Variance-based sensitivity analysis using harmonic analysis

Jean-Yves TISSOT
(Université de Grenoble/INRIA)

PhD supervisors: Clémentine PRIEUR and Éric BLAYO

Outline

Motivation

Background

FAST and RBD revisited

Cubature error in FAST

Bias in RBD

Numerical application

Rectangle rule in dimension 1

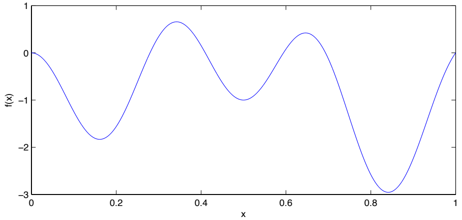
$$c_0(f) = \int_0^1 f(x)dx = \ ?$$

- $RR(f,n) = \frac{1}{n}\sum_{i=0}^{n-1} f(x_i)$

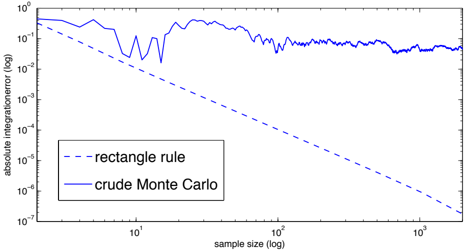
  with $x_i = \frac{i}{n}$ (equispaced nodes)

- $MC(f,n) = \frac{1}{n}\sum_{i=0}^{n-1} f(X_i)$

  with $X_i \sim \mathcal{U}([0,1])$, indep.

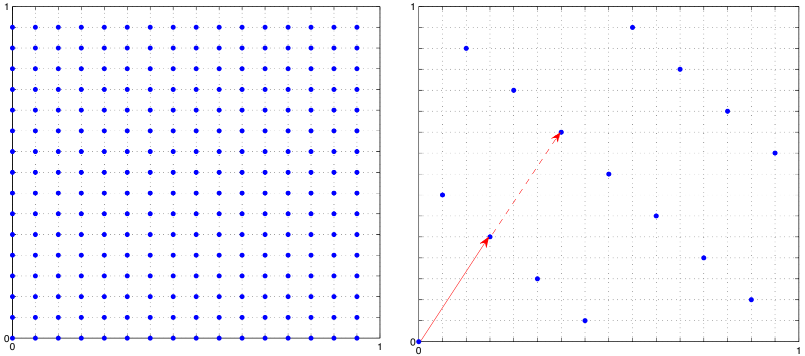


(a) smooth function f



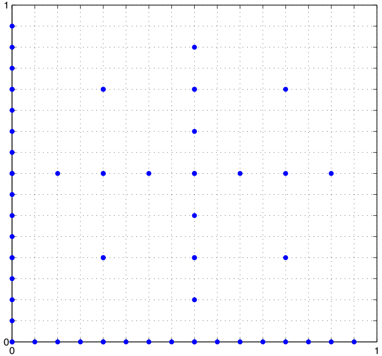
(b) absolute integration error (log/log)

How to generalize the design of experiments with $d > 1$...

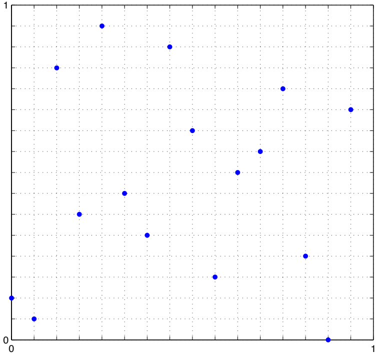


(a) regular grid (b) finite group of the torus $\mathbb{T}^2$

How to generalize the design of experiments with $d > 1$...



(c) sparse grid



(d) orthogonal array

Orthogonal arrays: a very quick introduction

Consider $A = \mathrm{OA}(n, d, q, t)$

n: number of points

d: number of factors

q: number of levels, here $l_1 = 0$, $l_2 = \frac{1}{q}, \ldots, l_q = \frac{q-1}{q}$

i.e. $A \subseteq \left\{0, \frac{1}{q}, \ldots, \frac{q-1}{q}\right\}^d$ and $|A| = n$

t: strength, $t \in \{0, 1, \ldots, d\}$

Definition

A has strength t if each of its $n \times t$ submatrices contains each t-uple in $\left\{0, \frac{1}{q}, \ldots, \frac{q-1}{q}\right\}^t$ the same number of times.

$\Longrightarrow$ "every projection of A onto any t variables is a q-levels regular grid"

So, what are we interested in?

Two existing methods of estimating variance-based sensitivity indices (SI)

- Fourier Amplitude Sensitivity Test (FAST): Cukier, Schaibly, Shuler, Levine... (4 papers in the 1970's), Saltelli et al. (1999)
- Random Balance Designs method (RBD): Tarantola et al. (2006)

Links to existing numerical integration theories

- lattice rules (discrete Fourier transform (DFT) on finite subgroups of the torus), mountains of papers and books beginning from the 1960's
- randomized orthogonal arrays sampling, Patterson (1954), Owen (1994)

Generalizations, improvements, error analysis...?

Tissot J.Y., Prieur C., Variance-based sensitivity analysis using harmonic analysis (*submitted*).
Preprint available at http://hal.archives-ouvertes.fr/hal-00680725

Background: harmonic ANOVA

$X_1, \ldots, X_d$ indep. random variables

f such that $\mathbb{E}\left[f(\mathbf{X})^2\right] < +\infty$

- Hoeffding decomposition:

$$f(\mathbf{X}) = \sum_{\mathfrak{u} \subseteq \{1,\ldots,d\}} f_{\mathfrak{u}}(\mathbf{X}_{\mathfrak{u}})$$

where $f_\emptyset = \mathbb{E}\left[f(\mathbf{X})\right]$ and
$\forall \mathfrak{v} \subset \mathfrak{u},\ \mathbb{E}\left[f_{\mathfrak{u}}(\mathbf{X}_{\mathfrak{u}}) \middle| \mathbf{X}_{\mathfrak{v}}\right] = 0.$

- ANOVA decomposition:

$$Var\left[f(\mathbf{X})\right] = \sum_{\mathfrak{u} \subseteq \{1,\ldots,d\}} Var\left[f_{\mathfrak{u}}(\mathbf{X}_{\mathfrak{u}})\right]$$

- Variance-based SI:

$$S_{\mathfrak{u}}(f, \mathbf{X}) := \frac{Var\left[f_{\mathfrak{u}}(\mathbf{X}_{\mathfrak{u}})\right]}{Var\left[f(\mathbf{X})\right]}$$

If $X_1, \ldots, X_d \sim \mathcal{U}([0,1])$

- Hoeffding decomposition:

$$\begin{aligned} f(\mathbf{X}) &= \sum_{\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{k}}(f) exp(2i\pi \mathbf{k} \cdot \mathbf{X}) \\ &= \sum_{\mathfrak{u} \subseteq \{1,\ldots,d\}} \underbrace{\sum_{\mathbf{k} \in \mathbb{Z}^d_{\mathfrak{u}}} c_{\mathbf{k}}(f) exp(2i\pi \mathbf{k} \cdot \mathbf{X})}_{f_{\mathfrak{u}}(\mathbf{X}_{\mathfrak{u}})} \end{aligned}$$

where $c_{\mathbf{k}}(f) = \mathbb{E}\left[f(\mathbf{X}) exp(-2i\pi \mathbf{k} \cdot \mathbf{X})\right]$ and

$\mathbb{Z}^d_{\mathfrak{u}} := \{\mathbf{k} \mid \forall i \in \mathfrak{u},\ k_i \in \mathbb{Z}^* \text{ and } \forall i \notin \mathfrak{u},\ k_i = 0\}$

- Parseval's identity $\Longrightarrow$

$$S_{\mathfrak{u}}(f) = \frac{V_{\mathfrak{u}}(f)}{V(f)} = \frac{\sum_{\mathbf{k} \in \mathbb{Z}^d_{\mathfrak{u}}} \left|c_{\mathbf{k}}(f)\right|^2}{\sum_{\mathbf{k} \in (\mathbb{Z}^d)^*} \left|c_{\mathbf{k}}(f)\right|^2}$$

Background: harmonic estimator of the SI's

Let D be a finite subset of $[0,1]^d$. Define

$$\widehat{c}_{\mathbf{k}}(f,D) := \frac{1}{|D|}\sum_{\mathbf{x}\in D} f(\mathbf{x})exp(-2i\pi\mathbf{k}\cdot\mathbf{x})$$

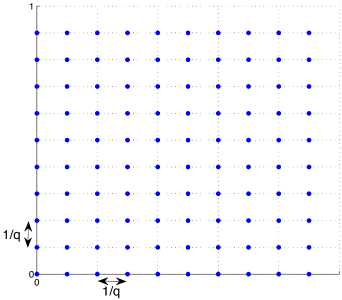
$$\widehat{V}_{\mathfrak{u}}(f,K_{\mathfrak{u}},D) := \sum_{\mathbf{k}\in K_{\mathfrak{u}}} |\widehat{c}_{\mathbf{k}}(f,D)|^2, \quad K_{\mathfrak{u}} \text{ is a finite truncation subset of } \mathbb{Z}_{\mathfrak{u}}^d$$

$$\widehat{V}(f,D) := \widehat{c}_{\mathbf{0}}(f^2,D) - \widehat{c}_{\mathbf{0}}(f,D)^2, \text{ note that } V(f) = c_{\mathbf{0}}(f^2) - c_{\mathbf{0}}(f)^2$$

$$\widehat{S}_{\mathfrak{u}}(f,K_{\mathfrak{u}},D) := \frac{\widehat{V}_{\mathfrak{u}}(f,K_{\mathfrak{u}},D)}{\widehat{V}(f,D)}$$

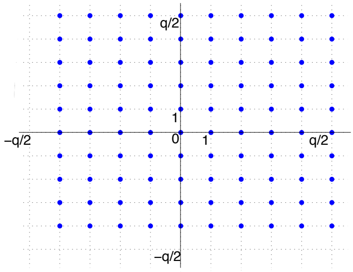
$\longrightarrow$ efficient design of experiments (DOE) D?

Background: a first example (multidimensional DFT and trigonometric interpolation)



(a) $D = \left\{0, \frac{1}{q}, \ldots, \frac{q-1}{q}\right\}^d$

(spatial domain)



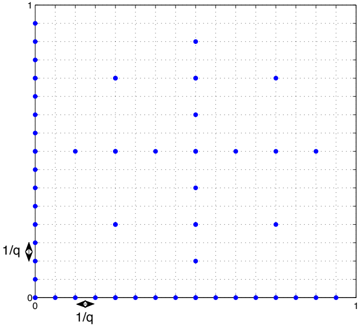
(b) $D^* = \mathbb{Z}^d \cap (-\frac{q}{2}, \frac{q}{2}]^d$

(frequency domain)

- $\forall \mathbf{k} \in D^*$, $\widehat{c}_{\mathbf{k}}(f, D)$ is the $\mathbf{k}$-th discrete complex Fourier coefficient
- $\widetilde{f}(\mathbf{x}) := \sum_{\mathbf{k} \in D^*} \widehat{c}_{\mathbf{k}}(f, D) exp(2i\pi \mathbf{k} \cdot \mathbf{x})$ is the trigo. interp. poly. (TIP) of f
- $\widehat{S}_{\mathfrak{u}}(f, D^* \cap \mathbb{Z}^d_{\mathfrak{u}}, D) = S_{\mathfrak{u}}(\widetilde{f})$ $\longrightarrow$ metamodel approach

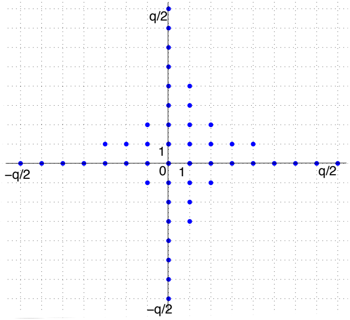
Background: remark on multidimensional DFT

- generally unfeasible (curse of dimensionality)
- possible generalization: Smolyak algorithm on hyperbolic crosses (i.e. interpolation on sparse grids) $\Longrightarrow$ become ill-conditioned as $q \uparrow$ and $d \uparrow$, Kämmerer & Kunis (2011)



(a) sparse grid

(spatial domain)



(b) hyperbolic cross

(frequency domain)

FAST revisited

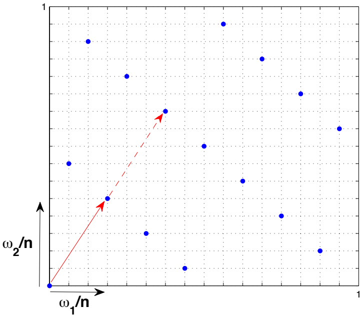
$$\widehat{S}_{\mathfrak{u}}^{FAST}(f, K_{\mathfrak{u}}, \mathbf{x}^*(\varphi, \boldsymbol{\omega})) = \widehat{S}_{\mathfrak{u}}((\mathcal{T}_{\varphi} \circ \mathcal{R}_1)f, K_{\mathfrak{u}}, G(\boldsymbol{\omega}))$$

- $\mathbf{x}^*(\varphi, \boldsymbol{\omega})$: DOE in classic FAST

  φ: random shift in $[0, 2\pi)^d$

  $\boldsymbol{\omega}$: "generator" of the DOE, $\in (\mathbb{N}^*)^d$
- $\mathcal{T}_{\varphi}$ and $\mathcal{R}_1$: linear operators on $L^2([0,1)^d)$
- $G(\boldsymbol{\omega})$: cyclic group with generator $\boldsymbol{\omega}/n$
- $\widehat{c}_{\mathbf{k}}(f, G(\boldsymbol{\omega})) = \dfrac{1}{n} \displaystyle\sum_{\mathbf{x} \in G(\boldsymbol{\omega})} f(\mathbf{x}) exp(-2i\pi \mathbf{k} \cdot \mathbf{x})$:

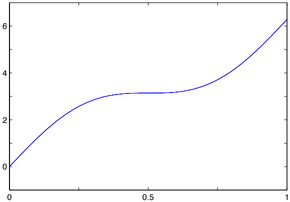
  DFT on cyclic group / rank-1 lattice rule



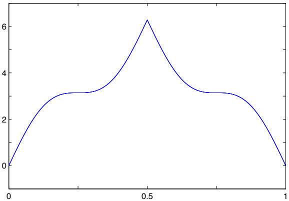
(a) DOE in revisited FAST

Shift and regularization operators on $L^2([0,1]^d)$

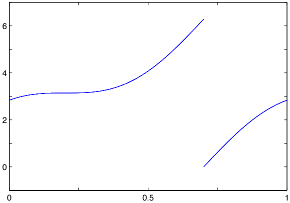
Define $\mathcal{R}_1$ and $\mathcal{T}_\varphi$, $\varphi \in [0,2\pi)^d$; and also $\mathcal{R}_p = \mathcal{R}_1 \circ \cdots \circ \mathcal{R}_1$ (p times)



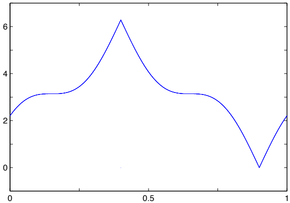
(b) Plot of $f : x \mapsto x + \sin(x)$



(c) Plot of $\mathcal{R}_1 f$



(d) Plot of $\mathcal{T}_{0.3} f$



(e) Plot of $(\mathcal{T}_{0.1} \circ \mathcal{R}_1) f$

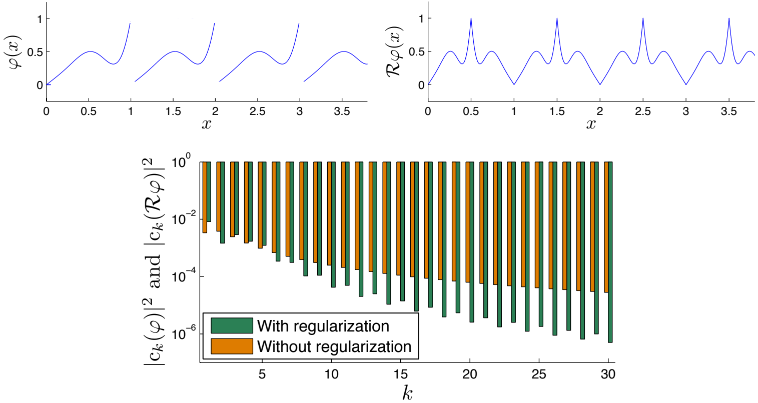
$\mathcal{R}_p$ and $\mathcal{T}_\varphi$ (properties)

- ANOVA decomposition is $\mathcal{R}_p$ and $\mathcal{T}_\varphi$-invariant

  $\longrightarrow \; S_{\mathfrak{u}}((\mathcal{T}_\varphi \circ \mathcal{R}_p)f) = S_{\mathfrak{u}}(f)$

- Riemann-Lebesgue lemma: $|c_{\mathbf{k}}(f)| \to 0$ as $||\mathbf{k}|| \to \infty$

  "the smoother the function, the faster the convergence"



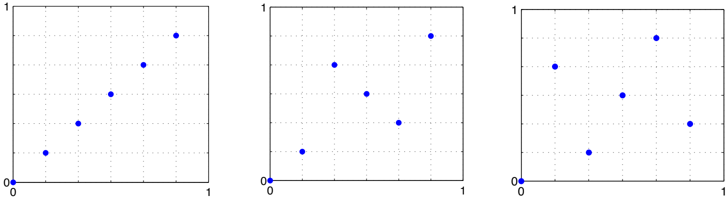
RBD revisited

$$\widehat{S}^{RBD}_{\{i\}}(f, K_{\{i\}}, \mathbf{x}^{\times}(\boldsymbol{\pi}, p)) = \widehat{S}_{\{i\}}((\mathcal{T}_{\tilde{\boldsymbol{p}}} \circ \mathcal{R}_p)f, pK_{\{i\}}, A(\boldsymbol{\pi}))$$

- $\mathbf{x}^{\times}(\boldsymbol{\pi}, p)$: DOE in classic RBD

  $p \in \mathbb{N}^*$ (generally set to 1) / $\boldsymbol{\pi}$: random permutation in $(\mathfrak{S}_n)^d$ /

  $\tilde{\boldsymbol{p}} = ((1-p)\pi/2p, \ldots, (1-p)\pi/2p)$
- $A = \left\{\left(\frac{i}{n}, \ldots, \frac{i}{n}\right),\ i = 0, \ldots, n-1\right\}$: OA$(n, d, n, 1)$
- $A(\boldsymbol{\pi})$: randomized orthogonal array based on A



(a) A with $n = 6$ (b) $A((35) \times ())$ (c) $A((35) \times (25)(36))$

Potential generalizations

(Operators $\mathcal{R}_p$ and $\mathcal{T}_\varphi$ are now omitted — p is set to 1 in RBD)

- $\widehat{S}_{\mathfrak{u}}(f, K_{\mathfrak{u}}, G(\boldsymbol{\omega}))$ with a group G of any rank $r \leq d$ not so easy in practice
- $\widehat{S}_i(f, K_{\{i\}}, A(\boldsymbol{\pi}))$ with a SI of any order: $\widehat{\mathrm{S}}_{\mathfrak{u}}(f, K_{\mathfrak{u}}, A(\boldsymbol{\pi}))$ OK
  $\longrightarrow$ already applied in Xu & Gertner (2011)
- $\widehat{S}_{\mathfrak{u}}(f, K_{\mathfrak{u}}, A(\boldsymbol{\pi}))$ with any orthogonal array $A =$OA(n, d, q, t) OK

Cubature error in FAST

Recall that $$\widehat{S}_{\mathfrak{u}}(f, K_{\mathfrak{u}}, G(\boldsymbol{\omega})) = \frac{\sum_{\mathbf{k}\in K_{\mathfrak{u}}} \left|\widehat{c}_{\mathbf{k}}(f, G(\boldsymbol{\omega}))\right|^2}{\widehat{c}_{\mathbf{0}}(f^2, G(\boldsymbol{\omega})) - \widehat{c}_{\mathbf{0}}(f, G(\boldsymbol{\omega}))^2}$$

- truncation error
- integration error (Poisson summation formula)

$$\widehat{c}_{\mathbf{k}}(f, G(\boldsymbol{\omega})) - c_{\mathbf{k}}(f) = \sum_{\mathbf{h}\in G(\boldsymbol{\omega})^{\perp}\setminus\{\mathbf{0}\}} c_{\mathbf{k}+\mathbf{h}}(f)$$

where $G(\boldsymbol{\omega})^{\perp} = \{\mathbf{h} \in \mathbb{Z}^d \mid \mathbf{h}\cdot\boldsymbol{\omega} \equiv 0 \pmod{n}\}$ is the infinite subgroup of $\mathbb{Z}^d$ orthogonal to G.

$\longrightarrow$ main issue: "constructing $G(\boldsymbol{\omega})$ which minimizes the error in some sense"

Classic approach: error minimization=avoiding interferences

- interference between $\mathbf{k}$ and $\mathbf{h}$ $\Longleftrightarrow \widehat{c}_{\mathbf{k}}(f, G(\omega)) = \widehat{c}_{\mathbf{h}}(f, G(\omega))$
- criterion to avoid the interference between $\mathbf{k}$ and $\mathbf{h}$:

$$\mathcal{P}(\mathbf{k}, \mathbf{h}, \omega) \;:\; (\mathbf{k} - \mathbf{h}) \cdot \omega \not\equiv 0 \pmod{n}$$

Proposition 1

Let $K = \cup_{\mathfrak{u} \neq \emptyset} K_{\mathfrak{u}}$ $(K_{\mathfrak{u}} \subseteq \mathbb{Z}^d_{\mathfrak{u}})$ *and* $G(\omega)$ *of order* n *such that*

$$\forall \mathbf{k}, \mathbf{h} \in K, \; \mathbf{k} \neq \mathbf{h}, \; \mathcal{P}(\mathbf{k}, \mathbf{h}, \omega) \text{ is true.}$$

If $n = |K|$ *then*

$\widetilde{f}_K(\mathbf{x}) = \sum_{\mathbf{k} \in K} \widehat{c}_{\mathbf{k}}(f, G(\omega)) exp(2i\pi \mathbf{k} \cdot \mathbf{x})$ *is the t.i.p. of* f *at the nodes* $\mathbf{g} \in G(\omega)$

and $\widehat{S}_{\mathfrak{u}}(f, K_{\mathfrak{u}}, G(\omega)) = S_{\mathfrak{u}}(\widetilde{f}_K) \longrightarrow$ *metamodel approach*

Remarks: (1) $|K| < n \Longrightarrow$ metamodel approach but weaker conclusion
(2) computational complexity $O(n^d)$ (basic exhaustive algorithm)

New approach: error minimization=achieving the (optimal) rate of convergence in a space of smooth functions

Weighted Korobov space $\mathcal{H}_{\alpha,\boldsymbol{\gamma},\boldsymbol{d}}$, $\alpha > 1$, $\gamma = (\gamma_{\mathfrak{u}}) \geq 0$ (RKHS)

- $f \in \mathcal{H}_{\alpha,\boldsymbol{\gamma},\boldsymbol{d}} \Longrightarrow |\mathrm{c}_{\mathbf{k}}(f)| \leq \dfrac{\gamma_{\mathfrak{u}_{\mathbf{k}}} ||f||_{\mathcal{H}_{\alpha,\boldsymbol{\gamma},\boldsymbol{d}}}}{\prod_{i \in \mathfrak{u}_{\mathbf{k}}} |k_i|^{\alpha/2}}$, $\quad (\mathfrak{u}_{\mathbf{k}} = \{i \mid k_i \neq 0\})$

Constructing $\boldsymbol{\omega}$ in rank-1 lattice rules

- $B^{opt}_{\alpha,\boldsymbol{\gamma},\boldsymbol{d}}(n) = \min\limits_{\boldsymbol{\omega} | o(G(\boldsymbol{\omega}))=n} \sup\limits_{||f||_{\mathcal{H}_{\alpha,\boldsymbol{\gamma},\boldsymbol{d}}} \leq 1} \left|\widehat{\mathrm{c}}_{\mathbf{0}}(f, G(\boldsymbol{\omega})) - \mathrm{c}_{\mathbf{0}}(f)\right|$
- objective: constructing $\boldsymbol{\omega}$ such that $G(\boldsymbol{\omega})$ achieves a nearly optimal rate of convergence $B_{\alpha,\boldsymbol{\gamma},\boldsymbol{d}}(n) \geq B^{opt}_{\alpha,\boldsymbol{\gamma},\boldsymbol{d}}(n)$
- Korobov-type construction $O(dn^2)$ (Korobov, 1960)

  component-by-component construction $O(d^2 n^2)$ (Sloan & Reztsov, 2002)

  fast CBC construction $O(dn\log(n))$ (using FFT) (Nuyens & Cools, 2004)

  fast CBC construction for embedded lattice rules $O(dn\log(n)^2)$ (Cools et al., 2006)

New approach (cont.)

Proposition 2

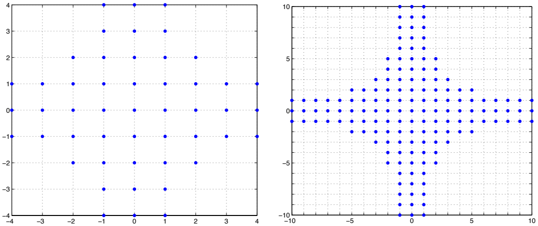
Let $f \in \mathcal{H}_{\alpha,\gamma,d}$ and $B_{\alpha,\gamma,d}(n)$ defined as previously. There exists $G(\boldsymbol{\omega})$ of order n such that:

- *if* $f^2 \in \mathcal{H}_{\alpha',\gamma',d}$, $\left|\widehat{V}\big(f, G(\boldsymbol{\omega})\big) - V(f)\right| = O\big(B_{\alpha,\gamma,d}(n)\big) + O\big(B_{\alpha',\gamma',d}(n)\big)$

and $\beta = \beta(n)$ such that

- *for* $|\mathfrak{u}| = 1$, $\left|\widehat{V}_{\mathfrak{u}}\big(f, \mathcal{Z}_{d,\beta} \cap \mathbb{Z}^d_{\mathfrak{u}}, G(\boldsymbol{\omega})\big) - V_{\mathfrak{u}}(f)\right| = O\big(B_{\alpha,\gamma,d}(n)^{1-1/\alpha}\big)$
- *for* $|\mathfrak{u}| = 2$, $\left|\widehat{V}_{\mathfrak{u}}\big(f, \mathcal{Z}_{d,\beta} \cap \mathbb{Z}^d_{\mathfrak{u}}, G(\boldsymbol{\omega})\big) - V_{\mathfrak{u}}(f)\right| =$
$$O\Big(\log\big(B_{\alpha,\gamma,d}(n)^{-1/\alpha}\big) B_{\alpha,\gamma,d}(n)^{1-1/\alpha}\Big)$$

where $\mathcal{Z}_{d,\beta} = \left\{\mathbf{k} \mid \prod_{i=1}^{d} \max(1, |k_i|) \leq \beta\right\}$: Zaremba cross (=hyperbolic cross)



(a) $\mathcal{Z}_{2,4}$ (b) $\mathcal{Z}_{2,10}$

Bias in RBD

Recall that

$$\widehat{S}_{\mathfrak{u}}(f, K_{\mathfrak{u}}, A(\pi)) = \frac{\widehat{V}_{\mathfrak{u}}(f, K_{\mathfrak{u}}, A(\pi))}{\widehat{V}(f, A(\pi))} = \frac{\sum_{\mathbf{k} \in K_{\mathfrak{u}}} \left| \widehat{c}_{\mathbf{k}}(f, A(\pi)) \right|^2}{\widehat{c}_{\mathbf{0}}(f^2, A(\pi)) - \widehat{c}_{\mathbf{0}}(f, A(\pi))^2}$$

Let A be an OA(n, d, q, t) and μ the normalized counting measure on $(\mathfrak{S}_n)^d$

- we are interested in $\mathbb{E}_\mu\left[\widehat{V}(f, A(\pi))\right]$ and $\mathbb{E}_\mu\left[\widehat{V}_{\mathfrak{u}}(f, K_{\mathfrak{u}}, A(\pi))\right]$
- let $f \in \mathcal{H}_\alpha = \mathcal{H}_{\alpha,1}$ (unweighted Korobov space), we have

$$\mathbb{E}_\mu\left[\left|\widehat{c}_{\mathbf{k}}(f, A(\pi))\right|^2\right] = |c_{\mathbf{k}}(f)|^2 + Var_\mu\left[\widehat{c}_{\mathbf{k}}(f, A(\pi))\right] + O(q^{-\alpha/2})$$

Bias in RBD (cont.)

Theorem 1 ([Owen, 1994] revisited as a duality relation)

Denote $D = \left\{0, \frac{1}{q}, \dots, \frac{q-1}{q}\right\}^d$, we have

$$Var_\mu\left[\widehat{c}_0(f, A(\pi))\right] = \frac{1}{n^2}\sum_{|\mathfrak{u}|>t}\left(\sum_{r=0}^{|\mathfrak{u}|} B(\mathfrak{u}, r)(1-q)^{r-|\mathfrak{u}|}\right)\left(\sum_{\mathbf{k}\in\mathbb{Z}^d_{\mathfrak{u}}\cap(-\frac{q}{2},\frac{q}{2}]^d} |\widehat{c}_{\mathbf{k}}(f, D)|^2\right)$$

where

$$B(\mathfrak{u}, r) = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{1}_{|\{l\in\mathfrak{u},\ A_{il}=A_{jl}\}|=r}$$

consists of the number of pairs of rows (A_i, A_j) that match on exactly r of the axes in $\mathfrak{u}$.

Remark: note that $t = 0$ and $n = 1$ leads to Parseval's identity.

Bias in RBD (cont.)

Proposition 3

Let $A = OA(n, d, q, t)$ free of the coincidence defect (= no two rows of A agree in any $t+1$ columns). If there exists $\alpha > 2t+1$ such that $f \in \mathcal{H}_\alpha$ then

(1) *if $f^2 \in \mathcal{H}_\alpha$, then* $\mathbb{E}_\mu\left[\widehat{V}(f, A(\boldsymbol{\pi}))\right] = V(f) - \frac{1}{n}\sum_{|\mathfrak{u}|>t} V_{\mathfrak{u}}(f) + o(n^{-1})$

(2) $\mathbb{E}_\mu\left[\widehat{V}_{\mathfrak{u}}(f, K_{\mathfrak{u}}, A(\boldsymbol{\pi}))\right] = V_{\mathfrak{u}}(f) + \frac{B}{n} + \varepsilon_{trunc}(f, K_{\mathfrak{u}}) + o(n^{-1})$

where $B \leq |K_{\mathfrak{u}}|\left(V(f) + c_0(f)^2\right)$

In practice,

- *no bias correction needed for* $\widehat{V}(f, A(\boldsymbol{\pi}))$
- $\frac{B}{n}$ *could be greater than* $V_{\mathfrak{u}}(f)$

Bias correction in RBD

(1) $A = \mathrm{OA}(n, d, q, 1)$

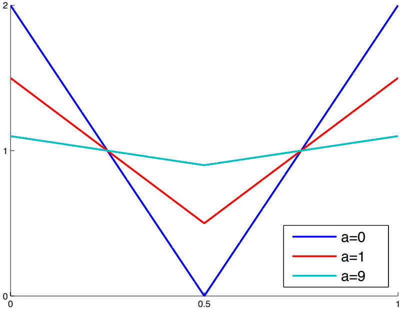
$$
\begin{aligned}
\widehat{V}_i^c(f, K_{\{i\}}, A(\pi)) &= \frac{n}{n - |K_{\{i\}}|}\widehat{V}_i(f, K_{\{i\}}, A(\pi)) - \frac{|K_{\{i\}}|}{n - |K_{\{i\}}|}\widehat{V}(f, A(\pi)) \\
\widehat{V}_{ij}^c(f, K_{\{i,j\}}, A(\pi)) &= \frac{n}{n+1}\widehat{V}_{ij}(f, K_{\{i,j\}}, A(\pi)) \\
&- \frac{|K_{\{i,j\}}|}{n+1}\left(\widehat{V}(f, A(\pi)) + \widehat{c}_0(f, A(\pi))\right)
\end{aligned}
$$

(2) $A = \mathrm{OA}(n, d, q, 2)$

$$
\begin{aligned}
\widehat{V}_i^c(f, K_{\{i\}}, A(\pi)) &= \widehat{V}_i(f, K_{\{i\}}, A(\pi)) \\
\widehat{V}_{ij}^c(f, K_{\{i,j\}}, A(\pi)) &= \frac{n}{n - |K_{\{i,j\}}|}\widehat{V}_{ij}(f, K_{\{i,j\}}, A(\pi)) \\
&- \frac{|K_{\{i,j\}}|}{n - |K_{\{i,j\}}|}\left(\widehat{V}(f, A(\pi)) - \widehat{V}_i - \widehat{V}_j\right)
\end{aligned}
$$

Analytical test case: Sobol's g-function

$$f(\mathbf{x}) = \prod_{i=1}^{d} f_i(x_i)\ , \quad \textit{where} \quad f_i(x_i) = \frac{|4x_i - 2| + a_i}{1 + a_i}\ , \quad a_i > 0$$



$$\textit{For any } \mathbf{k} \in \mathbb{Z}^d,\ c_{\mathbf{k}}(f) = \begin{cases} 0 & \textit{if } \exists i \mid k_i \neq 0 \textit{ and even} \\ \dfrac{\prod_{i \mid k_i \neq 0} 4\pi^{-2}(1+a_i)^{-1}}{\prod_{i \mid k_i \neq 0} k_i^2} & \textit{otherwise} \end{cases}$$

Analytical test case (cont.)

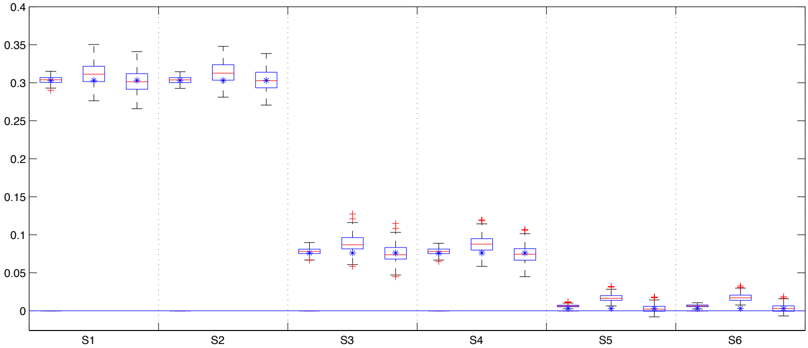
First test case: RBD + bias correction

- $d = 6$, $\mathbf{a} = (0, 0, 1, 1, 9, 9)$
- $n = 1681$
  (a) $A = \mathrm{OA}(1681, 6, 1681, 1)$
  (b) $A = \mathrm{OA}(1681, 6, 41, 2)$ (Bush's construction)
- truncation sets based on a Zaremba cross ($\beta = 12$)
- boxplots of 200 independent replicates

Second test case: FAST (new approach)

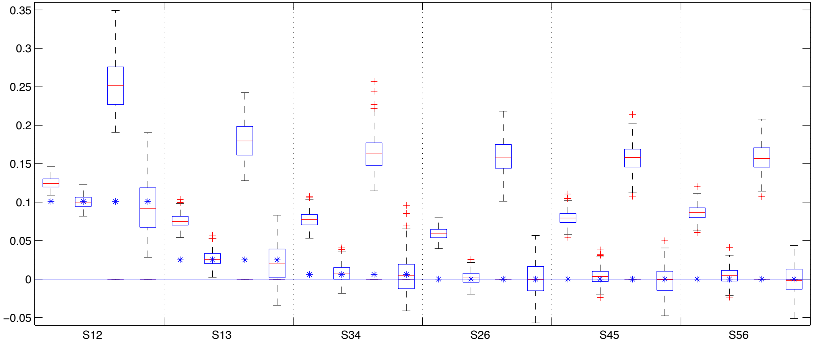
- $d = 30$, $\mathbf{a} = (0, 0.5, 1, 1.5, 2, \dots, 14.5)$
- embedded lattice rule $G(\boldsymbol{\omega}) = \big(G(\boldsymbol{\omega}, n)\big)_{n=2^{10}..2^{19}}$
  $n_{\max} = 2^{19} \approx 5.10^5$
  $j < k \Longrightarrow G(\boldsymbol{\omega}, 2^j) \subset G(\boldsymbol{\omega}, 2^k)$
- truncation sets based on a Zaremba cross $\big(\beta(n) = 0.8 n^{1/4}\big)$

RBD – first-order sensitivity indices



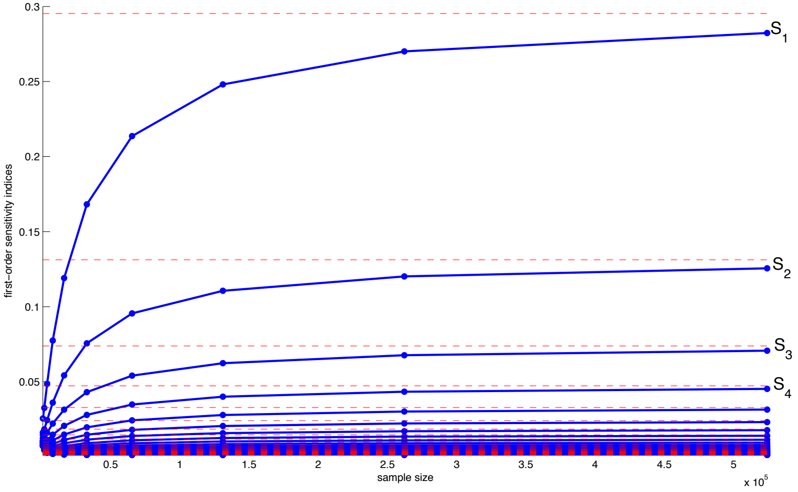
For each index, from the left to the right: strength 2, strength 1 and strength 1 with bias correction.

RBD – second-order sensitivity indices

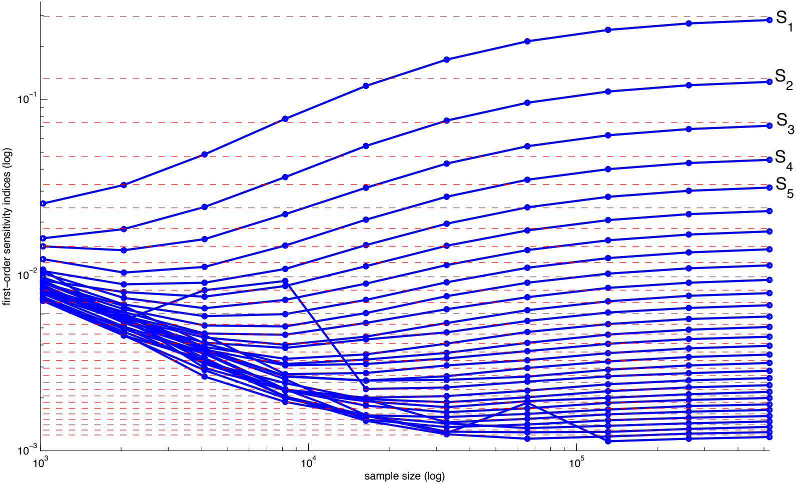


For each index, from the left to the right: strength 2, strength 2 with bias correction, strength 1 and strength 1 with bias correction.

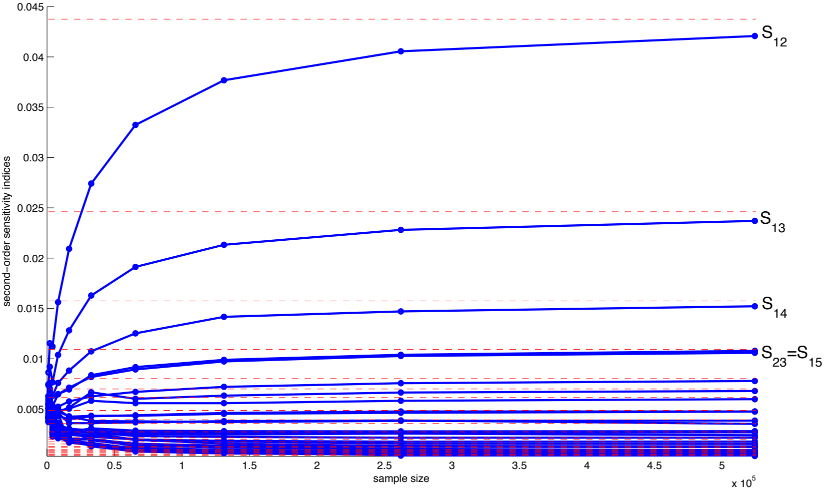
FAST – first-order sensitivity indices



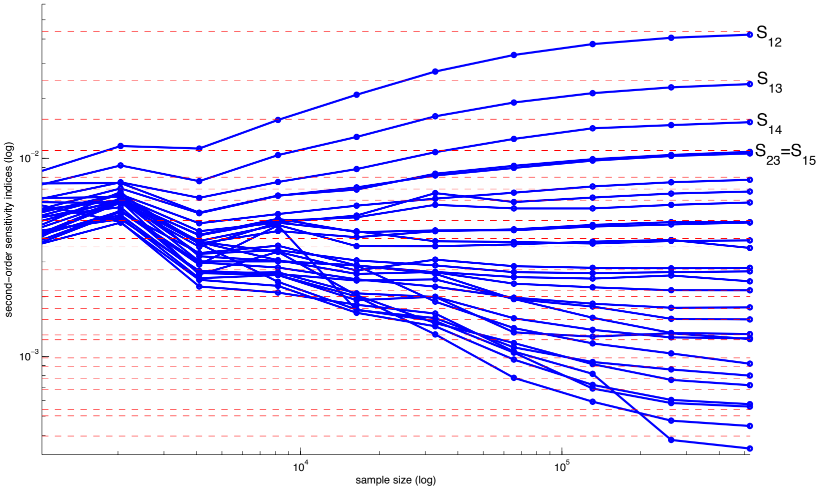
FAST – first-order sensitivity indices (log/log)



FAST – second-order sensitivity indices



FAST – second-order sensitivity indices (log/log)



Summary

We revisited FAST and RBD in light of

- DFT on cyclic groups/lattice rules
- randomized ortogonal array sampling

FAST: we proposed a new approach based on lattice rules

RBD: we generalized RBD to any orthogonal array
we proposed a bias correction method

That's all folks!