Uncertainty Propagation and Inverse Problem Resolution, Application to Hydrodynamics

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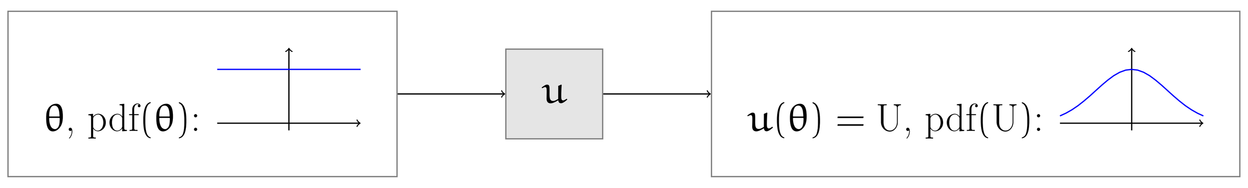
Uncertainty propagation: generalized Polynomial Chaos and iterative generalized Polynomial Chaos



Uncertainty propagation through a model u.

- **generalized Polynomial Chaos** :
  - $(\Omega, \mathcal{F}, \mathcal{P})$ is a random space. X is a random variable of law $d\mathcal{P}_X$.
  - $(\phi_k^X)_{k\in\mathbb{N}}$ is the polynomial orthonormal basis with respect to $d\mathcal{P}_X$: $\int \phi_k^X \phi_l^X \, d\mathcal{P}_X = \delta_{k,l}, \forall (k,l) \in \mathbb{N}^2$.
  - $u : \Omega \to \mathbb{R}$.
  - **Cameron-Martin theorem** : If $\int u(x)^2 \, d\mathcal{P}_X < \infty$, then $u_P(X) = \sum_{k=0}^{P} u_k \phi_k^X(X) \underset{P\to\infty}{\overset{L^2(\Omega,\mathcal{P})}{\longrightarrow}} u(X)$, where $u_k = \int u(X)\phi_k^X(X) \, d\mathcal{P}_X$.
  - $u(X) \simeq \sum_{k=0}^{P} u_k \phi_k^X(X)$ where $u_k = \int u(x)\phi_k^X(x) \, d\mathcal{P}_X(x)$.
- **iterative generalized Polynomial Chaos** (based on gPC and on the theory of moments):
  - Step 0 : gPC : $u(X) \approx u_P^X(X) = \sum_{k=0}^{P} u_k \phi_k^X(X)$.
  - Step 1 : $Z^1 = u_P^X(X)$, $\to$ gPC basis adapted to Z^1 (th. moments), i.e. $\int \phi_k^{Z^1}\phi_t^{Z^1} \, d\mathcal{P}_{Z^1} = \delta_{k,t}, \forall (k,t) \in \mathbb{N}^2$.
    u projection on this basis: $u_k^{Z^1} = \int u(Z^1)\phi_k^{Z^1}(Z^1) \, d\mathcal{P}_{Z^1} = \int u(X)\phi_k^{Z^1}(u_P^X(X)) \, d\mathcal{P}_X$
    New truncated polynomial approximation: $u(X) \simeq u_P^{Z^1}(Z^1) = \sum_{k=0}^{P} u_k^{Z^1}\phi_k^{Z^1}(Z^1)$.
  - Step k : Step 1 with $Z^k = u_P^{Z^{k-1}}(Z^{k-1})$.
  - End $\to u_{igPC}(X)$

Stabilization based on the theory of moments with the existence at each step of the polynomial orthonormal basis adapted to the preceding approximation.

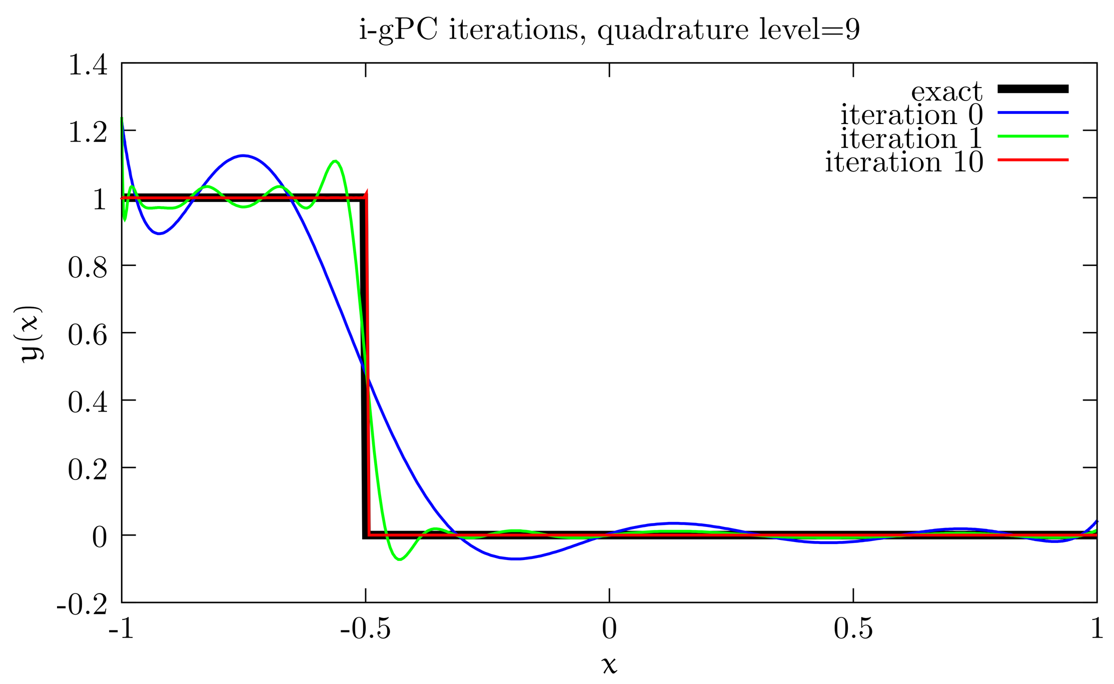


FIGURE 1: i-gPC behaviour with the iterations on a discontinuous test case.

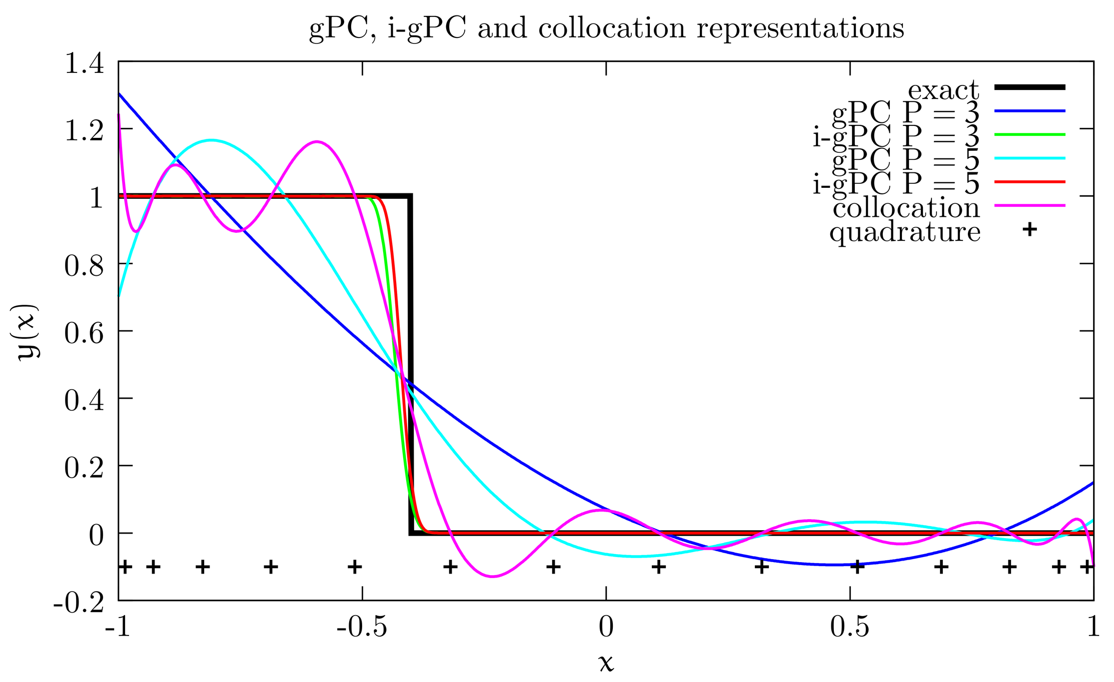


FIGURE 2: i-gPC compared with gPC and collocation methods.

Hydrodynamics

Euler equation in 2D space

$$\begin{cases} \partial_t\rho + \partial_x(\rho u_x) + \partial_y(\rho u_y) & = 0 \\ \partial_t(\rho u_x) + \partial_x(\rho u_x^2 + p) + \partial_y(\rho u_x u_y) & = 0 \\ \partial_t(\rho u_y) + \partial_x(\rho u_x u_y) + \partial_y(\rho u_y^2 + p) & = 0 \\ \partial_t(\rho e) + \partial_x(\rho u_x e + p u_x) + \partial_y(\rho u_y e + p u_y) & = 0 \\ \text{closing} & p = (\gamma - 1)\rho\varepsilon \end{cases} \quad (1)$$

$\varepsilon = e - \frac{1}{2}(u_x^2 + u_y^2)$

Shock tube problem: a shock is propagating in a tube containing a light and a heavy fluid separated by a preturbated interface $\Rightarrow$ amplification of the perturbations as t goes on.

Stochastic interpretation: the inital interface between the two fluids is modeled as a stochastic process [YZL+06].

$$x_{int}(y) = \sigma \sum_{n=8}^{N} Z_n \cos(2\pi b n y) \quad (2)$$

where the Z_n are iid $\mathcal{U}(-1;1)$, and b, σ are parameters.

Numerical resolution: Lagrange + projection, directional splitting, order 3 numerical flow.

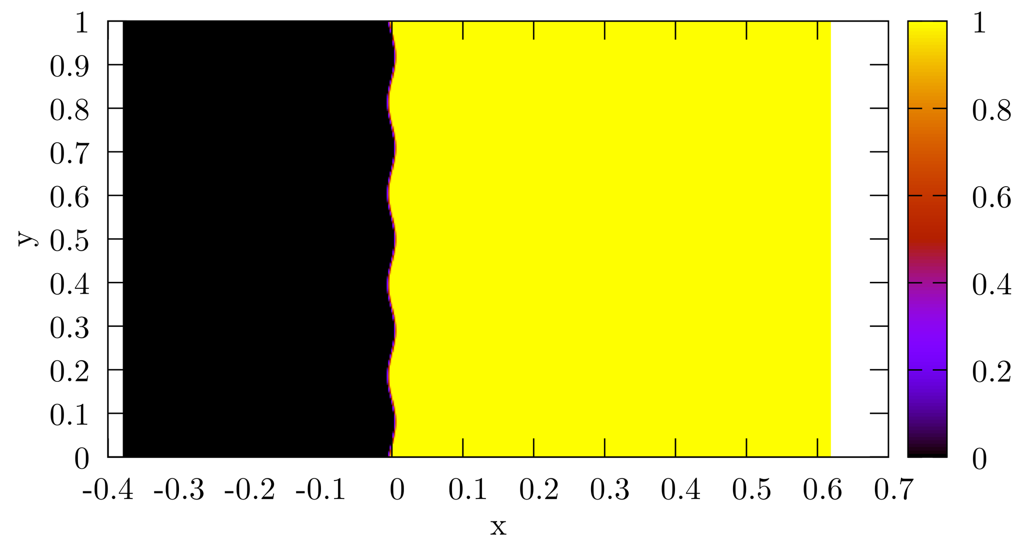


FIGURE 3: A solution at $t = 0$.

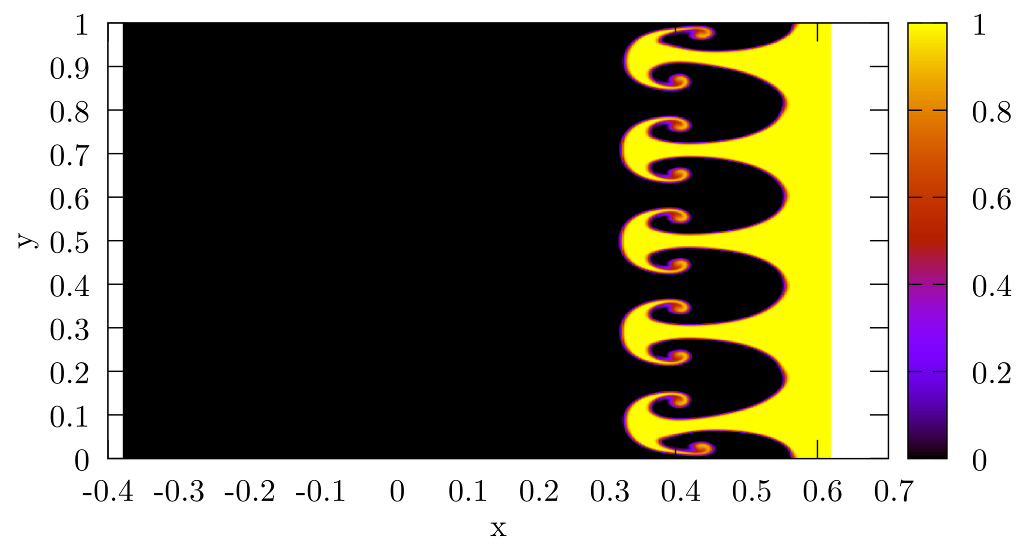


FIGURE 4: A solution at $t = t_{final}$.

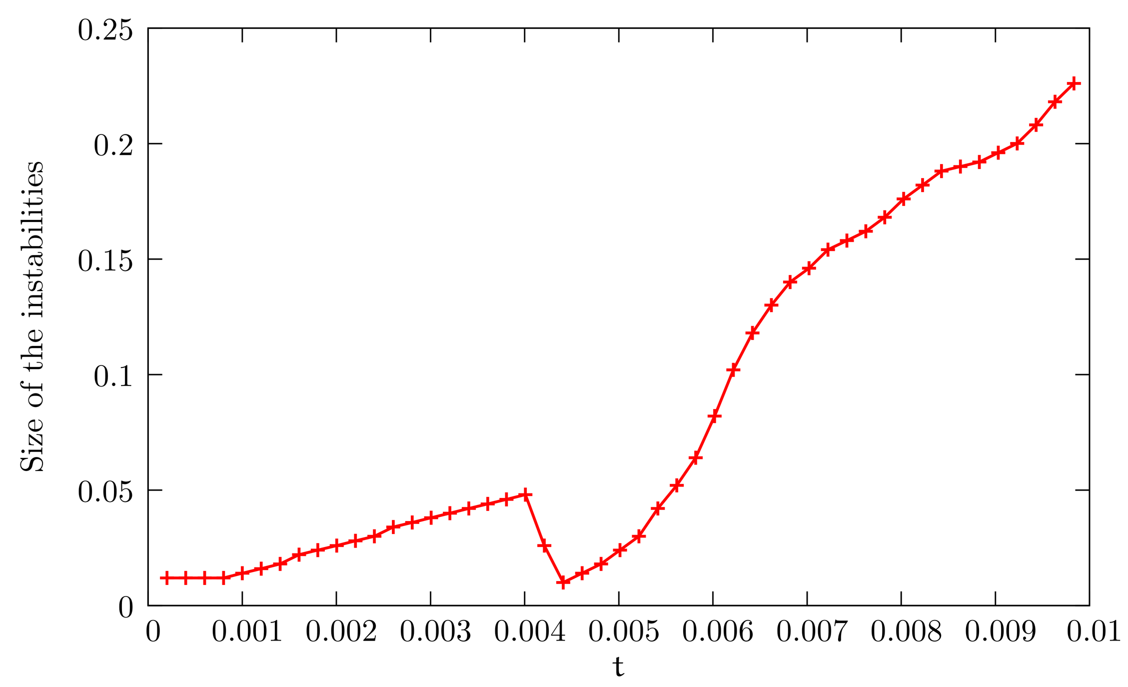


FIGURE 5: Mean growth of the instabilities size.

Statistical study of the size of the instabilities with the random initial perturbations of the interface between the two fluids.

Bayesian Inference

Inference on (b, σ) in th bayesian formalism:

$$\underbrace{\pi_{post}(b,\sigma)}_{\text{posterior}} \propto \underbrace{\pi_{pr}(b,\sigma)}_{\text{prior}} \underbrace{\mathbb{P}(b,\sigma|M)}_{\text{likelihood}} \quad (3)$$

where M is a set of measures.

The measures are from experimental shock tube experiments.

Conclusion

- i-gPC captures the discontinuities, i-gPC adapts the approximation to the solution, algorithm numerical analysis, stabilization $\Rightarrow$ [LBP12].
- Hydrodynamics shock tube numerical simulations, study of the size of the instabilities.
- Bayesian Inference with gPC and i-gPC on non smooth problems $\Rightarrow$ [LBP13].
- Future work: bayesian inference on the parameters of the stochastic process base on experimental shock tube experiments.

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