# Density modification based reliability sensitivity indices

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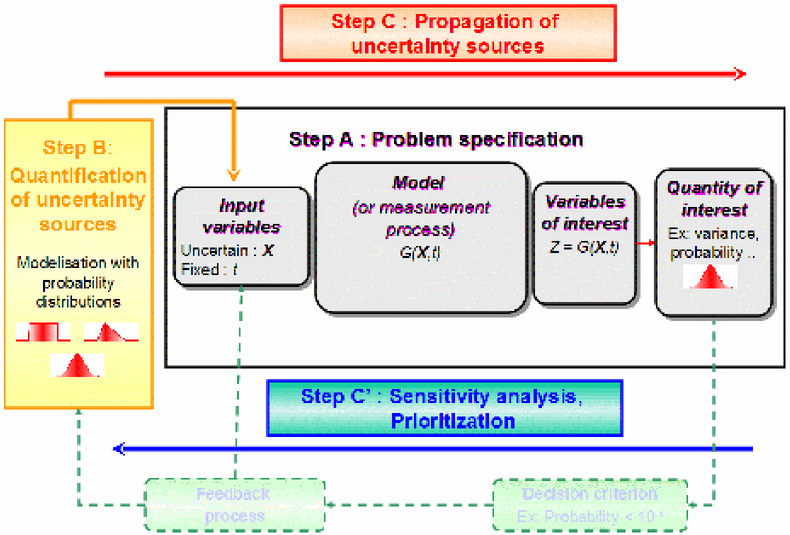
# Table of contents

# Plan de la présentation

# General formalism



# Step A

## Problem specification

- We study a **deterministic numerical model** denoted $G : \mathbb{R}^d \to \mathbb{R}$.
- The **uncertain inputs** are denoted $\mathbf{X}$ which is a $d$-dimensional random variable - of joint pdf $f_{\mathbf{X}}$.
  - We hereby consider independant inputs of marginal densities $f_{X_i}$ for $i = 1..d$.
- **Fixed inputs** are denoted $t$.
- The **uncertain output** is denoted $Z = G(\mathbf{X})$ and is a random variable.
- The **quantity of interest** is a function of $Z$ or of its pdf.
  - We will have a specific interest for binary output (reliability context). We consider the event $G(\mathbf{x}) < 0$ (system failure) and the complementary event $G(\mathbf{x}) \geq 0$ (system safe mode).
  - We therefore want to know the failure probability,
    $$P = \int 1_{G(\mathbf{x})<0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

# Step B and Step C

## Quantification & Propagation of uncertainty sources

### Step B

- Modelisation of the uncertainty sources
- Collaboration with the experts to fit the pdfs $f_{X_i}$

### Step C

- Propagation of uncertainty sources, by crude MC methods for instance
- Specific propagation methods in the case of a binary output

# Step C'

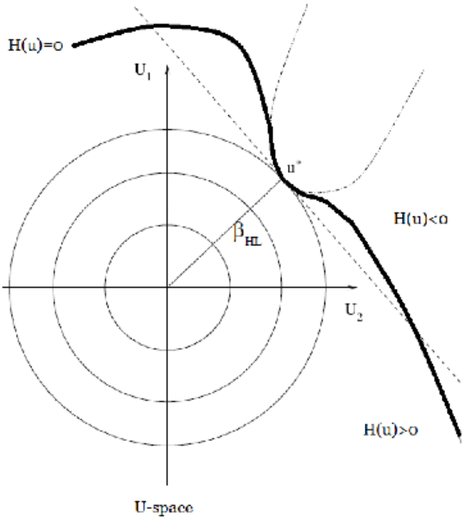
## Sensitivity Analysis (SA)

- The goal is to **explain** the **output uncertainties** regarding the **input uncertainties**.
- The uses can be :
  - **model simplification** by fixing non-influent inputs to a reference value
  - **variables ranking** for research priorization
  - **detection of interactions** between inputs.
- There is a **wide range** of SA techniques, regarding what type of problem the exprimenter is facing.
- Most SA methods focuses on **real-valued** numerical model.

# Former Work

## FORM/SORM Methods

- A way to perform step C and C' at the same time
- Based upon space transformation and minimization algorithm
- A hyperplane (FORM) or quadratic (SORM) approximation is fitted on the design point $u^*$
- The impact of every variable : its coordinates in the $U-$space
- Advantages : not costly
- Drawbacks : no error control, insufficient SA



# PhD objectives

## PhD Objectives:

- Identify the influence of each input on $P$
- With constraints :
  - *Estimate a very small failure probability*
  - *Make as few model call as possible*
  - *Provide an estimation of the error done*

# Plan de la présentation

# Density perturbation

## Idea beneath the SA method

- Proposal : for each input $X_i$ a perturbed input $X_{i\tau}$ with density "close" to the original one, controlled by a parameter $\tau$.
- Output $P_{i\tau}$ is computed : the probability of failure given that the $i-$th component of the input vector is $X_{i\tau}$.
- If $P_{i\tau}$ differs a lot from $P$ then the $i-$th input is influential.
- If $P_{i\tau}$ is close to $P$ the $i-$th input has a weak influence.
- 2 settings :
  - modify the density $X_{i\tau}$
  - make a sensitivity index from $P$ and $P_{i\tau}$.
- The idea of "close" leads to the choice of Kullback-Leibler divergence to control the range of the modification

# Density perturbation

## First draw

- The perturbed density is choosen in a given family :

$$f_{X_{i\tau}}(x) = \exp(\tau x - \psi(\tau)) f_{X_i}(x)$$

  - $\tau$ is a real parameter
  - $\psi(\tau)$ is a normalisation function, that is $\psi(\tau) = \log \int_{-\infty}^{+\infty} \exp(\tau t) f_{X_i}(t) dt$
  - $\psi(\tau)$ does not exists for every distribution
  - The idea beneath this transformation is that it corresponds to a mean shift for gaussian densities

- For a positive number $\gamma$ :

$$KL(f_{X_{i\tau}}, f_{X_i}) = \delta$$

  - $\tau_1^*$ and $\tau_2^*$ are the two solution of $\delta = \mathbb{E}[X_i]\tau - \psi(\tau)$
  - $\tau_1^*$ and $\tau_2^*$ are of opposite signs

# Density perturbation

## General modification

- Now nothing is assumed on $f_{X_{i\tau}}$.
- We set a collection of $K$ linear contraints written as :

$$\int g_k(x) f_{X_{i\tau}}(x) dx = 1 + \delta_k$$

  - where $g_k(x)$ is a constraint function and $\delta_k$ is a real value
- The optimal density is choosen:

$$f^*_{X_{i\tau}} = \underset{f_{X_{i\tau}}}{\operatorname{argmin}} KL(f_{X_{i\tau}}, f_{X_i})$$

- The information theory states that :

$$f^*_{X_{i\tau}}(x) = f_{X_i}(x) \exp\left(\sum_{k=1}^{K} \lambda_k g_k(x) - \psi(\lambda_1, ..., \lambda_K)\right)$$

  - where $\lambda_k$ are Lagrange multipliers and $\psi(\lambda_1, ..., \lambda_K)$ a regularization function

# Density perturbation

## Summary

- First approach :
  - the perturbed density is set in a specific family so that it corresponds to a mean shift for exponential laws
  - the optimal density is choosen by controlling the KL divergence
- General approach :
  - the wanted contraints are first set
  - the optimal density is the argmin of the KL divergence
- Advantages of a general modification :
  - contraints can be more general/relevant
  - relative contraints make more sense when comparing the impact of different inputs distribution
  - relative contraints are easier to explain to physics experts than KL divergence constraint

# Sensitivity index

## Index & Estimator

- Sensitivity index : a function of $P$ and $P_{i\tau}$
- Choice :

$$S_{i\tau} = \frac{P_{i\tau}}{P}\mathbf{1}_{P_{i\tau} > P} - \frac{P}{P_{i\tau}}\mathbf{1}_{P_{i\tau} < P}$$

- $P$ is estimated with MC method :

$$\widehat{P} = \frac{1}{N}\sum_{n=1}^{N} 1_{G(\mathbf{x}^n)<0} \qquad \text{with } \mathbf{x}^n \text{ i.i.d. to } f_{\mathbf{X}}$$

- $P_{i\tau}$ is estimated with the same function calls :

$$\widehat{P_{i\tau}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{G(\mathbf{x}^n)<0} \frac{f_{X_{i\tau}}(x_i^n)}{f_{X_i}(x_i^n)} \qquad \text{with } \mathbf{x}^n \text{ i.i.d. to } f_{\mathbf{X}}$$

# Sensitivity index

## Estimator properties

- The estimator of the sensitivity index is :

$$\widehat{S_{i\tau}} = \frac{\widehat{P_{i\tau}}}{\widehat{P}} \mathbf{1}_{\widehat{P_{i\tau}} > \widehat{P}} - \frac{\widehat{P}}{\widehat{P_{i\tau}}} \mathbf{1}_{\widehat{P_{i\tau}} < \widehat{P}}$$

- $\text{Cov}(\widehat{P}, \widehat{P_{i\tau}}) \neq 0$
  - $\mathbb{E}_{f_X}\left[\widehat{S_{i\tau}}\right]$ and $\text{Var}_{f_X}\left[\widehat{S_{i\tau}}\right]$ are not easy to express
- Using a Taylor approximation :

$$\widehat{S_{i\tau}} = S_{i\tau} + \frac{P\widehat{P_{i\tau}} - \widehat{P}P_{i\tau}}{P^2} \mathbf{1}_{P_{i\delta} > P} - \frac{\widehat{P}P_{i\tau} - P\widehat{P_{i\tau}}}{P_{i\tau}^2} \mathbf{1}_{P_{i\delta} < P} + R(\widehat{P_{i\tau}}, \widehat{P})$$

  - where $R(\widehat{P_{i\tau}}, \widehat{P})$ is a second order remainder

# Plan de la présentation

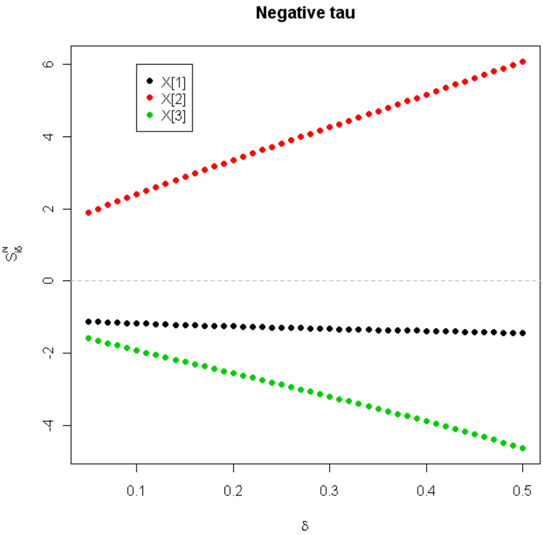
- First approach : KL divergence between the perturbed density and the original one is set
- 3 inputs, $f_{X_i} \sim \mathcal{N}(0,1)$ for $i = 1..3$
- Failure function :

$$G(\mathbf{x}) = \sum_{i=1}^{3} a_i x_i - k$$

  - with parameters $k = 16$ and $\mathbf{a} = (1, -6, 4)$
  - $P = 0.014$

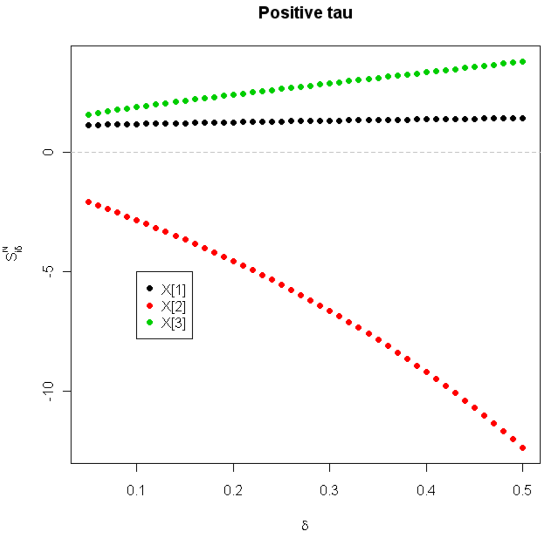
# Hyperplane

## Indices



# Hyperplane

## Indices



# Ishigami function

## Presentation

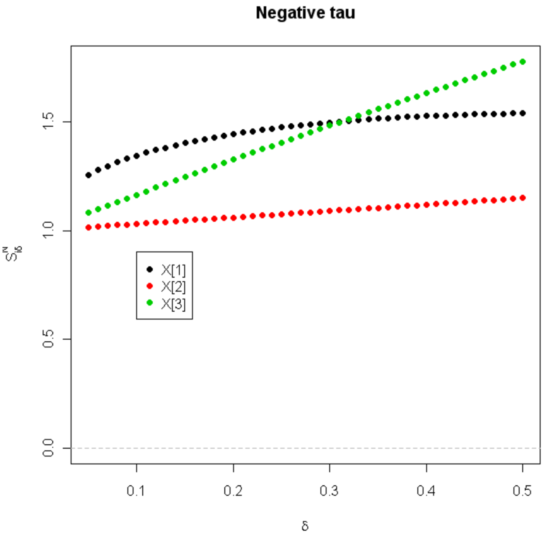
- First approach
- 3 inputs, $f_{X_i} \sim \mathcal{U}(-\pi, \pi)$ for $i = 1..3$
- Failure function :

$$G(\mathbf{x}) = \sin(x_1) + 7\sin(x_2)^2 + 0.1x_3^4\sin(x_1) + k$$

  - with parameter $k = 7$ so that $P = 5.9 * 10^{-3}$

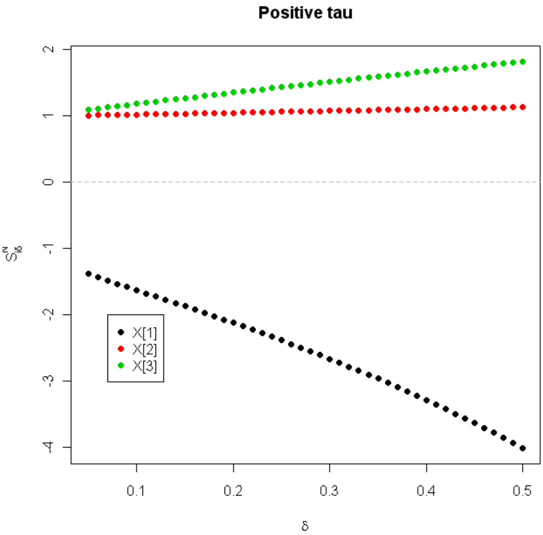
# Ishigami function

## Indices



# Ishigami function

## Indices



# Plan de la présentation

# Conclusion & Perspective

- A new approach for reliability sensitivity indices :
  - minimizing function calls
  - that can be used with importance sampling
- Several experiments yet to make to :
  - assess general method in the case of differents inputs
  - find general constraints
  - control the error made on the estimation