merit Order Plot



Necessary Conditions

Assume

- (A^*, S^*) is an equilibrium
- (θ^{*i}, ξ^{*i}) optimal strategy of agent $i \in I$

then

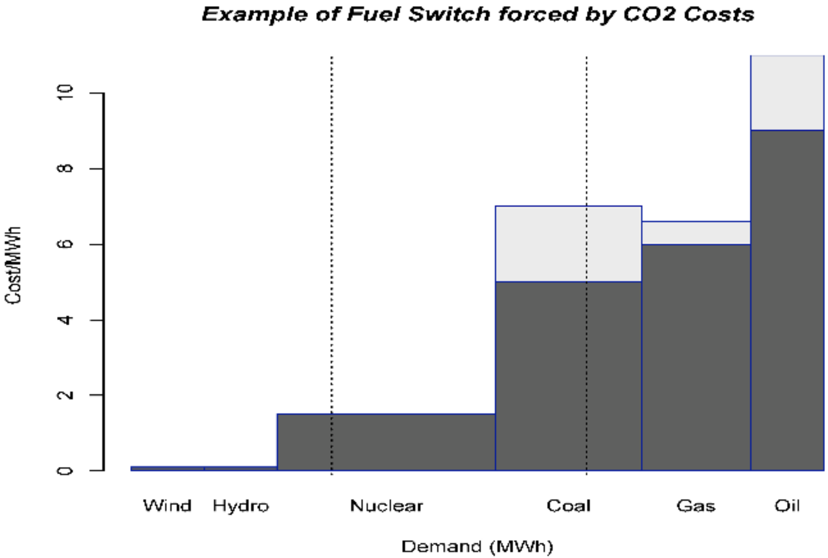
- The allowance price A^* is a **bounded martingale** in $[0, \pi]$
- Its terminal value is given by

$$A^*_T = \pi \mathbf{1}_{\{\Gamma^i + \Pi(\xi^{*i}) - \theta^{*i}_{T+1} \geq 0\}} = \pi \mathbf{1}_{\{\sum_{i \in \mathcal{I}} (\Gamma^i + \Pi(\xi^{*i}) - \theta^{*i}_0) \geq 0\}}$$

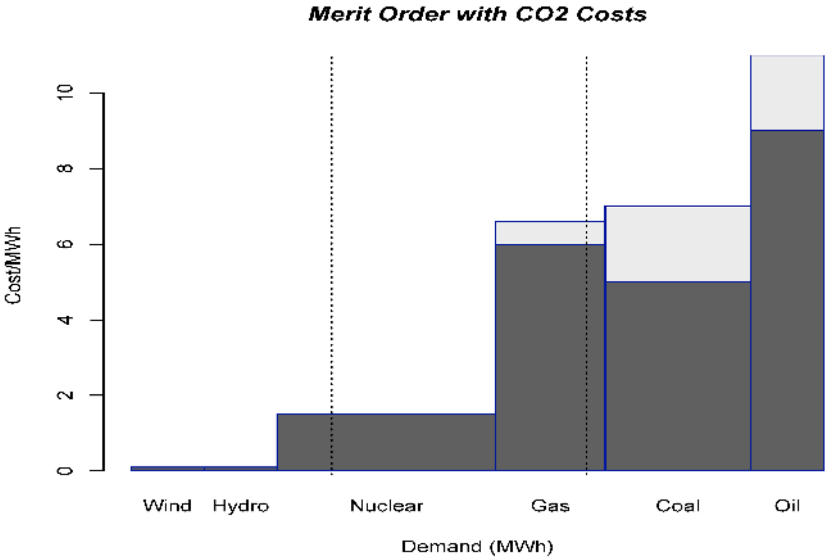
- The **spot prices** S^{*k} of the goods and the **optimal production strategies** ξ^{*i} are given by the **merit order** for the equilibrium with **adjusted costs**

$$\tilde{C}^{i,j,k}_t = C^{i,j,k}_t + e^{i,j,k} A^*_t$$

Example of a Fuel Switch forced by Regulation



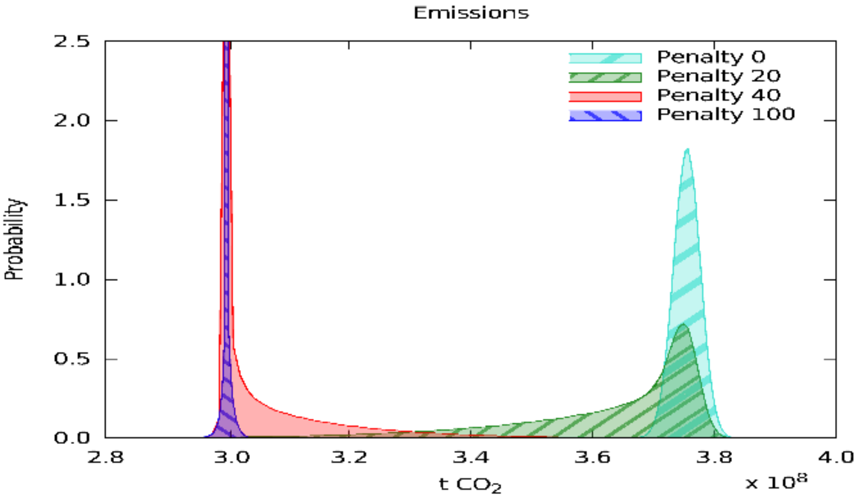
Example of a Merit Order Plot Including CO_2



Impact of the Penalty

- Trial Phase of EU ETS (2005 - 2007): 40 **Euros**
- First Phase of EU ETS (2008 - 2012): 100 **Euros**
- RGGI: Market Participants ***do not really pay attention***
- Option Data show Market Participants **DO NOT BELIEVE** the market will **EVER BE SHORT**
  - Influx of CERs
  - Hot Air (Russia, Poland .... excess allocation)
  - Lobbying & Political Pressure to put FLOORs and CIELINGs

Effect of the Penalty on Emissions



- **Consumer Burden**

$$SC = \sum_t \sum_k (S_t^{k,*} - S_t^{k,BAU*}) D_t^k.$$

- **Reduction Costs** (producers' burden)

$$\sum_t \sum_{i,j,k} (\xi_t^{i,j,k*} - \xi_t^{BAU,i,j,k*}) C_t^{i,j,k}$$

- **Excess Profit**

$$\sum_t \sum_k (S_t^{k,*} - S_t^{k,BAU*}) D_t^k - \sum_t \sum_{i,j,k} (\xi_t^{i,j,k*} - \xi_t^{BAU,i,j,k*}) C_t^{i,j,k} - \pi (\sum_t \sum_{ijk} \xi_t^{ijk} e_t^{ijk} - \theta_0)^+$$

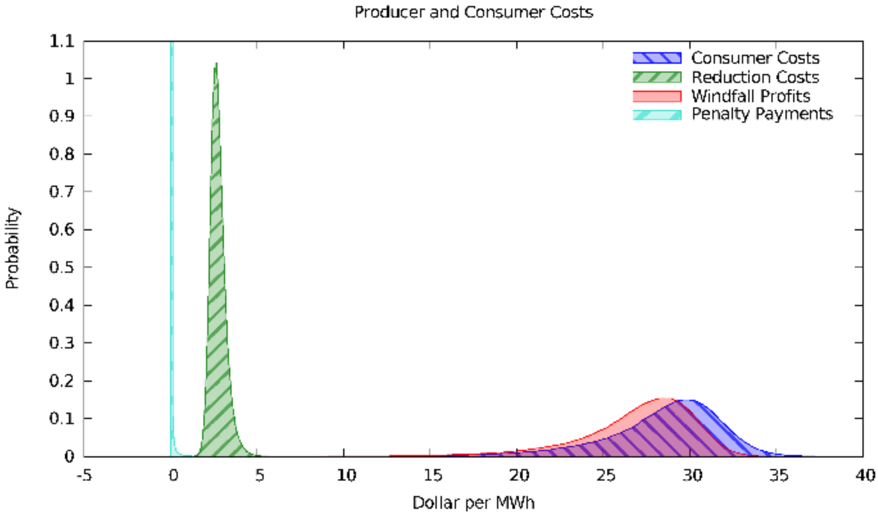
- **Windfall Profits**

$$WP = \sum_{t=0}^{T-1} \sum_{k \in K} (S_t^{*k} - \hat{S}_t^k) D_t^k$$

where

$$\hat{S}_t^k := \max_{i \in I, j \in J^{i,k}} C_t^{i,j,k} \mathbf{1}_{\{\xi_t^{*i,j,k} > 0\}}.$$

Costs in a Cap-and-Trade Scheme



Histograms of consumer costs, social costs, windfall profits and penalty payments of a standard cap-and-trade scheme calibrated to reach the emissions target with 95% probability and BAU.

One of many Possible Generalizations

Introduction of **Taxes** / **Subsidies**

$$\ddot{L}^{A,S,i}(\theta^i,\xi^i) = -\sum_{t=0}^{T-1} G_t^i + \sum_{k\in K}\sum_{j\in J^{i,k}}\sum_{t=0}^{T-1}(S_t^k - C_t^{i,j,k} - H_t^k)\xi_t^{i,j,k} + \sum_{t=0}^{T-1}\theta_t^i(A_{t+1}-A_t) - \theta_T^i A_T - \pi(\Gamma^i + \Pi^i(\xi^i) - \theta_T^i)^+.$$

In this case

- In equilibrium, **production** and **trading** strategies remain the same $(\theta^\dagger,\xi^\dagger) = (\theta^*,\xi^*)$
- **Abatement costs** and **Emissions reductions** are also the same
- New equilibrium prices $(A^\dagger, S^\dagger)$ given by

$$A_t^\dagger = A_t^* \quad \text{for all } t = 0,\dots,T \qquad (1)$$

$$S_t^{\dagger k} = S_t^{*k} + H_t^k \quad \text{for all } k\in K, t=0,\dots,T-1 \qquad (2)$$

- Cost of the tax passed along to the end consumer

Alternative Market Design

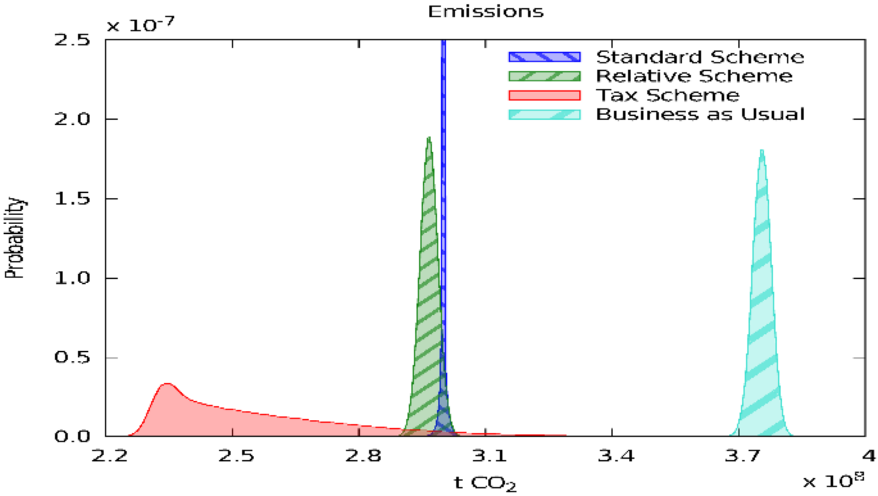
- **Currently Regulator Specifies**
  - Penalty π
  - Overall Certificate Allocation θ_0 $(= \sum_{i \in I} \theta_0^i)$
- **Alternative Scheme (Still) Controlled by Regulator**
  - (i) **Sets penalty level** π
  - (ii) **Allocates allowances**
    - θ_0' at inception of program $t = 0$
    - then **proportionally to production**

    $y\xi_t^{i,j,k}$ to agent i for producing $\xi_t^{i,j,k}$ of good k with technology j
  - (iii) **Calibrates** y, e.g. in **expectation**.

$$y = \frac{\theta_0 - \theta_0'}{\sum_{t=0}^{T-1} \sum_{k \in K} \mathbb{E}\{D_t^k\}}$$

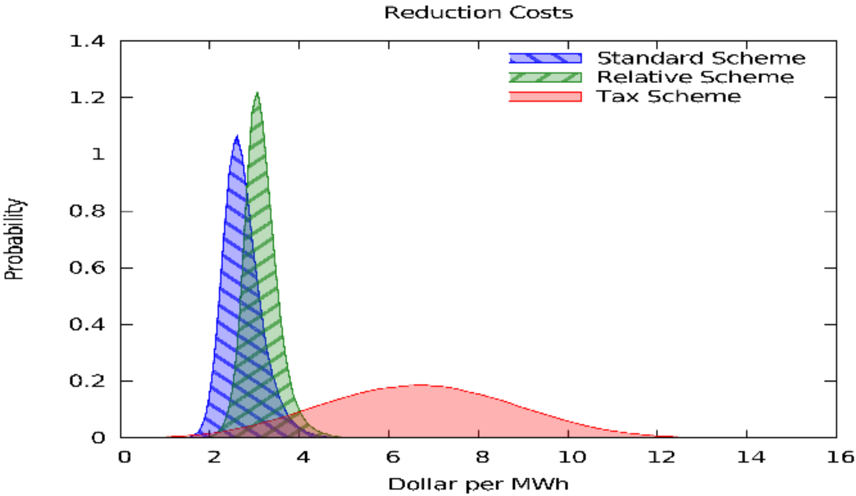
So total number of credit allowance is the same in expectation, i.e. $\theta_0 = \mathbb{E}\{\theta_0' + y \sum_{t=0}^{T-1} \sum_{k \in K} D_t^k\}$

Yearly Emissions Equilibrium Distributions



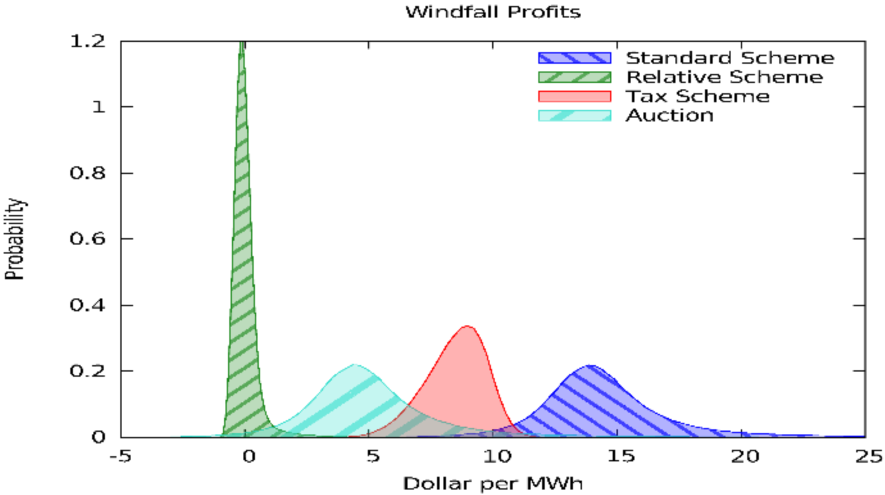
Yearly emissions from electricity production for the Standard Scheme, the Relative Scheme, a Tax Scheme and BAU.

Abatement Costs



Yearly abatement costs for the Standard Scheme, the Relative Scheme and a Tax Scheme.

Windfall Profits



Histograms of the yearly distribution of windfall profits for the Standard Scheme, a Relative Scheme, a Standard Scheme with 100% Auction and a Tax Scheme

What is Next?

- Why would we want to reduce **Windfall Profits**?
- Can one **Design** a cap-and-trade scheme to reach **Prescribed Distributions** for profits and costs?
- Optimizing **irreversible investment** decisions (installing **scrubbers, ...**)
- Need for **Partial Equilibrium** and/or **Reduced Form** Models
  - Require early **active trading**
  - Illustrate **Leakage** and/or **Market Exits**
  - Illustrate and identify **Market Impact** and/or **Manipulations**

Reduced Form Models & Option Pricing

- Emissions Cap-and-Trade Markets **SOON** to exist in the US (and Canada, Australia, Japan, ...)
- Need for **Formulae** (closed or approximate)
  - Equilibrium prices difficult to compute (only numerically)
  - Study effect of announcements (**Cetin-Verschuere, Grüll-Kiesel, ...**)
- Liquid **Option** Market **ALREADY** exists in Europe
  - Underlying $\{A_t\}_t$ non-negative martingale with **binary terminal value**
  - Think of A_t as of a binary option
  - Underlying of binary option should be *Cumulative Emissions*
- **Reduced Form Models** (**Uhrig-Homburg-Wagner, R.C - Hinz**)

The Option Market is **IMMATURE**

Option quotes on Jan. 3, 2008

Option Maturity	Option Type	Volume	Strike	Allowance Price	Implied Vol	Settlement Price
Dec-08	Call	150,000	24.00	23.54	50.50%	4.19
Dec-08	Call	500,000	26.00	23.54	50.50%	3.50
Dec-08	Call	25,000	27.00	23.54	50.50%	3.20
Dec-08	Call	300,000	35.00	23.54	50.50%	1.56
Dec-08	Call	1,000,000	40.00	23.54	50.50%	1.00
Dec-08	Put	200,000	15.00	23.54	50.50%	0.83

Could the Traders Be Using **BLACK**'s Formula?

Option quotes on Jan. 4, 2008

Option Maturity	Option Type	Volume	Strike	Allowance Price	Implied Vol	Settlement Price
Dec-08	Cal	200,000	22.00	23.55	51.25%	5.06
Dec-08	Call	150,000	26.00	23.55	51.25%	3.57
Dec-08	Call	450,000	27.00	23.55	51.25%	3.27
Dec-08	Call	100,000	28.00	23.55	51.25%	2.99
Dec-08	Call	125,000	29.00	23.55	51.25%	2.74
Dec-08	Call	525,000	30.00	23.55	51.25%	2.51
Dec-08	Call	250,000	40.00	23.55	51.25%	1.04
Dec-08	Call	700,000	50.00	23.55	51.25%	0.45
Dec-08	Put	1,000,000	14.00	23.55	51.25%	0.64
Dec-08	Put	200,000	15.00	23.55	51.25%	0.86
Dec-08	Put	200,000	15.00	23.55	51.25%	0.86
Dec-08	Put	400,000	16.00	23.55	51.25%	1.13
Dec-08	Put	100,000	17.00	23.55	51.25%	1.43
Dec-08	Put	1,000,000	18.00	23.55	51.25%	1.78
Dec-08	Put	500,000	20.00	23.55	51.25%	2.60
Dec-08	Put	200,000	21.00	23.55	51.25%	3.07
Dec-08	Put	200,000	22.00	23.55	51.25%	3.57

Chesney=Taschini

Allowance price should be of the form

$$A_t = \pi \mathbb{E}\{\mathbf{1}_N \,|\mathcal{F}_t\}$$

for a non-compliance set $N \in \mathcal{F}_t$. Choose

$$N = \{\Gamma_T \geq 1\}$$

for a random variable Γ_T representing the normalized emissions at compliance time. So

$$A_t = \pi \mathbb{E}\{\mathbf{1}_{\{\Gamma_T \geq 1\}} \,|\mathcal{F}_t\} = \pi \mathbb{P}\{\Gamma_T \geq 1 \,|\mathcal{F}_t\}, \qquad t \in [0, T]$$

We choose Γ_T in a parametric family

$$\Gamma_T = \Gamma_0 \exp\left[\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2\, ds\right]$$

for some square integrable deterministic function

$$(0, T) \ni t \hookrightarrow \sigma_t$$

Dynamic Price Model for $a_t = \frac{1}{\pi} A_t$

- a_t is given by

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T \sigma_s^2 ds} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \qquad t \in [0, T)$$

  where Φ is standard normal c.d.f.

- a_t solves the SDE

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{z_t} dW_t$$

  where the positive-valued function $(0, T) \ni t \hookrightarrow z_t$ is given by

$$z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}, \qquad t \in (0, T)$$

Risk Neutral Densities

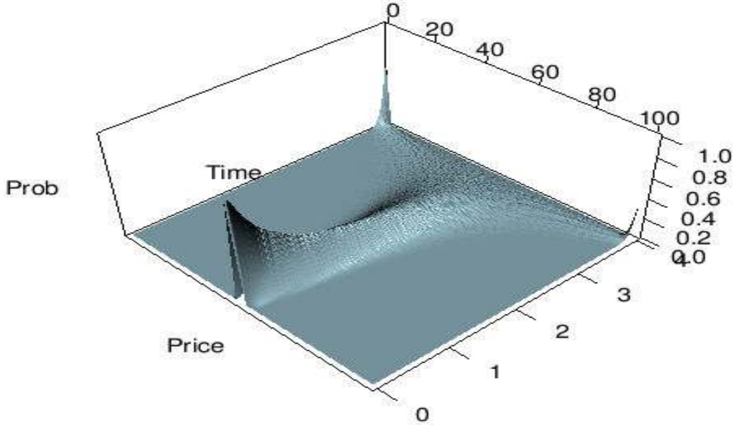


Figure: Histograms for each day of a 4 yr compliance period of 10^5 simulated risk neutral allowance price paths.

Calibration

Had to Be Historical !!!!

- Choose **Constant** Market Price of Risk
- **Two-parameter** Family for Time-change

$$\{z_t(\alpha,\beta) = \beta(T-t)^{-\alpha}\}_{t\in[0,T]}, \qquad \beta > 0, \alpha \geq 1.$$

Volatility function $\{\sigma_t(\alpha,\beta)\}_{t\in(0,T)}$ given by

$$\begin{aligned}
\sigma_t(\alpha,\beta)^2 &= z_t(\alpha,\beta)e^{-\int_0^t z_u(\alpha,\beta)du} \\
&= \begin{cases} \beta(T-t)^{-\alpha} e^{\beta\frac{T^{-\alpha+1}-(T-t)^{-\alpha+1}}{-\alpha+1}} & \text{for } \beta > 0, \alpha > 1 \\ \beta(T-t)^{\beta-1}T^{-\beta} & \text{for } \beta > 0, \alpha = 1 \end{cases}
\end{aligned}$$

Maximum Likelihood

Sample Data

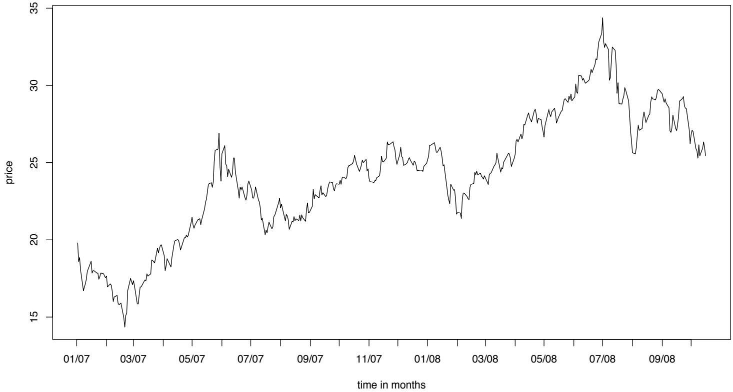


Figure: Future prices on EUA with maturity Dec. 2012

Call Option Price in One Period Model

for $\alpha = 1$, $\beta > 0$, the price of an European call with strike price $K \geq 0$ written on a one-period allowance futures price at time $\tau \in [0, T]$ is given at time $t \in [0, \tau]$ by

$$
\begin{aligned}
C_t &= e^{-\int_t^\tau r_s ds}\mathbb{E}\{(A_\tau - K)^+ \mid \mathcal{F}_t\} \\
&= \int (\pi\Phi(x) - K)^+ N(\mu_{t,\tau}, \nu_{t,\tau})(dx)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{t,\tau} &= \Phi^{-1}(A_t/\pi)\sqrt{\left(\frac{T-t}{T-\tau}\right)^\beta} \\
\nu_{t,\tau} &= \left(\frac{T-t}{T-\tau}\right)^\beta - 1.
\end{aligned}
$$

Price Dependence on T and Sensitivity to β

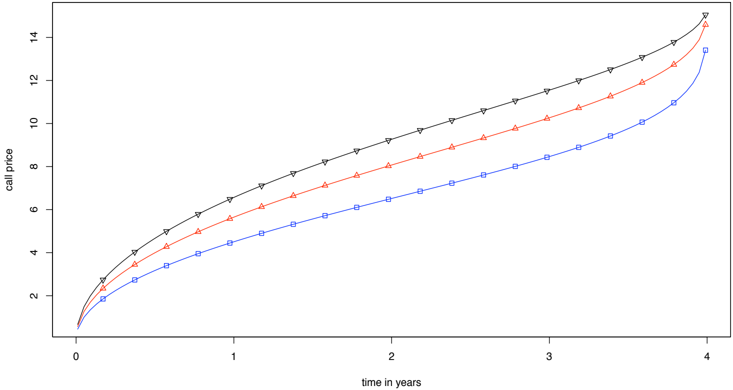
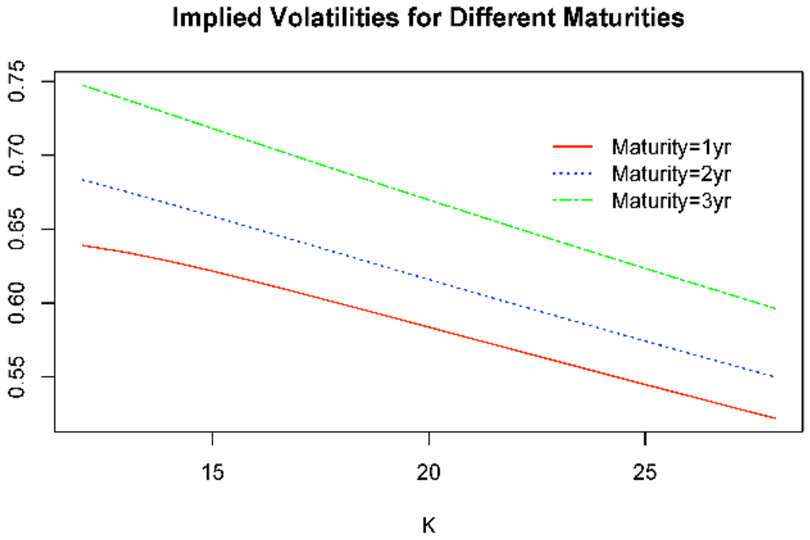
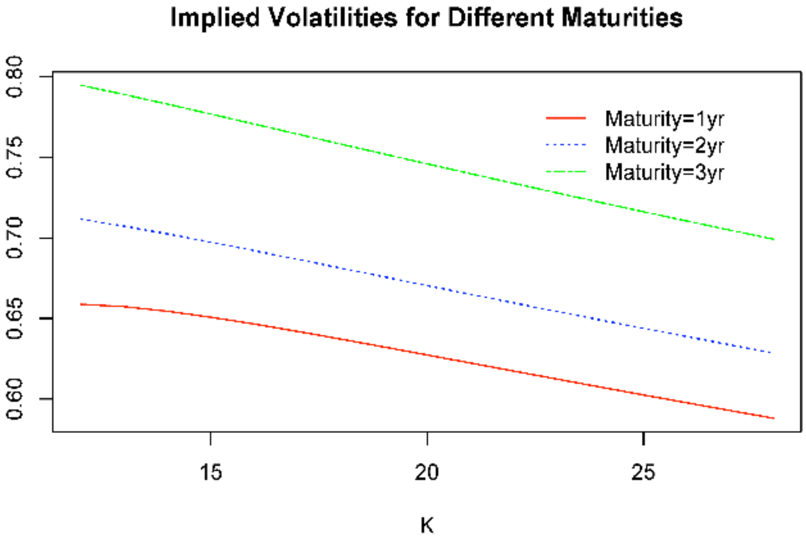


Figure: Dependence $\tau \mapsto C_0(\tau)$ of Call prices on maturity τ for $\alpha = 1$. Graphs $\square$, $\triangle$, and $\triangledown$ correspond to $\beta = 0.5$, $\beta = 0.8$, $\beta = 1.1$.

Implied Volatilities $\alpha = 1$, $\beta = 1.2$



Implied Volatilities $\alpha = 1$, $\beta = 0.6$, $\pi = 100$



Option quotes on April 9, 2010

With a Smile Now!

Option Maturity	Option Type	Volume	Strike	Allowance Price	Implied Vol	Settlement Price
Dec-10	Call	750,000	14.00	13.70	29.69	1.20
Dec-10	Call	150,000	15.00	13.70	29.89	0.85
Dec-10	Call	250,000	16.00	13.70	30.64	0.61
Dec-10	Call	250,000	18.00	13.70	32.52	0.34
Dec-10	Call	1,000,000	20.00	13.70	33.07	0.17
Dec-10	Put	1,000,000	10.00	13.70	37.42	0.29
Dec-10	Put	500,000	12.00	13.70	32.12	0.67
Dec-10	Put	500,000	13.00	13.70	30.37	1.01

Carbon Prices by PDE Methods

Model for Emissions

$$dE_t^i = [b_t^i - \eta_t^i]dt + \sigma_t^i dB_t^i$$

then in equilibrium

$$\begin{aligned} dE_t &= [b_t - (c')^{-1}(A_t)]dt + \sigma_t dB_t \\ dA_t &= Z_t dB_t \end{aligned}$$

with **terminal condition**

$$A_T = \pi \mathbf{1}_{[\kappa,\infty)}(E_T).$$

Existence & Uniqueness (**R.C. - Delarue - Espinoza-Touzi**)

(Comparison arguments for solutions of BSDEs)

Theoretical Existence and Uniqueness

Theorem

If $\sigma(t) \geq \underline{\sigma} > 0$ then for any $\lambda > 0$ and $\Lambda \in \mathbb{R}$, there exists a unique solution $(E, A, Z) \in M^2$. Moreover, A_t is nondecreasing w.r.t λ and nonincreasing w.r.t Λ.

Proof

- Approximate the singular terminal condition $\lambda \mathbf{1}_{[\Lambda,+\infty)}(E_T)$ by increasing and decreasing sequences $\{\varphi_n(E_T)\}_n$ and $\{\psi_n(E_T)\}_n$ of smooth monotone functions of E_T
- Use
  - comparison results for BSDEs
  - the fact that E_T has a density (implying $\mathbb{P}\{E_T = \Lambda\} = 0$)

  to control the limits

Carbon Prices by PDE Methods

Exogenous Model for Power Price

$$dP_t = \mu(P_t)dt + \sigma(P_t)dB_t$$

In equilibrium, aggregate cumulative emissions given by

$$dE_t = (c')^{-1}(P_t - eA_t)dt$$

with usual martingale condition

$$dA_t = Z_t dB_t \quad \text{with terminal condition} \qquad A_T = \pi \mathbf{1}_{[\kappa,\infty)}(E_T).$$

gives **degenerate Forward-Backward SDE**!

Carbon Prices by PDE Methods (cont.)

The corresponding FBSDE under $\mathbb{Q}$ reads

$$\textbf{FBSDE} \begin{cases} dP_t &= \sigma(t, P_t)dB_t, \qquad P_0 = p \\ dE_t &= f(P_t, A_t)dt, \qquad E_0 = 0 \\ dA_t &= Z_t dB_t. \end{cases}$$

with terminal condition $A_T = \lambda \mathbf{1}_{[\Lambda,+\infty)}(E_T)$

NB: The volatility of the forward equation is **degenerate!**

Still, **Natural Conjecture**: For $\lambda > 0$ and $\Lambda \in \mathbb{R}$, the above FBSDE has a unique solution (P, E, A, Z).

Existence and Uniqueness

Theorem

Assuming uniformly Lipschitz coefficients, there exists a unique progressively measurable quadruplet $(P_t, E_t, A_t, Z_t)_{0 \le t \le T}$ satisfying **FBSDE** on $[0, T]$ and

$$\mathbf{1}_{(\Lambda,\infty)}(E_T) \le A_T \le \mathbf{1}_{[\Lambda,\infty)}(E_T).$$

The terminal condition $A_T = \mathbf{1}_{[\Lambda,\infty)}(E_T)$ may not be satisfied!

Singularity of the Terminal Value (R.C. - Delarue)

Assume further **forward diffusion elliptic** $\delta^{-1} > \sigma(t, p) \geq \delta > 0$ then

Theorem

- E_t has a smooth density whenever $t < T$,
- The distribution of E_T has a (Dirac) point mass at Λ, i.e.

$$\mathbb{P}\{E_T = \Lambda\} > 0.$$

- The support of the conditional distribution of A_T given $\{E_T = \Lambda\}$ is the **WHOLE** interval $[0, 1]$!

Consequences

- **The terminal condition $A_T = \mathbf{1}_{[\Lambda,\infty)}(E_T)$ is not satisfied!**
- At time T, the **price A_T of one allowance is not determined** by the model **on the set** $\{E_T = \Lambda\}$ of positive probability!

Comments on the Existence of a Point Mass fot E_T

- Ruled out (by assumption) in early equilibrium studies

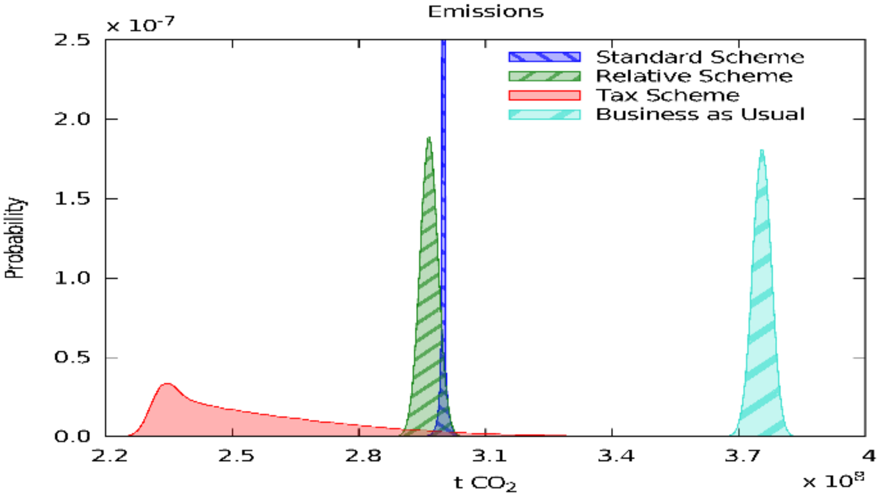
Assumption

the $\mathcal{F}_{T-1}$-conditional distribution of $\sum_{i\in I}\Delta^i$ possesses almost surely no point mass, or equivalently, for all $\mathcal{F}_{T-1}$-measurable random variables Z

$$\mathbb{P}\left\{\sum_{i\in I}\Delta^i = Z\right\} = 0$$

- Thought to be *innocent* !
- Should have known better!
  - **Numerical Evidence** from case studies shows high emission concentration near (below) Λ

Yearly Emissions Equilibrium Distributions



Yearly emissions from electricity production for the Standard Scheme, the Relative Scheme, a Tax Scheme and BAU.

Talk based on

1. R.C., M. Fehr and J. Hinz: Mathematical Equilibrium and Market Design for Emissions Markets Trading Schemes. *SIAM J. Control and Optimization* (2009)
2. R.C., M. Fehr, J. Hinz and A. Porchet: Mathematical Equilibrium and Market Design for Emissions Markets Trading Schemes. *SIAM Review* (2010)
3. R.C., and J. Hinz: Risk-Neutral Modeling of Emission Allowance Prices and Option Valuation *Management Science* (to appear)
4. R.C., F. Delarue, G.E. Espinosa and N. Touzi: Singular BSDEs and Emission Derivative Valuation (submitted for publication)
5. R.C., F. Delarue: Singular FBSDEs and Scalar Conservation Laws Driven by Diffusion Processe (in preparation)