Adaptive stochastic coupling in the Arlequin method

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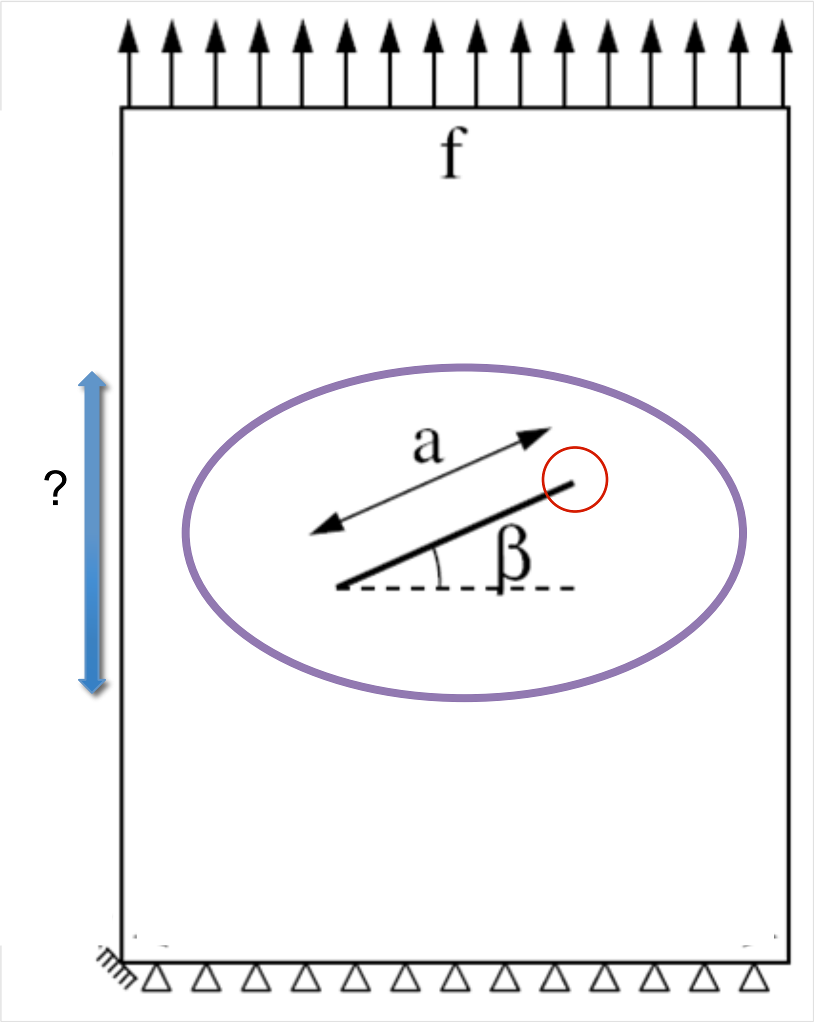
Context



- Structure with local behavior modification (cuts, holes, different processes…)

- Multiscale: close to singularities (load, BC…), close to / far from the defect…

Global homogeneous behavior with:

- Strong variability of parameters
- Insufficient knowledge of material parameters
- Specific and complex physics (cracking…)

→ Local Stochastic model superposed on a Deterministic one

Summary

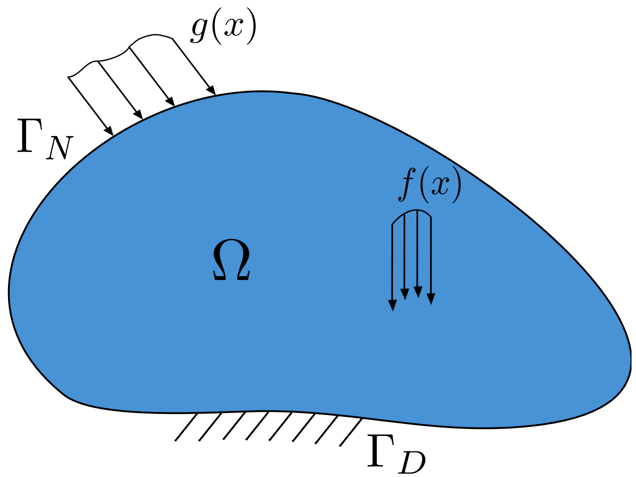
1. Definition of the reference model
2. Reduced model with the Arlequin Method
   - The Arlequin method
   - Specificity of the coupling
   - Raw results on a simple case
3. Goal-oriented error estimation
   - Quantity of interest
   - Definition of the adjoint problem
   - Error estimation and adaptivity
4. Example of adaptive model with stochastic coupling

1. Definition of the reference model

Considering $(\Theta, \mathscr{F}, P)$ a complete probability space,

- Equilibrium equation

$$\forall x \in \Omega,$$
$$-\nabla \cdot (\boldsymbol{K}(x,\theta)\nabla \boldsymbol{u}(x,\theta)) = f(x)$$



- Boundary conditions

$$\begin{cases} u = 0 \text{ on } \Gamma_D \\ K(x,\theta)\nabla u = g(x) \text{ on } \Gamma_N \end{cases} \quad \text{a.s.}$$

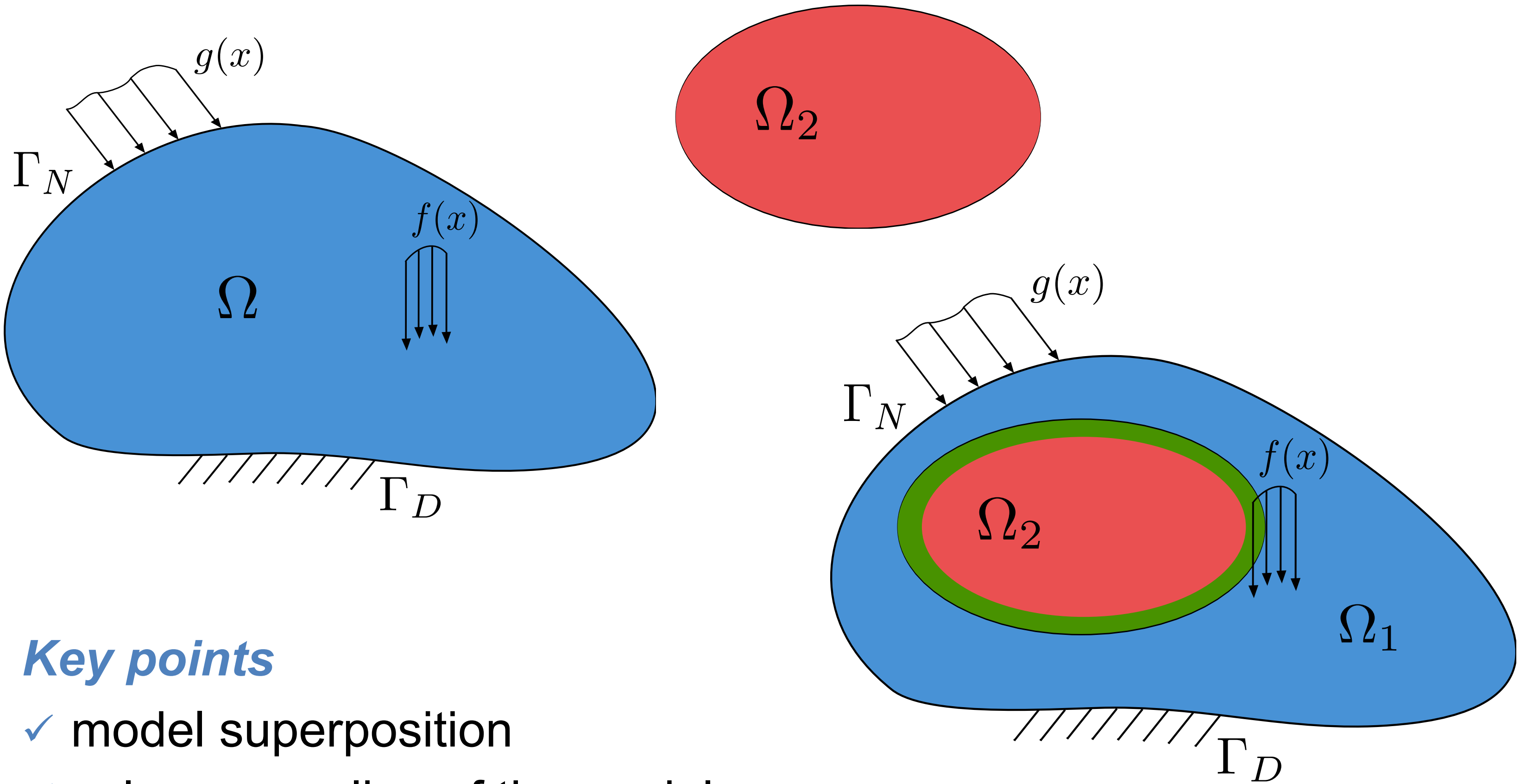
- Stochastic material property

$$\boldsymbol{K}(x,\theta) \in \mathscr{L}^2(\Theta, \mathscr{C}^0(\Omega))$$

$$0 < K_{\min} \leq \boldsymbol{K}(x,\theta) \leq K_{\max} < \infty, \ \forall x \in \Omega \quad \text{a.s.}$$

In practice, the solution is unavailable.

2. Reduced model using the Arlequin method [Ben Dhia 1998-2008]



Key points

- ✓ model superposition
- ✓ volume coupling of the models
- ✓ distribution of the mechanical energy $(\alpha_1(x), \alpha_2(x))$

Equilibrium equation [Cottereau 2010]

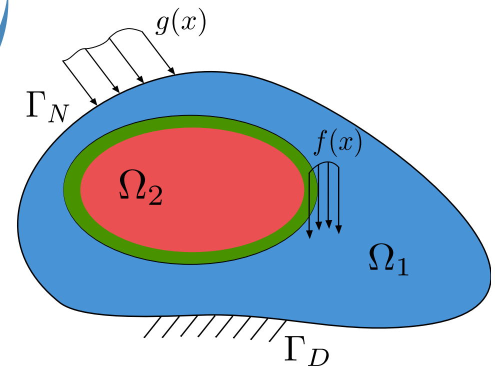
Find $(u_d, \boldsymbol{u_s}, \boldsymbol{\lambda}) \in \mathcal{V}_d \times \mathcal{W}_s \times \mathcal{W}_c$ such that:

$$\begin{cases} a_d(u_d, v) + C(\boldsymbol{\lambda}, v) & = & \ell_d(v), & \forall v \in \mathcal{V}_d \\ \boldsymbol{a_s}(\boldsymbol{u_s}, \boldsymbol{v}) - C(\boldsymbol{\lambda}, \boldsymbol{v}) & = & \boldsymbol{\ell_s}(\boldsymbol{v}), & \forall \boldsymbol{v} \in \mathcal{W}_s \\ C(\boldsymbol{\mu}, \Pi u_d - \boldsymbol{u_s}) & = & 0, & \forall \boldsymbol{\mu} \in \mathcal{W}_c \end{cases}$$

Equilibrium of model 1

Coupling of the models

Equilibrium of model 2



Internal works

$$a_d(u, v) = \int_{\Omega_1} \alpha_1(x) K_d(x) \nabla u \nabla v \, d\Omega$$

$$\boldsymbol{a_s}(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{E}\left[\int_{\Omega_2} \alpha_2(x) \boldsymbol{K_s}(x, \theta) \nabla \boldsymbol{u} \nabla \boldsymbol{v} \, d\Omega\right]$$

External works

$$\boldsymbol{\ell_d}(v) = \int_{\Omega_1} \alpha_1(x) f v \, d\Omega$$

$$\boldsymbol{\ell_s}(\boldsymbol{v}) = \mathrm{E}\left[\int_{\Omega_2} \alpha_2(x) f \boldsymbol{v} \, d\Omega\right]$$

Coupling operator and space [Cottereau 2011]

- Coupling operator $C : \mathcal{W}_c \times \mathcal{W}_c \to \mathbb{R}$

$$C(\boldsymbol{u},\boldsymbol{v}) = \mathrm{E}\left[\int_{\Omega_c} \kappa_0 \boldsymbol{u}\boldsymbol{v} + \kappa_1 \nabla\boldsymbol{u}\nabla\boldsymbol{v}\, d\Omega\right]$$

- Coupling space

$$\mathcal{W}_c = \{\underline{v}(x) + \boldsymbol{\theta}\mathbb{I}_c(x) | \underline{v} \in \mathscr{H}^1(\Omega_c), \boldsymbol{\theta} \in \mathscr{L}^2(\Theta,\mathbb{R}), \int_{\Omega_c} \underline{v}(x)\ d\Omega = 0\}$$

Random functions, perfectly spatially correlated

Deterministic

Random field with a spatially varying mean

Meanings of the coupling

$$
\begin{aligned}
C(\mu, \Pi u_d - \boldsymbol{u_s}) &= 0, \ \forall \boldsymbol{\mu} \in \mathcal{W}_c \\
&= \underline{C}(\mathrm{E}[\boldsymbol{\mu}], u_d - \mathrm{E}[\boldsymbol{u_s}]) + \mathrm{E}\left[\boldsymbol{\theta} \int_{\Omega_c} (u_d - \boldsymbol{u_s}) d\Omega\right]
\end{aligned}
$$

With

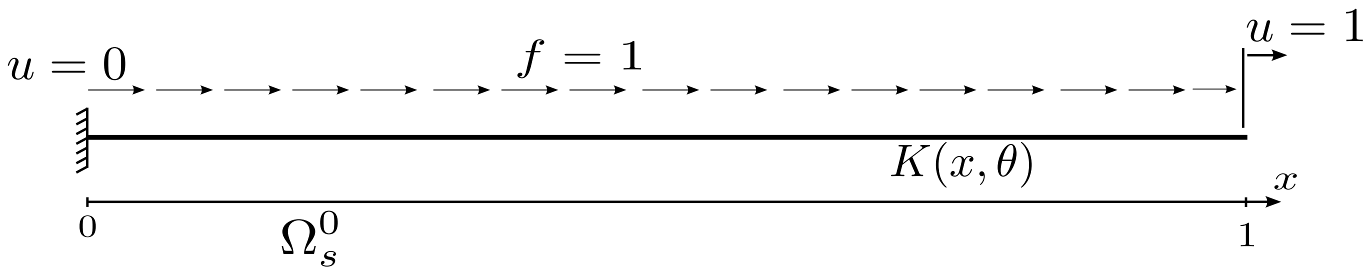
$$\underline{C}(u, v) = \int_{\Omega_c} \kappa_0 uv + \kappa_1 \nabla u \nabla v \, d\Omega$$

Equality between the mean of the stochastic field and the deterministic one

Average cancelling of the stochastic field variability

Simple application

- Monodimensional application (reference model)

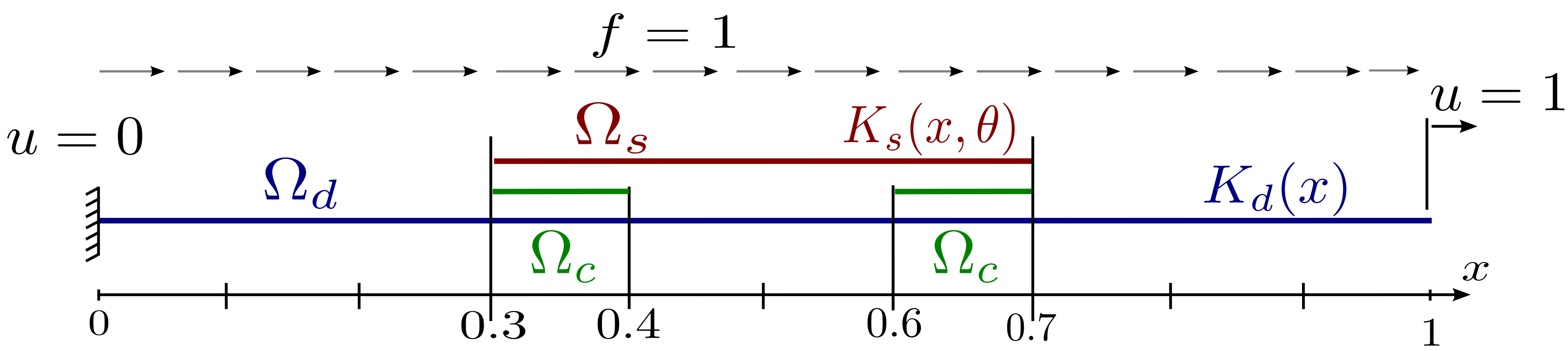


- Stochastic distributed following a uniform law of bounds [0.2294 ; 1.7706], with parameters:

$$\begin{cases} \mathrm{E}\left[\boldsymbol{K}(x,\theta)\right] = 1 \\ L_{\text{correlation}} = 0.01 \\ \sigma = 0.2 \end{cases}$$

Arlequin approximation

- Monodimensional application



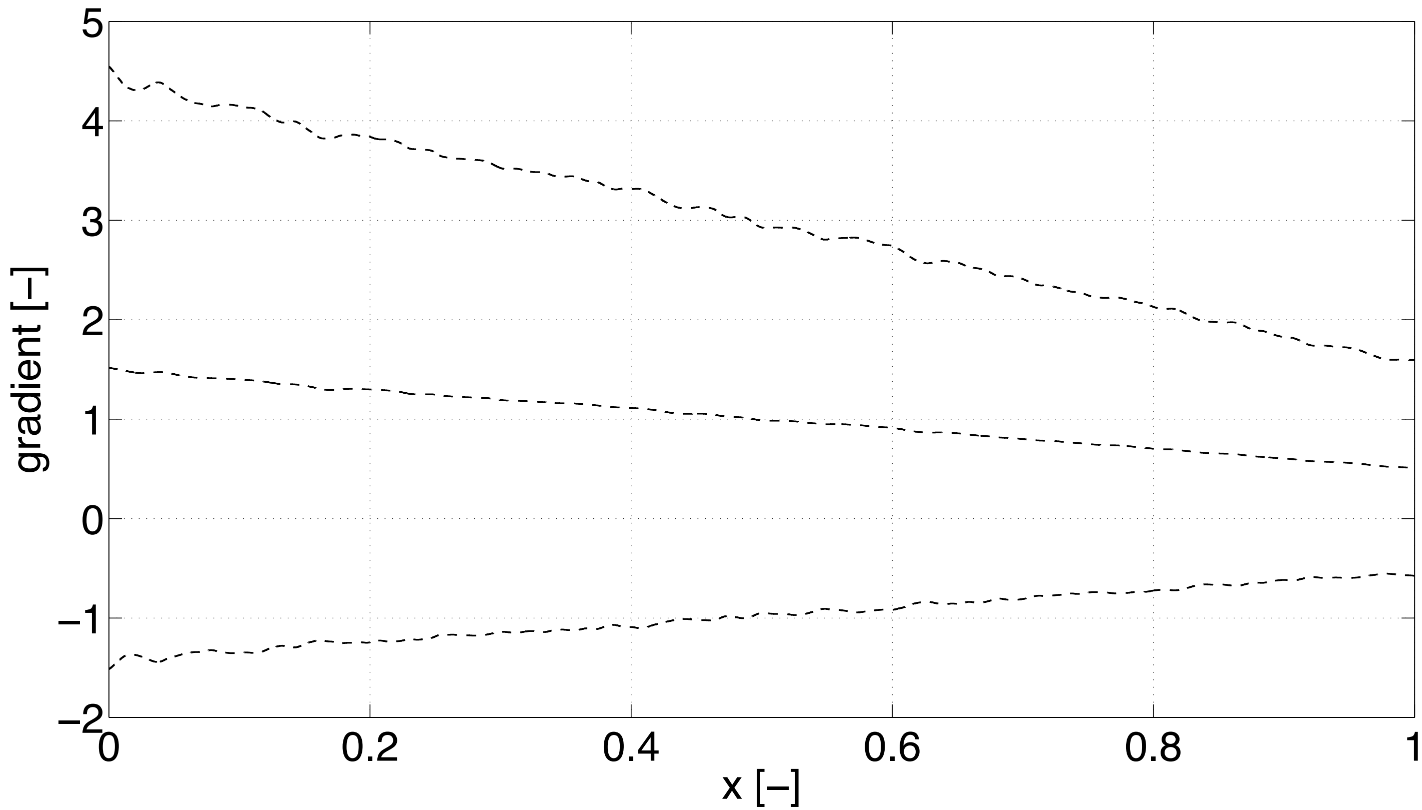
- Deterministic model described by: $K_d(x) = 0.7537$
- $K_s(x,\theta)$ distributed following a uniform law with parameters: $\begin{cases} \mathrm{E}\left[\boldsymbol{K}_s(x,\theta)\right] = 1 \\ L_{\text{correlation}} = 0.01 \\ \sigma = 0.2 \end{cases}$

Remark :

To ensure the physical meaning of the coupling

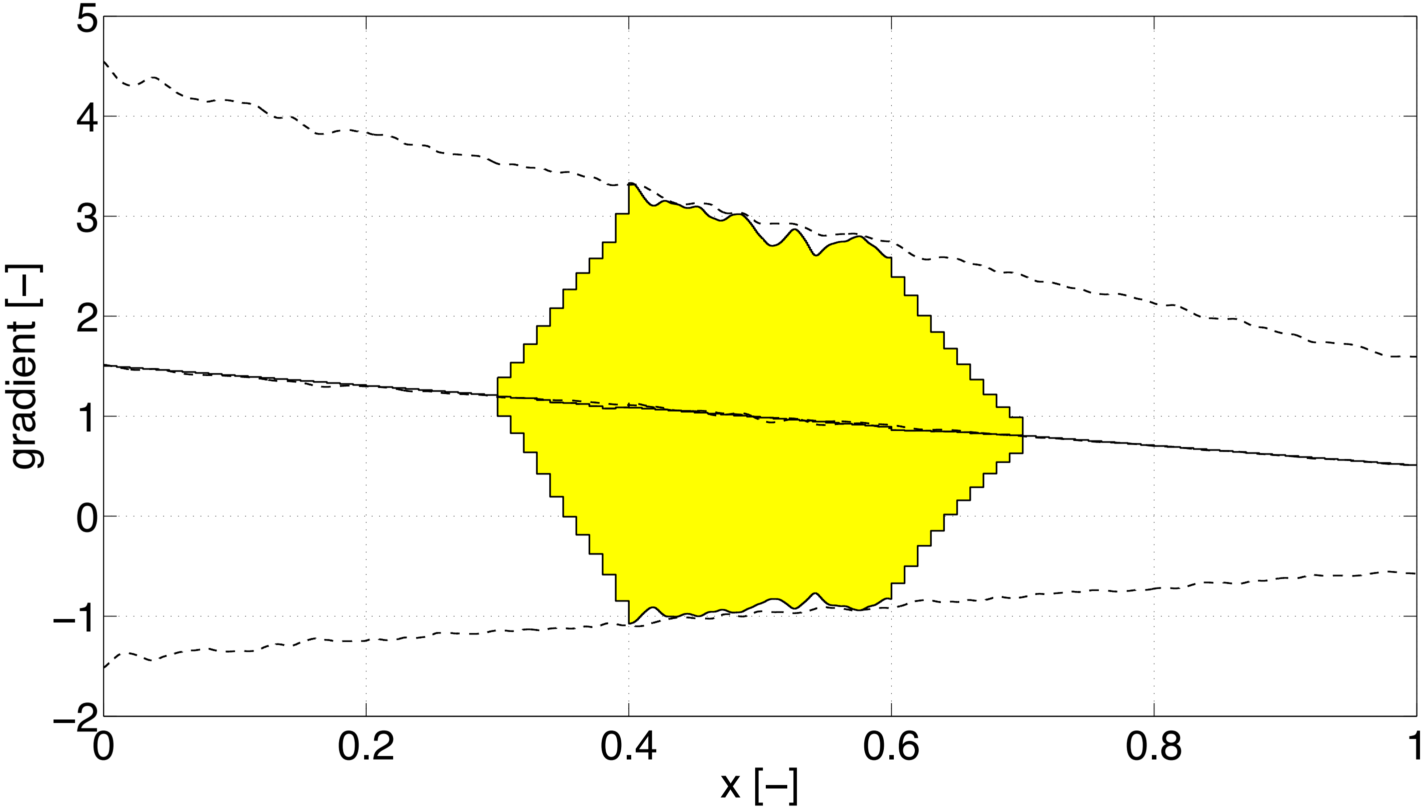
$$K_d^{-1} = \mathrm{E}\left[K_s^{-1}\right]$$

Solution : gradient of the displacement



Dashed black lines: mean and 90% confidence interval with a full stochastic monomodel

Solution : gradient of the displacement

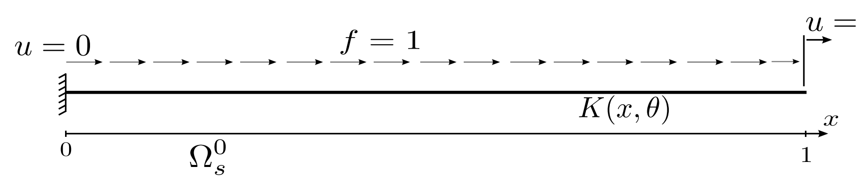


Continued black lines: deterministic solution, and mean of the stochastic one

Yellow zone: 90% confidence interval (representation of the fluctuation)

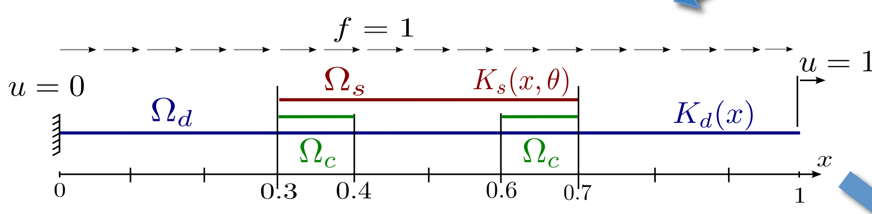
Dashed black lines: mean and 90% confidence interval with a full stochastic monomodel

3. Error estimation



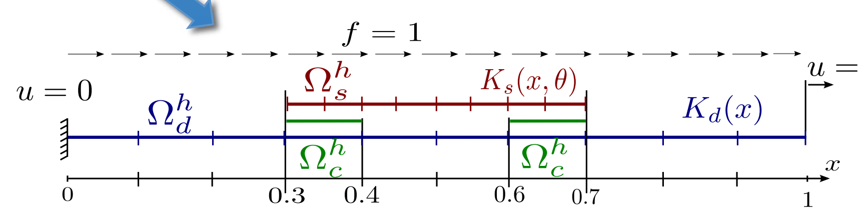
Modeling, reducing

Modeling error



Numerical approximation

Numerical error



Interest:

- ✓ Control the quality of the solution
- ✓ Drive an adaptive model

Goal-oriented error estimation [Prudhomme 1999] [Oden 2001]

- Local quantity of interest: $q(u)$
  - Mean of the displacement of a point
  - Local average of the standard deviation of the stress field

- Use of global error estimation with Extraction techniques:

  Example with the mathematical expectation of a displacement

$$q(u) = \mathrm{E}\left[u(x = x_m)\right]$$
$$q(u) = \mathrm{E}\left[\int_\Omega \delta(x - x_m)\, u\, d\Omega\right]$$

- Related error

$$\eta = q(u^{ex}) - q(u^0)$$

- Parameters : L_s, h_d, h_s

Definition of the adjoint problem

- Primal reference problem (1)

$$\text{Find } u \in \mathcal{V} \text{ such that:}\quad a(u,v) = \ell(v), \qquad \forall v \in \mathcal{V}$$

- Adjoint problem if q is linear:

$$\text{Find } p \in \mathcal{V} \text{ such that:}\quad a(v,p) = q(v), \qquad \forall v \in \mathcal{V}$$

$$q(u) = \mathrm{E}\left[\int_\Omega \delta(x - x_m)\, u \, d\Omega\right]$$

- The adjoint problem is still defined on the reference model.

Approximation of the adjoint problem

Using the Arlequin method

Primal:

Find $(u_d, \boldsymbol{u_s}, \boldsymbol{\lambda}) \in \mathcal{V}_d \times \mathcal{W}_s \times \mathcal{W}_c$ such that:

$$\left\{\begin{array}{rcll} a_d(u_d, v) + C(\boldsymbol{\lambda}, v) & = & \ell_d(v), & \forall v \in \mathcal{V}_d \\ \boldsymbol{a_s}(\boldsymbol{u_s}, \boldsymbol{v}) - C(\boldsymbol{\lambda}, \boldsymbol{v}) & = & \ell_s(\boldsymbol{v}), & \forall \boldsymbol{v} \in \mathcal{W}_s \\ C(\boldsymbol{\mu}, \Pi u_d - \boldsymbol{u_s}) & = & 0, & \forall \boldsymbol{\mu} \in \mathcal{W}_c \end{array}\right.$$

Adjoint with q linear:

Find $(\tilde{p}_{u_d}, \tilde{\boldsymbol{p}}_{\boldsymbol{u_s}}, \tilde{\boldsymbol{p}}_{\boldsymbol{\lambda}}) \in \tilde{\mathcal{V}}_d \times \tilde{\mathcal{W}}_s \times \tilde{\mathcal{W}}_c$ such that:

$$\left\{\begin{array}{rcll} a_d(v, \tilde{p}_{u_d}) + C(v, \tilde{\boldsymbol{p}}_{\boldsymbol{\lambda}}) & = & q_d(v), & \forall v \in \mathcal{V}_d \\ \boldsymbol{a_s}(\boldsymbol{v}, \tilde{\boldsymbol{p}}_{\boldsymbol{u_s}}) - C(\boldsymbol{v}, \tilde{\boldsymbol{p}}_{\boldsymbol{\lambda}}) & = & q_s(\boldsymbol{v}), & \forall v \in \mathcal{W}_s \\ C(\Pi \tilde{p}_{u_d} - \tilde{\boldsymbol{p}}_{\boldsymbol{u_s}}, \boldsymbol{\mu}) & = & 0, & \forall \boldsymbol{\mu} \in \mathcal{W}_c \end{array}\right.$$

Quality control by estimation of the global error

Error estimation

- Estimation of the error for linear quantity of interest:

$$\eta = q(u^{ex}) - q(u^0) = \mathscr{R}(u,p) \approx \mathscr{R}(u,\tilde{p})$$

Where the residual $\mathscr{R} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is defined by:

$$\mathscr{R}(u,v) = \ell(v) - a(u,v)$$

and where u and $\tilde{p}$ are projections of (u_d, u_s, λ) and of $(\tilde{p}_{u_d}, \tilde{p}_{u_s}, \tilde{p}_\lambda)$ in $\mathcal{V}$ respectively.

For instance:
[Prudhomme 2008]

$$u = \begin{cases} u_d & \text{in } \Omega_d \setminus \Omega_s \\ u_s & \text{in } \Omega_s \end{cases}$$

Error sources

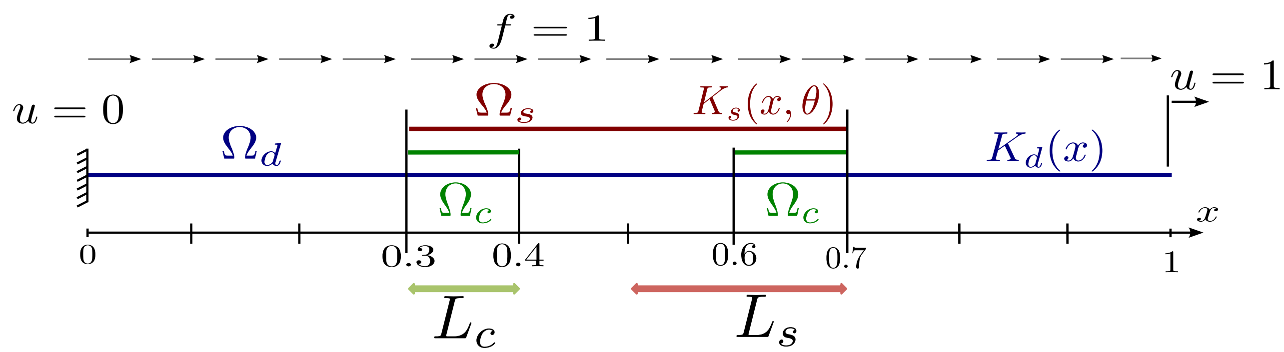
- Several error sources:
  - ✓ Modeling error (Arlequin and Stochastic homogenization)
  - ✓ Spatial discretization error (FEM)
  - ✓ Stochastic discretization (Truncation of the Monte Carlo method)
- Introducing intermediate models
  - ➢ Continuum deterministic-stochastic Arlequin model (arl)
  - ➢ Arlequin model only discretized in space (arl^h)
  - ➢ Arlequin model discretized in space and using Monte Carlo ($arl^{h\theta}$)
- Decomposition of the error:

$$\begin{aligned}\eta &= q(u^{ex}) - q(u^0)\\ &= \underbrace{(q(u^{ex}) - q(u^{arl}))}_{\eta^m} + \underbrace{(q(u^{arl}) - q(u^h))}_{\eta^h} + \underbrace{(q(u^h) - q(u^0))}_{\eta^\theta}\end{aligned}$$

modeling error — discretization error — stochastic error

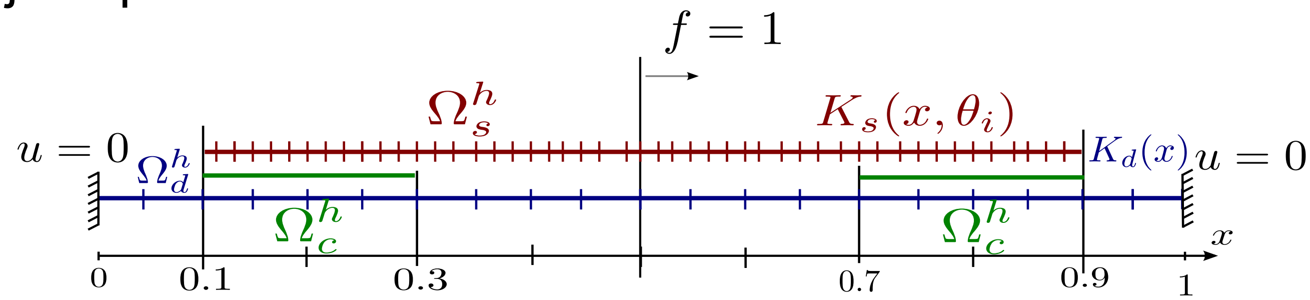
4. Example of adaptive coupling

- Approximated model (primal)

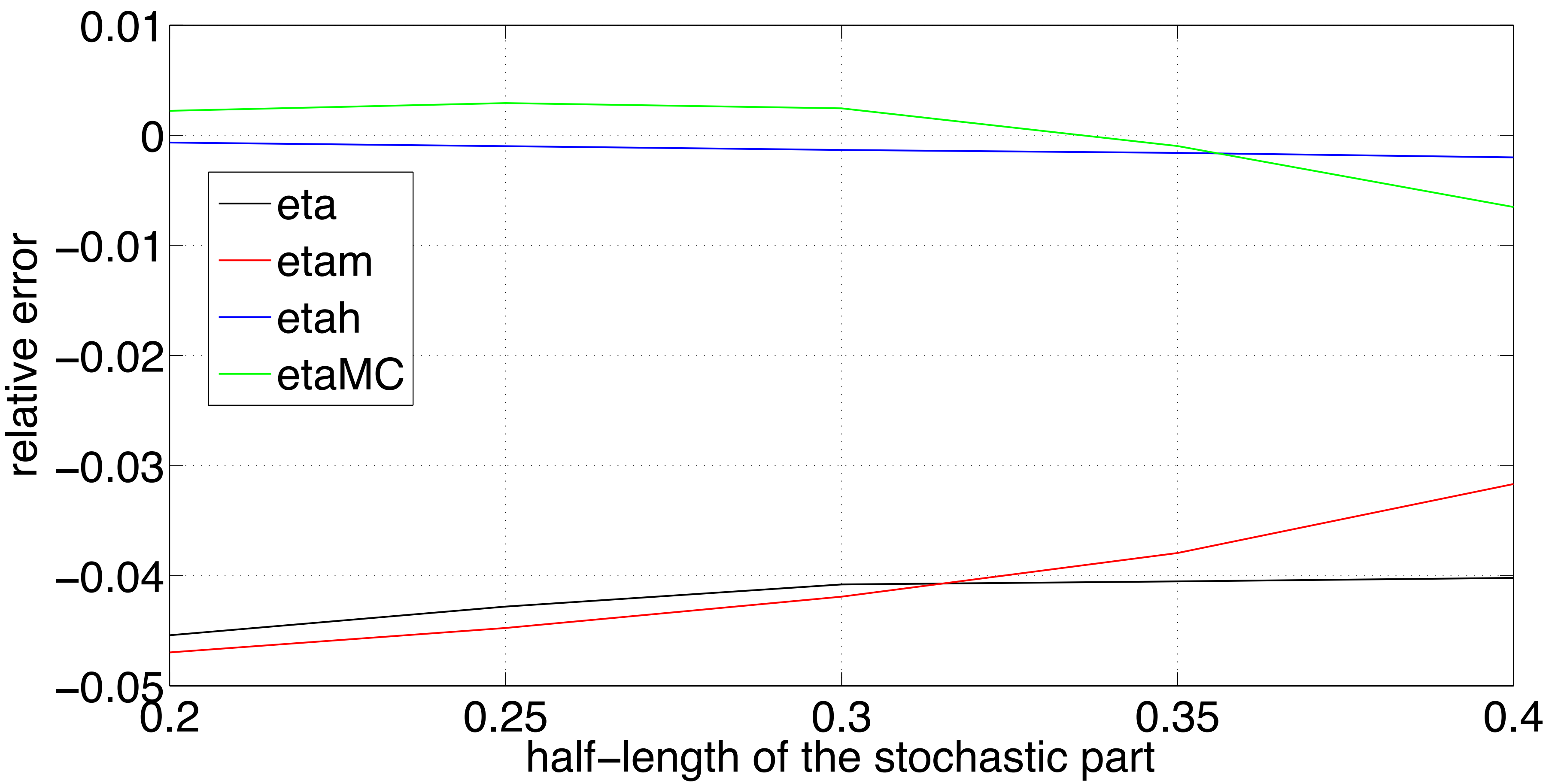


Numerical model : discretization by FEM and Monte Carlo techniques
h_d h_s nMC

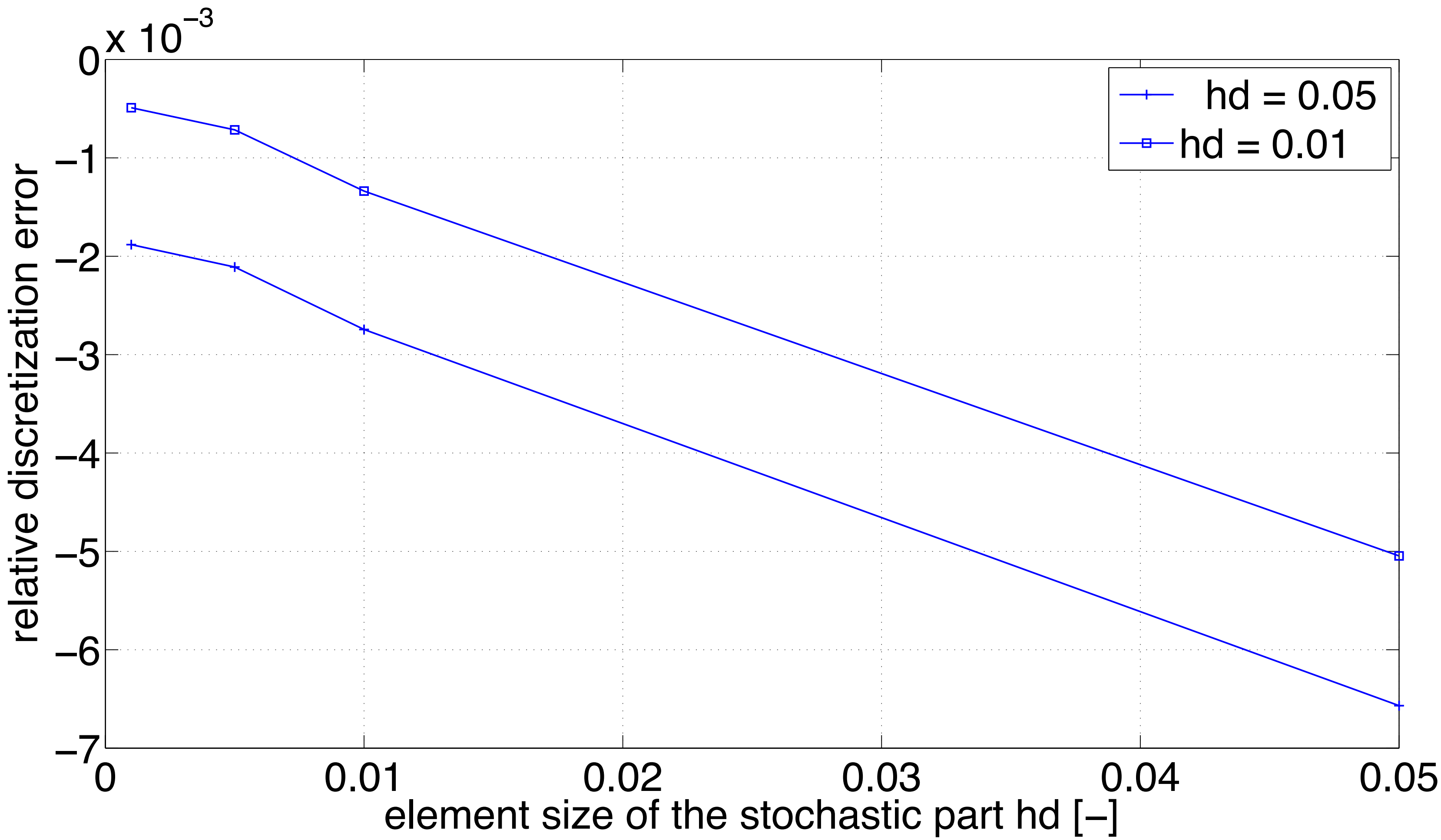
- Adjoint problem



Evolution of the modeling error for different L_s

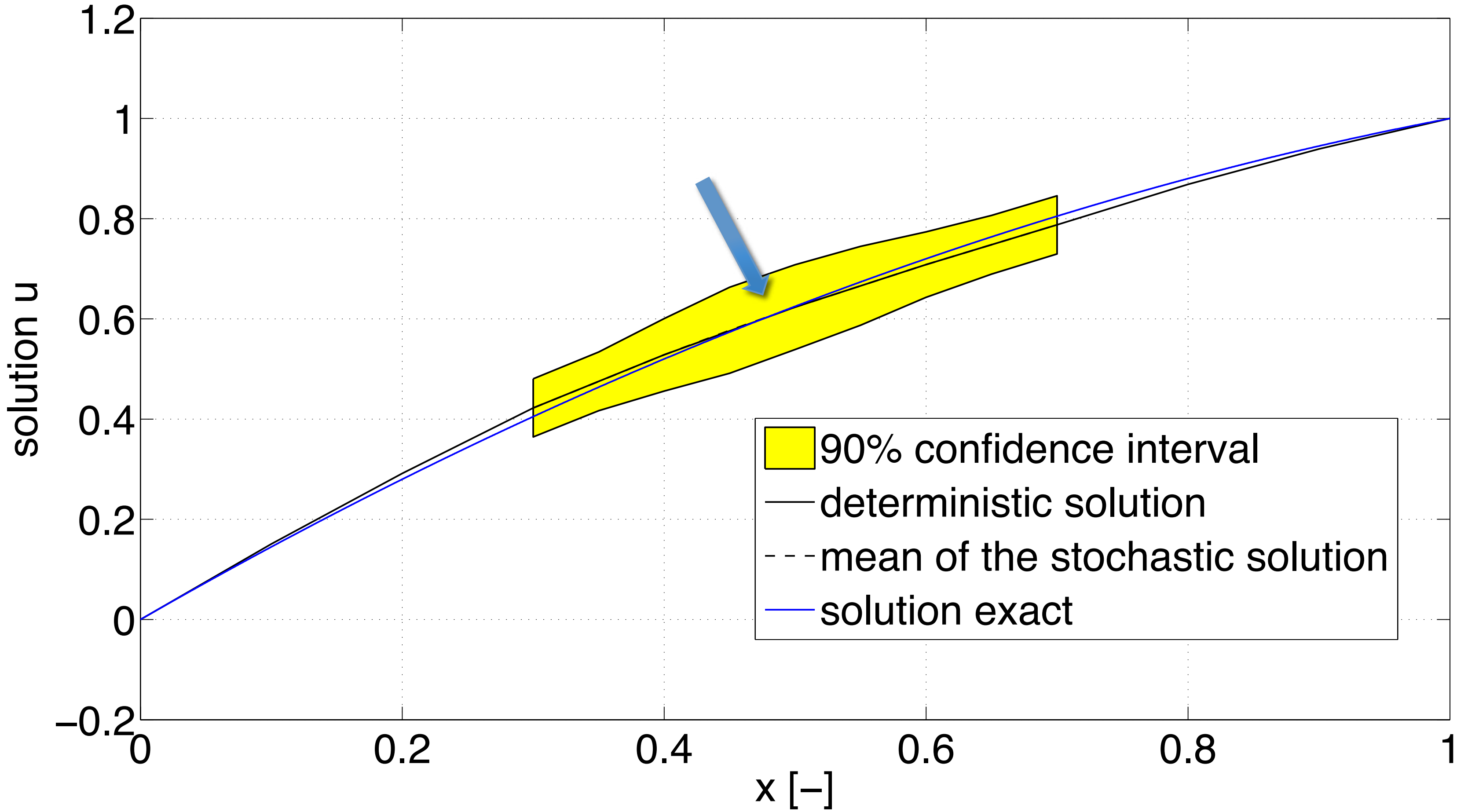


Evolution of the discretization error for different h_d, h_s



Remark about the example

Errors on the example are relatively small



Conclusion and extensions

Outcomes

✓ Efficient deterministic - stochastic coupling

✓ Goal oriented error estimation with a linear quantity of interest

✓ Error sources

Outlines

➢ Other intermediate problems (sto-sto,semi-discretized…)

➢ Importance of the projection into the reference admissible space

➢ Extension to the coupling of particular stochastic model with a deterministic continuum one

Thank you for your attention

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