Maximum likelihood for *MA* processes over graphs

T. Espinasse, F. Gamboa, J-M. Loubes

Université Paul Sabatier

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Outline

Origin of the problem

Traffic : Predict the speed of the vehicles with missing values

For now : Spatial dependency is not exploited

Aims

- Give a model which uses spatial dependency
- Estimate the spatial correlation
- Spatial filtering

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Model : Speed process $(X_i)_{i\in G}$ indexed by the vertices G of a graph **G**.

Definition (Unoriented weighted graph)

$\mathbf{G} = (G, W)$:

- *G set of vertices (infinite countable)*
- *$W \in [-1,1]^{G\times G}$ Weighted adjacency operator (symmetric)*

Neighbors : $i \sim j$ if $W_{ij} \neq 0$
Degree of a vertex : $D_i = \sharp\{j, i \sim j\}$.

Assumption (H_0)

- $D := \sup_{i\in G} D_i < +\infty$, **G** *has bounded degree*
- $\forall i \in G, \sum_{j\in G} |W_{ij}| \leq 1$ *even renormalize*

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Problem

Remark :

- Our work is robust to renormalization
- For $\mathbb{Z}$, for instance : $W_{ij}^{(\mathbb{Z})} = \frac{1}{2}\mathbf{1}_{|i-j|=1}$

W acts on $l^2(G)$:

$$\forall u \in l^2(G), \forall i \in G, (Wu)_i := \sum_{j \in G} W_{ij} u_j$$

Under H_0

W is bounded as operator of $B_G := l^2(G) \to l^2(G)$:

$$\|W\|_{2,op} \leq 1$$

H_0' : W takes a finite number of values

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Observation : Speed correlations independent of the position and the orientation in the traffic network

Aim : Get a model “stationary” and ”isotropic” :
$(X_i)_{i \in G}$ Gaussian zero-mean with covariance operator $K \in \mathbb{R}^{G \times G}$:
$\Rightarrow$ Characterized by K

Aim : Extension of the time-series
$\Rightarrow$ Construction of *MA* processes with respect to the adjacency operator

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For $\mathbb{Z}$: $(\epsilon_n)_{n \in \mathbb{Z}}$ Gaussian i.i.d.

$$X_n = \sum_{k \in \mathbb{Z}} a_k \epsilon_{n-k}$$

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Aim : Identification of maximum likelihood
$\Rightarrow$ Generalize Whittle’s approximation

A few bibliography

Spectral representation of stationary processes :

- $\mathbb{Z}^d$: X. Guyon
- Homogeneous tree : J-P. Arnaud
- Distance-transitive graphs : H. Heyer

Maximum likelihood

- $\mathbb{Z}$: here R. Azencott et D. Dacunha-Castelle
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Extension to any graph

$\mathbb{Z}$ **:** *MA* processes

$$K = g(W^{(\mathbb{Z})})$$

Definition

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$$\textit{If } K = g(W),$$

- *g polynomial : $MA_q^{(W)}$*
- *$\frac{1}{g}$ polynomial : $AR_p^{(W)}$ $\cdots$*

Remarks :

- Conditions about g
- Equivalent to $\mathbb{Z}$

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Problem :

- $\Theta \subset \mathbb{R}$ compact
- $(f_\theta)_{\theta\in\Theta}$ parametric family of densities associated to $K(f_\theta) = f_\theta(W)$
- Asymptotic $(\mathbf{G}_n)_{n\in\mathbb{N}}$ nested subgraphs
  Example $G = \mathbb{Z}$: $G_n = [1, n]$.
- $\theta_0 \in \mathring{\Theta}$, $\mathbf{X} \sim \mathcal{N}\left(0, K(f_{\theta_0})\right)$
- We observe the restriction X_n of $\mathbf{X}$ to $\mathbf{G}_n$, cov : $K_n(f_\theta)$
- $m_n = \sharp G_n$

Aim : Estimate θ_0 by maximum likelihood :

$$L_n(\theta) := -\frac{1}{2}\left(m_n \log(2\pi) + \log\det\left(K_n(f_\theta)\right) + X_n^T\left(K_n(f_\theta)\right)^{-1} X_n\right)$$

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Whittle's approximation

Whittle's approximation for $\mathbb{Z}$, log det

$$\frac{1}{n}\log\det\left(T_n(f_\theta)\right) \to \int_{[0,2\pi]} \log\left(f_\theta\right) \mathrm{d}t$$

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Moreover, $T(f)T(g) = T(fg)$ so that $\left(T(f)\right)^{-1} = T(\frac{1}{f})$

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$$\frac{1}{n}\left(X_n^T \left(T_n(f_\theta)\right)^{-1} X_n - X_n^T T_n\left(\frac{1}{f_\theta}\right) X_n\right) \to 0, \ \text{a.s.}$$

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Introduce $\delta_n = \sharp\delta G_n$.
Example $G = \mathbb{Z}$: $\delta_n = 2$

Approximation log det for any graph

Definition (Local spectral measure)

$$\forall g, h \in G, \mu_{gh}^{(k)} = \left(W^k\right)_{gh}$$

Assumption (Existence of a spectral measure)

$H_1 : \exists \mu, \frac{1}{m_n} \sum_{g \in G_n} \mu_{gg} \to \mu$

Assumption (Boundary problems)

$H_2 : \delta_n = o(m_n)$

Whittle's approximation for **G**, log det

$$\frac{1}{m_n} \log\det\left(K_n(f_\theta)\right) \to \int_{\mathrm{Sp}(W)} \log\left(f_\theta\right) \mathrm{d}\mu(t)$$

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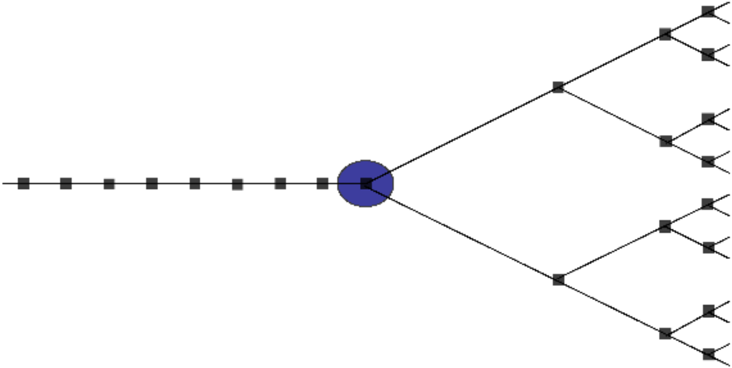
Sufficient condition for existence of μ

l-type for the vertices: $\forall g \in G, \forall l \geq 1, t^l(g) = W^l_{gg}$

Assumption (Homogeneity assumption)

$$\forall l \geq 0, \forall n \geq 1, \forall g \in G_n, \frac{\sharp \left\{ j \in G_n, t^{(l)}(j) = t^{(l)}(g) \right\}}{\sharp G_n} \to p^{(l)}_{t^{(l)}(g)}$$

Sufficient condition for existence of μ



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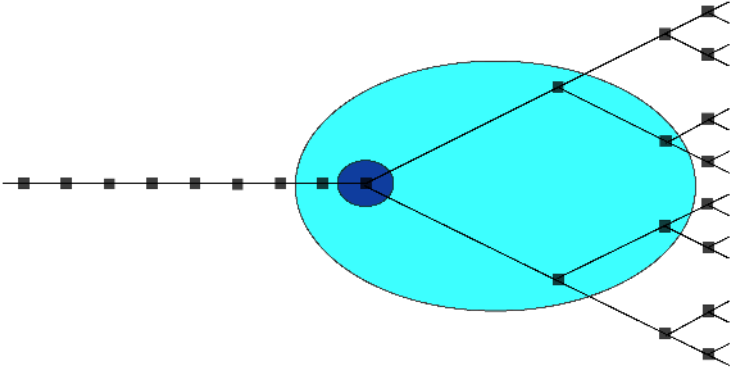

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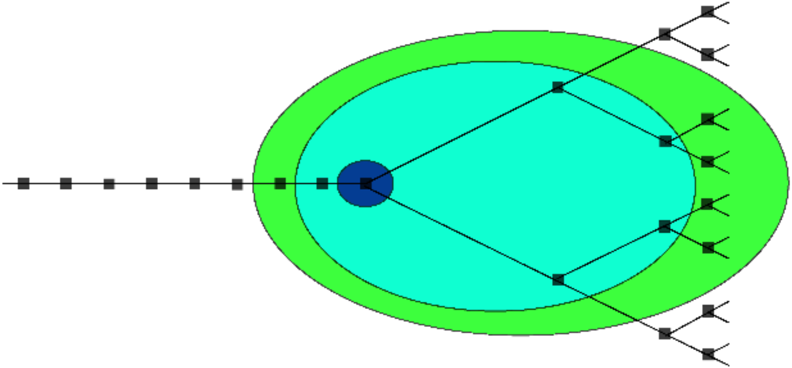

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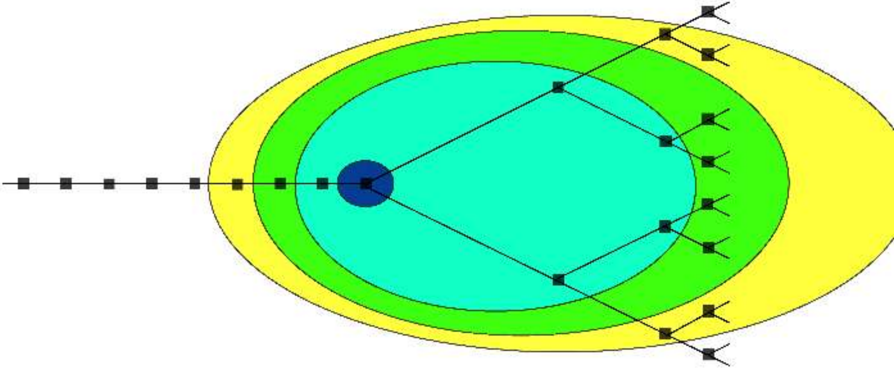

Quadratic form

Close to the weak version for $\mathbb{Z}$

Regularity criterion

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$$\forall A \in M_n(\mathbb{R}), b_n(A) = \frac{1}{\delta_n} \sum_{ij} |A_{ij}|$$

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Lemma

Asymptotic homomorphism

$$b_n \left(K_n(f) K_n(g) - K_n(fg) \right) \leq \frac{1}{2} \alpha(f) \alpha(g)$$

Convergence

Let $\theta_n, \bar{\theta}_n, \tilde{\theta}_n$ be the respective arg max of

$$
\begin{aligned}
L_n(\theta) &:= -\frac{1}{2}\left(m_n \log(2\pi) + \log\det\left(K_n(f_\theta)\right) + X_n^T \left(K_n(f_\theta)\right)^{-1} X_n\right) \\
\bar{L}_n(\theta) &:= -\frac{1}{2}\left(m_n \log(2\pi) + m_n \int \log(f_\theta(x))\mathrm{d}\mu(x) + X_n^T \left(K_n(f_\theta)\right)^{-1} X_n\right) \\
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Assumption (H_3)

- $\theta \to f_\theta$ *injective*
- $\forall \lambda \in Sp(W), \theta \to f_\theta(\lambda)$ *continuous.*
- $\forall \theta \in \Theta, \alpha(\log(f_\theta)) \leq \rho < +\infty$

Convergence

Let $\theta_n, \bar{\theta}_n, \tilde{\theta}_n$ be the respective arg max of

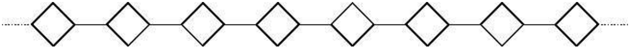
$$
\begin{aligned}
L_n(\theta) &:= -\frac{1}{2}\left(m_n \log(2\pi) + \log\det\left(K_n(f_\theta)\right) + X_n^T \left(K_n(f_\theta)\right)^{-1} X_n\right) \\
\bar{L}_n(\theta) &:= -\frac{1}{2}\left(m_n \log(2\pi) + m_n \int \log(f_\theta(x))\mathrm{d}\mu(x) + X_n^T \left(K_n(f_\theta)\right)^{-1} X_n\right) \\
\tilde{L}_n(\theta) &:= -\frac{1}{2}\left(m_n \log(2\pi) + m_n \int \log(f_\theta(x))\mathrm{d}\mu(x) + X_n^T \left(K_n\left(\frac{1}{f_\theta}\right)\right) X_n\right)
\end{aligned}
$$

Theorem (Convergence of the maximum likelihood)

The estimates θ_n,$\bar{\theta}_n$, $\tilde{\theta}_n$ converge $P_{f_{\theta_0}}$-a.s. to the true value θ_0 for any $\theta_0 \in \Theta$.

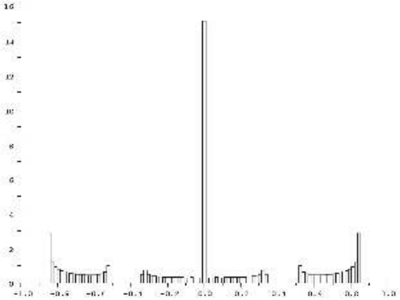
Applications

Figure: Graph G



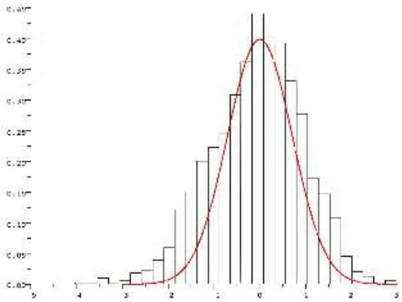
Applications

Figure: Empirical spectrum



Applications

Figure: Empirical distribution



Estimation of the real-life spectral measure

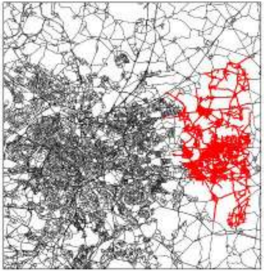
Data come from *Mediamobile* (specialized in prediction of travel time)

Estimation of the real-life spectral measure

Data come from *Mediamobile* (specialized in prediction of travel time)
This work in progress is made in collaboration with J.N. Kien (Mediamobile).

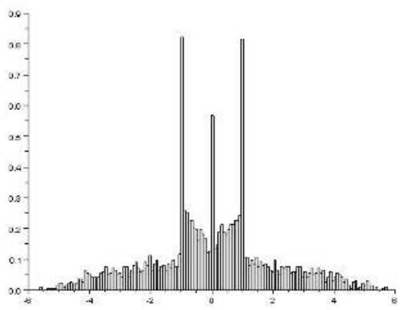
Estimation of the real-life spectral measure

Figure: Traffic network



Estimation of the real-life spectral measure

Figure: Empirical spectrum



Forthcoming works

- Long memory
- Prediction
- ...

Other questions

- Causal process
- Tree (boundary problems)
- Random graphs

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Thank you