Global sensitivity analysis for models with correlated input parameters

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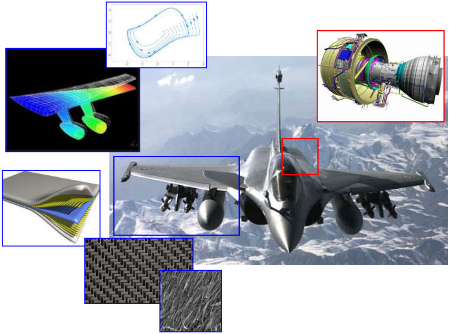
Context

- Modern engineering makes a massive use of numerical simulations (CAE, FE-models) for the robust design of complex systems.
- The structure can be decomposed into *submodels* representing a component, a scale of modelling or a physical discipline.
- The *computational workflow* consists in an imbrication of submodels.



Global sensitivity analysis for robust engineering

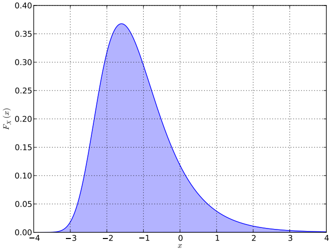
Robust design optimization

Uncertainties that may affect the system response, such as environmental loading, material properties or manufacturing tolerances, are taken into account.

Uncertainty quantification

Parameters of the system are modelled by a *random vector* $\boldsymbol{X} \sim f_{\boldsymbol{X}}(\boldsymbol{x})$.

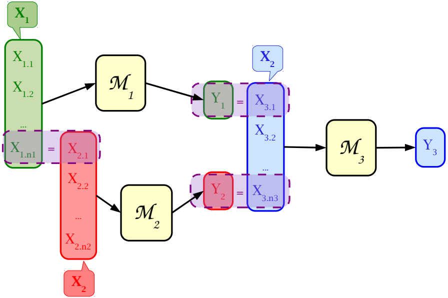
The corresponding parameter of interest $Y = \mathcal{M}(\boldsymbol{X})$ is studied.



Sensitivity analysis aims at identifying the important design variables through sensitivity measures :

- *Importance factors* in reliability analysis (FORM/SORM),
- *Sobol' indices* are derived from the decomposition of the variance of the system output.

Issues of nested modelling



- Of interest is often the sensitivity of the last level parameter, or *final performance*, to output parameters from the previous levels.
- The joint distribution of these parameters, *margins + copula* (correlation), is implicitly defined by uncertainty propagation.

> There is a need to develop sensitivity analysis tools for complex computational workflows.

Outline

Outline

The ANOVA decomposition

[Sobol' (1993)]

Let us first consider an independent input random vector $\boldsymbol{X}$ of dimension n and a variable of interest Y defined by $Y = \mathcal{M}(\boldsymbol{X})$. The model $\mathcal{M}$ can be uniquely decomposed by :

$$\begin{aligned} Y &= \mathcal{M}(\boldsymbol{X}) \\ &= \mathcal{M}_0 + \sum_{i=1}^{n} \mathcal{M}_i(X_i) + \sum_{1 \leqslant i < j \leqslant n} \mathcal{M}_{i,j}(X_i, X_j) + \ldots + \mathcal{M}_{1,\ldots,n}(X_1, \ldots, X_n) \\ &= \sum_{u \subseteq \{1,\ldots,n\}} \mathcal{M}_u(\boldsymbol{X}_u) \end{aligned}$$

where the summands have zero mean, are orthogonal and where the output variance can be written as :

$$\mathbb{V}[Y] = \sum_{u \subseteq \{1,\ldots,n\}} \mathbb{V}[\mathcal{M}_u(\boldsymbol{X}_u)]$$

Sobol' first order and total indices for the subset of variables $\boldsymbol{X}_u$ are defined by :

$$S_{1u} = \frac{\mathbb{V}[\mathbb{E}[Y|\boldsymbol{X}_u]]}{\mathbb{V}[Y]} \quad \text{and} \quad S_{Tu} = 1 - \frac{\mathbb{V}[\mathbb{E}[Y|\boldsymbol{X}_{\bar{u}}]]}{\mathbb{V}[Y]}$$

Estimation of the sensitivity indices

[Janon et al. (2012)]

Let us denote by $Y^u = \mathcal{M}(\boldsymbol{X}_u, \boldsymbol{X}'_{\bar{u}})$ where $\boldsymbol{X}'_{\bar{u}}$ is an independent copy of $\boldsymbol{X}_{\bar{u}}$. The first order Sobol' indices can also be written :

$$S_{1u} = \frac{\mathrm{Cov}\left[Y, Y^u\right]}{\mathbb{V}\left[Y\right]}$$

We now consider N-samples $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Y}^u$ of X, Y and Y^u. Sobol' first order indices can be estimated by :

$$S_u^N = \frac{\frac{1}{N}\sum y_i {y_i}^u - \left(\frac{1}{N}\sum y_i\right)\left(\frac{1}{N}\sum {y_i}^u\right)}{\frac{1}{N}\sum {y_i}^2 - \left(\frac{1}{N}\sum y_i\right)^2} \qquad (1)$$

or more efficiently by :

$$T_u^N = \frac{\frac{1}{N}\sum y_i {y_i}^u - \left(\frac{1}{N}\sum\left[\frac{y_i + {y_i}^u}{2}\right]\right)^2}{\frac{1}{N}\sum\left[\frac{{y_i}^2 + {y_i}^{u2}}{2}\right] - \left(\frac{1}{N}\sum\left[\frac{y_i + {y_i}^u}{2}\right]\right)^2} \qquad (2)$$

Confidence interval on sensitivity indices [Martinez (2011)]

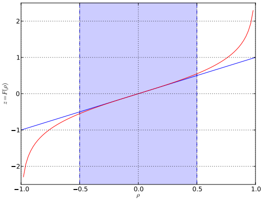
- The *Fisher transform* :

  The Fisher transformation of a correlation coefficient from a two Gaussian random variables N-sample $\boldsymbol{X}$ is given by :

$$z = \frac{1}{2}\ln\frac{1+\hat{\rho}}{1-\hat{\rho}} = \text{artanh}(\hat{\rho}), \quad \text{with} \quad \hat{\rho} = \frac{\text{Cov}\left[X_1, X_2\right]}{\sigma_1\sigma_2}$$

  and z as the following asymptotic behaviour :

$$z \sim \mathcal{N}\left(\frac{1}{2}\ln\frac{1+\rho}{1-\rho}, \frac{1}{\sqrt{N-3}}\right)$$



  By considering the sensitivity index S_{1u} as a linear correlation coefficient $\rho\left(\mathcal{M}(\boldsymbol{X}_u, \boldsymbol{X}_{\bar{u}}), \mathcal{M}(\boldsymbol{X}_u, \boldsymbol{X}'_{\bar{u}})\right)$, a α-confidence interval is given by :

$$\left[\tanh\left(\frac{1}{2}\ln\frac{1+\hat{S}_u}{1-\hat{S}_u} + \frac{\Phi^{-1}\left(\frac{\alpha}{2}\right)}{\sqrt{N-3}}\right), \tanh\left(\frac{1}{2}\ln\frac{1+\hat{S}_u}{1-\hat{S}_u} + \frac{\Phi^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{N-3}}\right)\right]$$

Polynomial chaos expansion

[Ghanem and Spanos (2003)]

- *Limitation* :

  Although estimators are efficient, they still require a large number ($N = 10^3 - 10^4$) of model evaluations for an accurate estimation of the indices

  $\longrightarrow$ *This is hardly achievable when a coupling procedure with a FE-model is performed.*

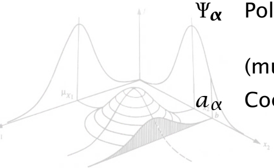
- *Polynomial chaos expansion* :

$$Y = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$$

where :

$\Psi_{\boldsymbol{\alpha}}$ Polynomial chaos basis, *e.g.* $\Psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) = \prod_{i=1}^{n} He_{\alpha_i}(\xi_i)$ (multivariate Hermite polynomials) for Gaussian variables

a_{α} Coefficients to be determined



Polynomial chaos expansion

- *In practice* :

  One only retains the polynomials $\Psi_{\boldsymbol{\alpha}}$ whose total degree is less than p :

$$Y \approx \sum_{|\boldsymbol{\alpha}| \leqslant \in p} a_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$$

- *Computation of the coefficients by regression* :

$$Y = \sum_{j=0}^{P-1} a_j \, \Psi_j(\boldsymbol{X}) + \epsilon_P \equiv \boldsymbol{a}^T \boldsymbol{\Psi}(\boldsymbol{X}) + \epsilon_P, \quad P = \binom{n+p}{p}$$

  Can be solve by least square regression :

$$\hat{\boldsymbol{a}} = \text{argmin} \frac{1}{N} \sum_{i=1}^{N} \left(\mathcal{M}(\boldsymbol{x}^{(i)}) - \boldsymbol{a}^T \boldsymbol{\Psi}(\boldsymbol{x}^{(i)}) \right)^2, \quad \boldsymbol{a} \in \mathbb{R}^P$$

Y is characterized by the coefficients $\hat{\boldsymbol{a}} = \{a_0, \ldots, a_{P-1}\}^T$ of the expansion.

PCE-based Sobol' indices (1)

[Sudret (2008)]

- *Polynomial chaos approximation* :

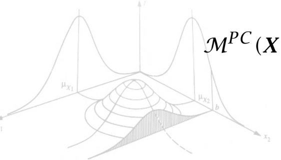
$$\mathcal{M}(\boldsymbol{X}) \approx \mathcal{M}^{PC}(\boldsymbol{X}) = \sum_{|\boldsymbol{\alpha}| \leq p} a_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) \tag{3}$$

- *Multi-indices notation* :

$\mathcal{I}_{i_1,\dots,i_s}$ such as only the indices $(i_1, \dots, i_s)$ are non zero :

$$\mathcal{I}_{i_1,\dots,i_s} = \left\{ \boldsymbol{\alpha} : \begin{array}{lll} \alpha_k > 0 & \forall k = 1, \dots, M & k \in (i_1, \dots, i_s) \\ \alpha_k = 0 & \forall k = 1, \dots, M & k \notin (i_1, \dots, i_s) \end{array} \right\} \tag{4}$$

Summands can be obtained by gathering functions of the subset of variables $\mathcal{I}_{i_1,\dots,is}$:



$$\mathcal{M}^{PC}(\boldsymbol{X}) = a_0 + \sum_{i=1}^{n} \sum_{\boldsymbol{\alpha} \in \mathcal{I}_i} a_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(X_i) + \ldots + \sum_{\boldsymbol{\alpha} \in \mathcal{I}_{1,2,\dots,n}} a_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) \tag{5}$$

PCE-based Sobol' indices (2)

[Sudret (2008)]

Summands of the expansion $\mathcal{M}^{PC}(\boldsymbol{X})$ can be compared to those from the Sobol' decomposition :

$$\mathcal{M}_{i_1,\dots,i_s}\left(x_{i_1},\dots,x_{i_s}\right) = \sum_{\boldsymbol{\alpha}\in\mathcal{I}_{i_1,\dots,i_s}} a_{\boldsymbol{\alpha}}\Psi_{\boldsymbol{\alpha}}\left(x_{i_1},\dots,x_{i_s}\right) \tag{6}$$

PCE-based Sobol' first order and total indices are given by :

$$SU_{i_1,\dots,i_s} = \frac{1}{\sigma_Y^{2,PC}} \sum_{\boldsymbol{\alpha}\in\mathcal{I}_{i_1,\dots,i_s}} a_{\boldsymbol{\alpha}}^2 = \frac{\sum_{\boldsymbol{\alpha}\in\mathcal{I}_{i_1,\dots,i_s}} a_{\boldsymbol{\alpha}}^2}{\sum_{0<|\boldsymbol{\alpha}|\leq p} a_{\boldsymbol{\alpha}}^2} \tag{7}$$

and

$$SU_{j_1,\dots,j_t}^T = \sum_{(i_1,\dots,i_s)\subset(j_1,\dots,j_t)} SU_{i_1,\dots,i_s} \tag{8}$$

Sobol' indices are given by the coefficients of the expansion without any extra numerically expensive simulations.

Outline

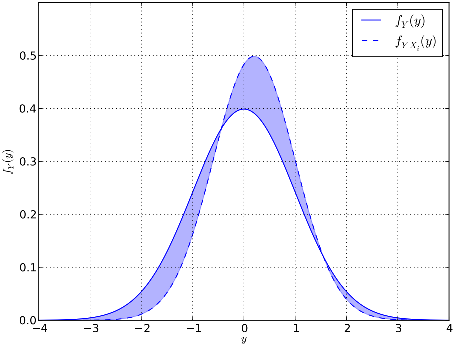
The δ-sensitivity measure

[Borgonovo (2007)]

- *Principle* :

  If the model output $Y = \mathcal{M}(\boldsymbol{X})$ is affected by a input variable X_i, then the conditional distribution $f_{Y|X_i}(y)$ significantly differs from $f_Y(y)$.

- *The δ-sensitivity measure* :



Let us define the *shift* (blue area left) between the two distributions by :

$$s(X_i) = \int_{D_Y} \left| f_Y(y) - f_{Y|X_i}(y) \right| dy.$$

We define by δ_i the following index :

$$\begin{aligned}\delta_i &= \frac{1}{2} \mathbb{E}\left[s(X_i) \right] \\ &= \frac{1}{2} \int_{D_{X_i}} \left[\int_{D_Y} \left| f_Y(y) \right. \right. \\ &\quad \left. \left. - f_{Y|X_i}(y) \right| dy \right] f_{X_i}(X_i) dx_i\end{aligned}$$

Computational aspects

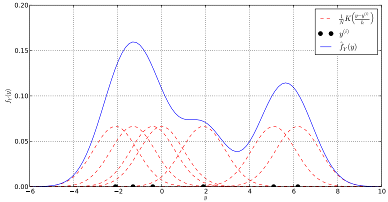
[Wand and Jones (1995)]

- *Kernel smoothing estimation* for the distributions :

Approximation of the PDF using a N-sample :

$$\hat{f}_Y(y) = \frac{1}{h}\sum_{i=1}^{N} K\left(\frac{y - y^{(i)}}{h}\right), \quad K(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right)$$

where K is the Gaussian kernel and $h = \left(\frac{4}{3}\frac{\hat{\sigma}}{N}\right)^{\frac{1}{5}}$ its optimal bandwidth.



- *Double quadrature loop* for the integrals :

1. $s(X_i) = \int_{D_Y} \left| f_Y(y) - f_{Y|X_i}(y) \right| dy \approx \sum_{q_1=1}^{n_Q} \omega_{q_1} \left| f_Y(y_{q_1}) - f_{Y|X_i}(y_{q_1}) \right|$
2. $\mathbb{E}\left[s(X_i)\right] = \int_{D_{X_i}} s(X_i) f_{x_i}(x_i) dx_i \approx \sum_{q_2=1}^{n_Q} \omega_{q_2} s(x_{i,q_2})$

δ-sensitivity measure based on the CDFs [Borgonovo (2011)]

δ-measures require large samples ($N = 10^5$) for an accurate estimation of the densities.

- *Alternative computation scheme* :

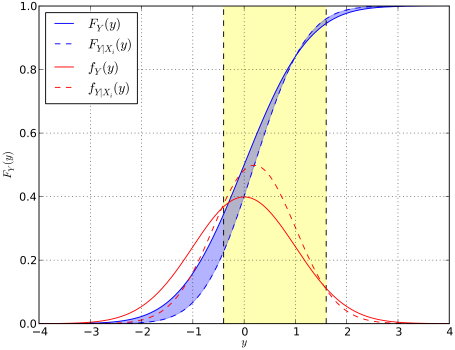
$$s(X_i) = 2\mathbb{P}_Y\left(F_Y(y) > F_{Y|X_i}(y)\right) - 2\mathbb{P}_Y\left(F_Y(y) < F_{Y|X_i}(y)\right)$$

That is :

$$\delta_i = \mathbb{E}\left[F_Y(y_1) - F_{Y|X_i}(y_2) - F_Y(y_2) - F_{Y|X_i}(y_1)\right]$$

where $y1$, $y2$ are the solutions of :

$$f_Y(y) - f_{Y|X_i}(y) = 0, \quad y_1 < y_2.$$



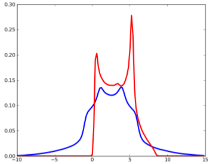
Test case : the Ishigami function

The *Ishigami function* is defined by :

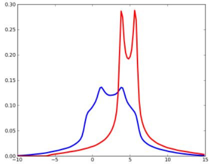
$$Y = \sin(X_1) + a\,\sin(X_2)^2 + b\,X_3^4 \sin(X_1)$$

with $a = 7$, $b = 0.1$ and $X_i \sim \mathcal{U}[-\pi, \pi]$, $i = 1, 2, 3$. We use $N = 10^5$ because of the shape of the output PDF.

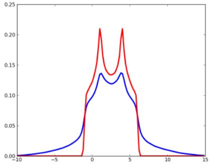
Parameters	δ_i	$S1_i$	ST_i
X_1	0.23	0.40	0.71
X_2	0.37	0.29	0.29
X_3	0.10	0.00	0.31
Total	0.70	0.69	1.31



$x_1 = 0.328,\ s(x_1) = 0.506$



$x_2 = -1.353,\ s(x_2) = 0.826$



$x_3 = 0.792,\ s(x_3) = 0.406$

The ANCOVA decomposition

[Li and Rabitz (2010)]

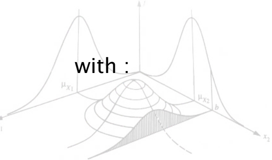
ANCOVA stands for **AN**alysis of **COVA**riance.

By definition :

$$\begin{aligned}\mathbb{V}[Y] &= \mathrm{Cov}[Y, Y] \\ &= \mathrm{Cov}\left[Y, \sum_{i=1}^{n} \mathcal{M}_i(X_i) + \sum_{1 \leqslant i < j \leqslant n} \mathcal{M}_{i,j}(X_i, X_j) + \ldots + \mathcal{M}_{1,\ldots,n}(\boldsymbol{X})\right] \\ &= \mathrm{Cov}\left[Y, \sum_{u \subseteq \{1,\ldots,n\}} \mathcal{M}_u(\boldsymbol{X}_u)\right]\end{aligned}$$

It requires a functional decomposition of the model. Fortunately, one is given by the *polynomial chaos expansion*!

$$Y = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} a_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X})$$



with :

$$\sum_{\boldsymbol{\alpha} \in u} a_{\boldsymbol{\alpha}} \, \Psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) \equiv \mathcal{M}_u(\boldsymbol{X}_u)$$

Sensivity indices for dependent variables [Li and Rabitz (2010)]

Given :

$$\mathbb{V}[Y] = \mathrm{Cov}\left[Y, \sum_{u \subseteq \{1,\dots,n\}} \mathcal{M}_u(\boldsymbol{X}_u)\right]$$

We define a *triplet of sensitivity indices* :

$$S_u = \frac{\mathrm{Cov}[Y, \mathcal{M}_u(\boldsymbol{X}_u)]}{\mathbb{V}[Y]} \approx \frac{\sum_{i=1}^N (y^{(i)} - \bar{y})(\mathcal{M}_u(\boldsymbol{x}_u^{(i)}))}{\sum_{i=1}^N (y^{(i)} - \bar{y})^2}$$

$$S_u^S = \frac{\mathbb{V}[\mathcal{M}_u(\boldsymbol{X}_u)]}{\mathbb{V}[Y]} \approx \frac{\sum_{i=1}^N (\mathcal{M}_u(\boldsymbol{x}_u^{(i)}))^2}{\sum_{i=1}^N (y^{(i)} - \bar{y})^2}$$

$$S_u^C = \frac{\mathrm{Cov}\left[\sum_{v \subseteq \{1,\dots,n\}, u \cap v = \{0\}} \mathcal{M}_v(X_v), \mathcal{M}_u(\boldsymbol{X}_u)\right]}{\mathbb{V}[Y]} \approx \frac{\sum_{i=1}^N (\sum_v \mathcal{M}_v(\boldsymbol{x}_v^{(i)}))(\mathcal{M}_u(\boldsymbol{x}_u^{(i)}))}{\sum_{i=1}^N (y^{(i)} - \bar{y})^2}$$

- S_u : *total* contribution of $\boldsymbol{X}_u$ to $\mathbb{V}[Y]$
- S_u^S : *structural* contribution of $\boldsymbol{X}_u$ $\mathbb{V}[Y]$
- S_u^C : *correlative* contribution of $\boldsymbol{X}_u$ $\mathbb{V}[Y]$

$$S_u = S_u^S + S_u^C$$

A didactic example : model

Let us examine the following polynomial function :

$$Y = \mathcal{M}(\boldsymbol{X}) = X_1 + X_2 + X_2^2 + X_1 X_2 + 3$$

with $X_i \sim \mathcal{N}(0,1)$, $i = 1,2$.

Choosing $p = 2$, the *functional decomposition* reads :

$$\mathcal{M}(\boldsymbol{x}) = \mathcal{M}_0 + \mathcal{M}_1(x_1) + \mathcal{M}_2(x_2) + \mathcal{M}_{1,2}(x_1,x_2)$$

with :

$$\begin{array}{rcccl} \mathcal{M}_0 & = & 4 \times \Psi_{0,0}(x_1,x_2) & = & 4 \\ \mathcal{M}_1(x_1) & = & 1 \times \Psi_{1,0}(x_1,x_2) & = & x_1 \\ \mathcal{M}_2(x_2) & = & 1 \times \Psi_{0,1}(x_1,x_2) + \sqrt{2} \times \Psi_{0,2}(x_1,x_2) & = & x_2 + x_2^2 - 1 \\ \mathcal{M}_{1,2}(x_1,x_2) & = & 1 \times \Psi_{1,1}(x_1,x_2) & = & x_1 x_2 \end{array}$$

A didactic example : independent variables

Let us first consider X_1, X_2 as *independent variables*, *i.e.* $F_{\boldsymbol{X}}(\boldsymbol{x}) = F_{X_1}(x_1)F_{X_2}(x_2)$.
Sobol' PCE-based indices are :

$$
\begin{array}{lll}
S_{1_1} = 0.2, & S_{T_1} = 0.4 & (S_{1,2} = 0.2) \\
S_{1_2} = 0.6, & S_{T_2} = 0.8 & (S_{2,1} = 0.2)
\end{array}
$$

$\mathcal{X}$ and $\mathcal{Y}$ are N-samples used for the evaluation of the moments. Example with $N = 1000$:

$$S_1 \approx \frac{\sum_{i=1}^{N} (x_1^{(i)})(y^{(i)} - \bar{y})}{\sum_{i=1}^{N} (y^{(i)} - \bar{y})^2}, \quad S_1^S \approx \frac{\sum_{i=1}^{N} (x_1^{(i)})^2}{\sum_{i=1}^{N} (y^{(i)} - \bar{y})^2}, \quad S_1^C \approx S_1 - S_1^S$$

$\rho_S = 0.2$	S_i	S_i^S	S_i^C
X_1	0.20	0.20	0.00
X_2	0.60	0.60	0.00
$X_{1,2}$	0.20	0.20	0.00
$\sum$	1.00	1.00	0.00

$\longrightarrow$ *Structural contributions are equal to the Sobol' indices.*
$\longrightarrow$ *Correlative contributions are zero.*

A didactic example : correlated variables

Let us now consider X_1, X_2 as *correlated variables* with the following rank correlation matrix :

$$\boldsymbol{S} = \begin{bmatrix} 1 & \rho_S \\ \rho_S & 1 \end{bmatrix}$$

We now have $F_{\boldsymbol{X}}(\boldsymbol{x}) = C_{\boldsymbol{R}}(F_{X_1}(x_1), F_{X_2}(x_2))$, $R_{ij} = 2\sin\left(\frac{\pi}{6} S_{ij}\right)$, where $C_{\boldsymbol{R}}$ is the Gaussian copula of (X_1, X_2).

$\rho_S = 0.2$	S_i	S_i^S	S_i^C
X_1	0.19	0.17	0.02
X_2	0.59	0.51	0.08
$X_{1,2}$	0.22	0.18	0.04
$\sum$	1.00	0.86	0.14

$\rho_S = 0.8$	S_i	S_i^S	S_i^C
X_1	0.19	0.10	0.09
X_2	0.52	0.29	0.23
$X_{1,2}$	0.29	0.14	0.15
$\sum$	1.00	0.53	0.47

→ Correlative contributions increase with the correlation whereas structural contributions decrease.

Pros and cons

δ-measure

Advantages $+$ **:**

- moment-free
- no hypothesis on the dependence structure

Drawbacks $-$ **:**

- non-unity sum
- require accurate approximation of the conditional distributions
- PDF : 2-loop integration, CDF : 1-loop integration

ANCOVA decomposition

- consistent with Sobol' indices (**I**)
- a triplet of indices (S_i, S_i^S, S_i^C)
- separate structural and correlative contributions

- require a functional decomposition of the model
- complex to interpret
- 3 (at least 2) indices to compute

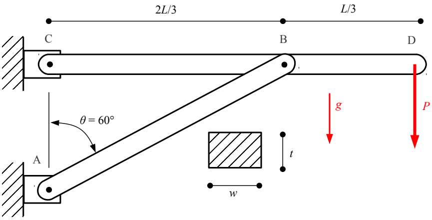
We now have two alternative methods for global sensitivity analysis for models with correlated input parameters.

Outline

Bracket Structure : presentation

[Dubourg (2011)]

- *Definition and probabilistic model* :



Parameter	Unity	Distribution	μ	$\frac{\sigma}{\mu}$
P	kN	Gumbel	100	10%
ρ	kg.m^{-3}	Weibull	7860	10%
L	m	Normal	5	5%
ω_{CD}	mm	Normal	125	10%
t	mm	Normal	250	10%

Bracket Structure : model

[Dubourg (2011)]

- The output of interest is the *bending stress* σ_B reading :

$$\sigma_B = \frac{6M_B}{\omega_{CD} t^2} \quad \text{with} \quad M_B = \frac{PL}{3} + \frac{\rho g \omega_{CD} t L^2}{18}$$

- Computing parameters : $p = 7$ $(Q^2 = 1.000)$, $N = 10^4$.

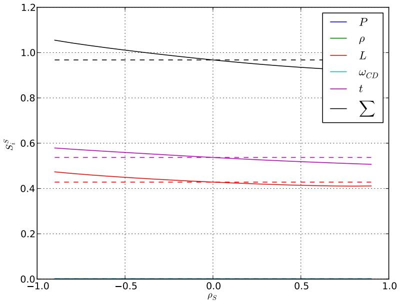
$$\boldsymbol{S} = \begin{array}{c} P \\ \rho \\ L \\ \omega_{CD} \\ t \end{array} \overset{\begin{array}{ccccc} P & \rho & L & \omega_{CD} & t \end{array}}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.8 \\ 0 & 0 & 0 & 0.8 & 1 \end{pmatrix}}$$

Parameter	S_i	S_i^S	S_i^C	δ_i
P	0.00	0.00	0.00	0.02
ρ	0.00	0.00	0.00	0.01
L	0.42	0.42	0.00	0.27
ω_{CD}	0.02	0.00	0.02	0.02
t	0.56	0.54	0.02	0.33
$\sum$	1.00	0.96	0.04	0.65

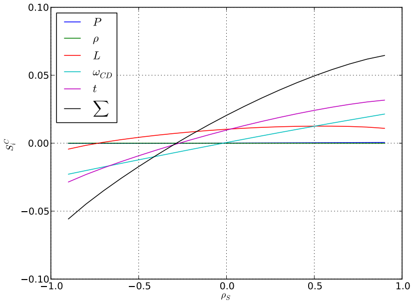
Bracket Structure : influence of ρ_S

We now study the influence of $\rho_S(\omega_{CD}, t)$ on the structural and correlative contributions.

Structural contributions



Correlative contributions



In this case :

→ *Correlative contributions of ω_{CD} and t strongly depends on ρ_S.*
→ *They are negative when $\rho_S < 0$ and positive when $\rho_S > 0$.*

Conclusion

General remarks on GSA :

1. Sensitivity indices represent the contribution of one (or more) input parameter on the variance of the model output.
2. *Polynomial chaos expansions* are a powerful metamodelling tool to deal with time-demanding models.
3. Sobol' indices can be directly computed from the coefficients of the expansion.

Correlated input parameters :

1. The nested modelling of complex structures involves correlation between the parameters.
2. Classical variance-based methods cannot be applied due to the hypothesis of independence.
3. A *moment-free method* has been presented to circumvent the issue of correlated input parameters.
4. The so-called *ANCOVA decomposition* reprensents a generalization of the ANOVA decomposition.
5. Polynomial chaos expansions offer a *functional decomposition* of the model.

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End of the presentation

Thank you for your attention.