Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

Multi-level Monte Carlo Finite Volume methods for stochastic systems of hyperbolic conservation laws

Jonas Šukys

Seminar for Applied Mathematics (SAM),
Department of Mathematics (D-MATH),
ETH Zürich, Switzerland.

MascotNum, ETH Zürich, Switzerland
April 23, 2014.

Outline

Hyperbolic nonlinear conservation laws

(Non)linear hyperbolic conservation laws

Conservation of the physical quantities (mass, momentum, energy):

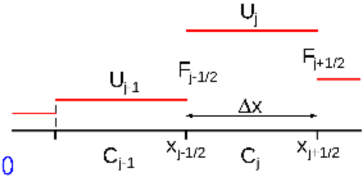
$$\begin{cases} \partial_t \mathbf{U}(\mathbf{x},t) + \operatorname{div} \mathbf{F}(\mathbf{U},\mathbf{x}) = 0, \\ \mathbf{U}(\mathbf{x},0) = \mathbf{U}_0(\mathbf{x}), \end{cases} \qquad \mathbf{x} \in \mathbb{R}^d, \ \ t > 0.$$

- Hyperbolicity: finite speed of propagation
- Nonlinearity: smooth initial data leads to solutions with shocks
- Weak solutions need to be considered (+ entropy conditions for uniqueness)
- No explicit solutions - numerical schemes

Numerical solution using Finite Volume Method (FVM):

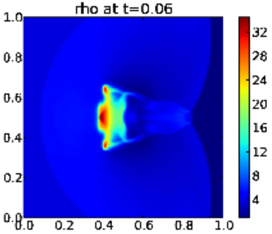
- **Cell averages:**

$$\mathbf{U}_j(t) \approx \frac{1}{|\mathcal{C}_j|} \int_{\mathcal{C}_j} \mathbf{U}(x,t)dx$$

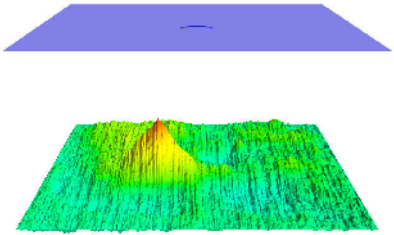


- **ODE:** $\partial_t \mathbf{U}_j(t) + \frac{1}{\Delta x}\left(\mathbf{F}_{j+\frac{1}{2}} - \mathbf{F}_{j-\frac{1}{2}}\right) = 0$
  - **Approximate Riemann fluxes:** HLL, Godunov (Roe)
  - **Explicit time stepping:** FE, SSP-RK2 with CFL: $\Delta t < \Delta x / (\text{max wave speed})$

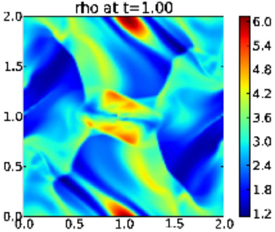
Examples of (non)linear hyperbolic conservation laws



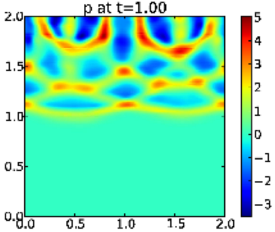
Euler cloud-shock interaction



Shallow water with varying topography



MHD Orszag-Tang vortex



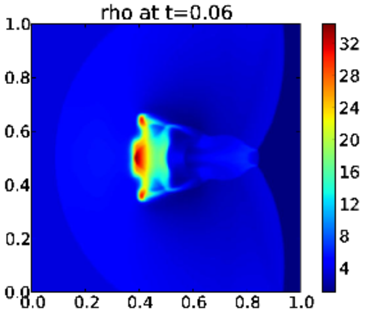
Wave propagation in porous medium

Compressible Euler equations of gas dynamics

Question: What is the time evolution of density/pressure/velocity fields in compressible fluids?

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u} \otimes \mathbf{u} + p\mathbf{ID}) = 0, \\ E_t + \mathrm{div}((E+p)\mathbf{u}) = 0. \end{cases}$$

$$E = \frac{p}{\gamma - 1} + \frac{\rho \mathbf{u}^2}{2}.$$



Density in cloud-shock interaction

- design of aircraft profiles
- gas turbines
- internal combustion engines
- ...

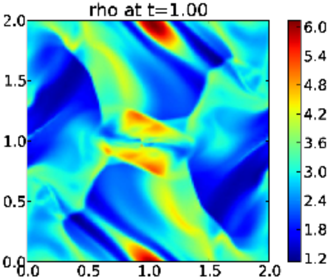
- uncertain cloud geometry/density
- uncertain shock size/location
- uncertain gas constant γ

Magnetohydrodynamics equations for plasma physics

Describes magnetic and density/pressure/velocity fields interaction in electrically conducting fluid.

$$
\begin{cases}
\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\
(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + (p + \frac{1}{2}|\mathbf{B}|^2)I - \mathbf{B} \otimes \mathbf{B}) = -\mathbf{B} \operatorname{div} \mathbf{B}, \\
\mathbf{B}_t + \operatorname{div}(\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) = -u \operatorname{div} \mathbf{B}, \\
E_t + \operatorname{div}((E + p + \frac{1}{2}|\mathbf{B}|^2)\mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\mathbf{B}) = -(\mathbf{u} \cdot \mathbf{B}) \operatorname{div} \mathbf{B}.
\end{cases}
$$

- plasmas (e.g. in the sun)
- liquid metals
- various electrolytes
- HLL 3-wave and 5-wave solvers
  - not strictly hyperbolic
  - non-convex fluxes
  - div constraint
- Godunov-Powell source term
- positivity preserving (W)ENO



Density in Orszag-Tang vortex

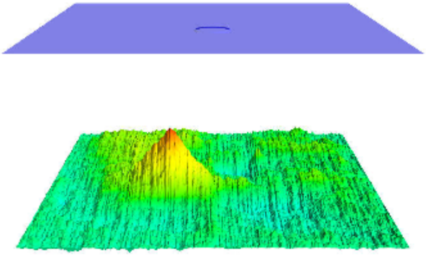
Shallow water equation with varying bottom topography

Question: what is the time evolution of a tsunami wave caused by an earthquake?

$$\begin{cases} h_t + \operatorname{div}(h\mathbf{u}) = 0 \\ (h\mathbf{u})_t + \operatorname{div}(h\mathbf{u} \otimes \mathbf{u}) = -\nabla(ghb + \frac{1}{2}gh^2) \end{cases}$$

Important for:

- avalanche modeling
- debris slides
- atmospheric flows of weather prediction
- risk assessment of region flooding (due to tsunami or dam break)
- ...



Water level above bottom topography

- uncertain initial perturbation
- uncertain bottom topography
- ...

Acoustic wave equation in heterogeneous medium

Question: What is the time evolution of the acoustic wave propagating through random medium?

$$p_{tt}(\mathbf{x}, t) - \nabla \cdot (c(\mathbf{x})\nabla p) = f(\mathbf{x})$$

$$\Downarrow \mathbf{U} = [p, \mathbf{u}]^\top$$

$$\begin{cases} p_t(\mathbf{x}, t) - \nabla \cdot (c(\mathbf{x})\mathbf{u}) = tf(\mathbf{x}), \\ \mathbf{u}_t(\mathbf{x}, t) - \nabla p(\mathbf{x}, t) = 0. \end{cases}$$

- sound/elastic wave propagation through geological layers
- structural mechanics

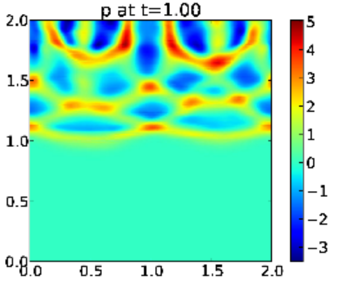
•••

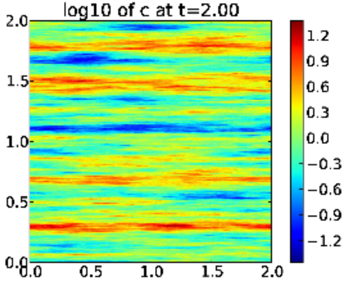
- $c(\mathbf{x})$ is often uncertain, e.g. **log-normal** with covariance

$$\mathrm{Cov}(\log c(\mathbf{x}), \log c(\mathbf{y})) = \sigma^2 e^{-\|\mathbf{x}-\mathbf{y}\|_\eta}$$

with anisotropic correlation lengths

$$\eta_1, \ldots, \eta_2.$$





Stochastic (non)linear systems of balance laws

$\mathbf{U}_0(\mathbf{x})$, $\mathbf{c}(\mathbf{x})$, $\mathbf{F}(\cdot,\cdot)$, $\mathbf{S}(\mathbf{x},t,\cdot)$ are uncertain $\longrightarrow$ solution $\mathbf{U}(\mathbf{x},t)$ is also uncertain:

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{U}(\mathbf{x},t,\omega) + \operatorname{div}\mathbf{F}(\mathbf{c}(\mathbf{x},\omega),\mathbf{U},\omega) = \mathbf{S}(\mathbf{x},t,\mathbf{U},\omega), \\ \mathbf{U}(\mathbf{x},0,\omega) = \mathbf{U}_0(\mathbf{x},\omega), \end{cases} \qquad \forall\omega\in\Omega, \quad (\Omega,\mathcal{F},\mathbb{P}).$$

Well-posedness

Determine required statistical regularity of uncertain input $\mathbf{I} = \{\mathbf{U}_0,\mathbf{c},\mathbf{F},\mathbf{S}\}$ such that **random entropy solution** $\mathbf{U}(\mathbf{x},t,\omega)$ has finite mean and variance.

Goals

- Theory for the existence of $\mathbf{U}(\mathbf{x},t,\omega)$ and its statistical moments
- Numerical methods for approximation of statistical moments (e.g. $\mathbb{E}[\mathbf{U}]$)
- Massively parallel implementation using efficient load balancing

Theory and numerical results on MLMC-FVM

for hyperbolic conservation laws

	Scalar stochastic PDE	System of stochastic PDE
Linear	▸ Linear advection Theory + numerical results [1]	▸ Acoustic wave ▸ Linear elasticity Theory + numerical results [2]
Nonlinear	▸ Burgers' ▸ Buckley-Leverett Theory + numerical results [1]	▸ Euler ▸ Magneto-hydrodynamics ▸ Shallow water [3] Theory[4], Numerical results[5] [6]

[1] Mishra, Schwab (Math. Comp., 2012)
[2] Šukys, Mishra, Schwab (MCQMC 2012, Springer Proc. Math. Stat., 2013)
[3] Mishra, Schwab, Šukys (SIAM J. Sci. Comput., 2012)
[4] Fjordholm, Käppeli, Mishra, Tadmor. *Entropy Measure Valued Solutions* (arXiv, 2014)
[5] Mishra, Schwab, Šukys (J. Comput. Phys., 2012)
[6] Mishra, Schwab, Šukys (Springer LNCSE, 2013)

Short review of
MC-FVM and MLMC-FVM

Monte Carlo FVM algorithm (MC-FVM)

We are interested in $\mathbb{E}[\mathbf{U}(\mathbf{x},t)]$ and $\mathbb{V}[\mathbf{U}(\mathbf{x},t)]$ with $(\mathbf{x},t) \in \mathbf{D} \times \mathbf{T} \subset \mathbb{R}^d \times \mathbb{R}_+$.

1. **Draw** M i.i.d. samples of random quantities (input data)

$$\mathbf{I}^i = \{\mathbf{U}_0^i(\cdot), \mathbf{c}^i(\cdot), \mathbf{F}^i(\cdot), \mathbf{S}^i(\cdot)\}, \quad i = 1, \dots, M.$$

2. For each draw, **solve** for approximate (FVM with Δx) entropy solutions

$$\mathbf{I}^i \quad \longrightarrow \quad \mathbf{U}_{\Delta x}^i(\mathbf{x}, t^n) = (\mathbf{U}_{\mathbf{j}}^n)^i, \quad \forall \mathbf{x} \in \mathcal{C}_{\mathbf{j}}.$$

3. **Estimate statistics** of $\mathbb{E}[\mathbf{U}]$ with

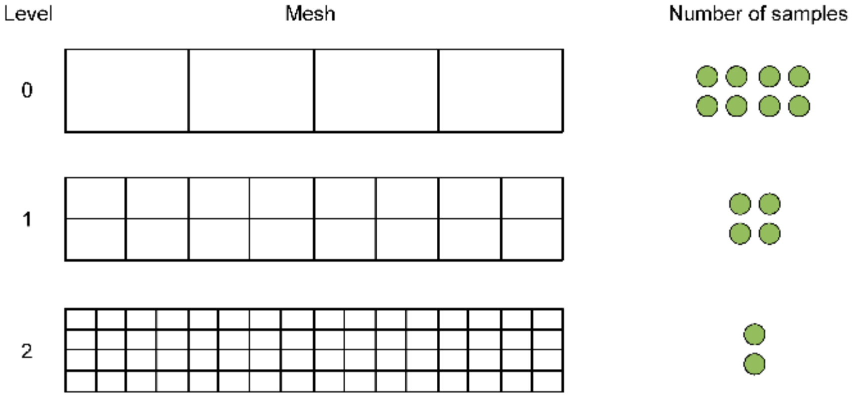
$$\boxed{\mathbb{E}[\mathbf{U}] \approx E_M[\mathbf{U}_{\Delta x}] := \frac{1}{M} \sum_{i=1}^{M} \mathbf{U}_{\Delta x}^i.}$$

Drawback: slow convergence + costly FVM $\longrightarrow$ extremely expensive for $d > 1$.

Multi-Level Monte Carlo[7] FVM method (MLMC-FVM)

- **Nested levels** of resolution

$$\Delta x_\ell = \mathcal{O}(2^{-\ell}\Delta x_0), \quad \ell \in \mathbb{N}_0.$$



[7] Introduced by Heinrich (1999); Giles (2008); Barth, Schwab, Zollinger (2011).

Multi-Level Monte Carlo FVM method (MLMC-FVM)

1. **Draw** M_ℓ i.i.d. samples of random quantities (input data) for each level ℓ

$$\mathbf{I}_\ell^i = \{\mathbf{U}_{0,\ell}^i(\cdot), \mathbf{c}_\ell^i(\cdot), \mathbf{F}_\ell^i(\cdot), \mathbf{S}_\ell^i(\cdot)\}, \qquad i = 1, \dots, M.$$

2. For each draw i and level ℓ, **solve** for approximate (FVM with Δx_ℓ) solutions

$$\mathbf{I}_\ell^i \quad \longrightarrow \quad \mathbf{U}_{\Delta x_\ell}^i.$$

3. **Estimate statistics:**

$$\mathbb{E}[\mathbf{U}] \approx \mathbb{E}[\mathbf{U}_{\Delta x_L}] = \mathbb{E}\left[\mathbf{U}_{\Delta x_0}\right] + \sum_{\ell=1}^{L} \mathbb{E}\left[\mathbf{U}_{\Delta x_\ell} - \mathbf{U}_{\Delta x_{\ell-1}}\right].$$

Estimate each term in the telescoping sum using MC-FVM

$$\boxed{E^L[\mathbf{U}_{\Delta x_L}] = E_{M_0}[\mathbf{U}_{\Delta x_0}] + \sum_{\ell=1}^{L} E_{M_\ell}[\underbrace{\mathbf{U}_{\Delta x_\ell} - \mathbf{U}_{\Delta x_{\ell-1}}}_{\text{variance}\to 0 \text{ as } \ell\to\infty}].}$$

Error vs. Work for Multi-Level Monte Carlo FVM

Theorem [8]

- **scalar** conservation laws: $\mathbf{I} = \{\mathbf{U}_0\}$, $E = L^1 \cap L^\infty(\mathbf{D})$, $H = TV(\mathbf{D})$.

$$\mathbf{U}_0 \in L^2 \cap L^\infty(\Omega, E \cap H), \quad \mathbf{F} \in L^\infty(\Omega, \mathbf{C}^1(-\|\mathbf{U}_0\|_\infty, \|\mathbf{U}_0\|_\infty)).$$

- **linear hyperbolic systems**: $\mathbf{I} = \{K, \mathbf{U}_0, \mathbf{S}\}$, $E = \mathbf{L}^2 \cap \mathbf{L}^\infty(\mathbf{D})$, $H = \mathbf{H}^s(\mathbf{D})$.

$$\mathbf{U}_0, \mathbf{S} \in L^2(\Omega, E \cap H), \quad \mathbf{A}_r \in L^0(\Omega, C^1(\mathbf{D})^{m\times m}) : K \in L^2(\Omega).$$

Denoting FVM convergence rate by s,

$$\left\|\mathbb{E}[\mathbf{U}] - E^L[\mathbf{U}_{\Delta x_L}]\right\|_{L^2(\Omega,E)} \leq C_1 \Delta x_L^s + C_2 \sum_{\ell=1}^{L} M_\ell^{-\frac{1}{2}} \Delta x_\ell^s + C_3 M_0^{-\frac{1}{2}}.$$

Constants $C_{1,2,3}$ depend on $\mathbf{D} \times \mathbf{T}$, s, $\mathbf{U}_0, \mathbf{A}_r, \mathbf{F}, \mathbf{S}$, but not on $L, \Delta x_\ell, M_\ell$.

[8] Mishra, Schwab (Math. Comp., 2012); Šukys, Mishra, Schwab (MCQMC 2012 Proc., 2013)

Choosing number of samples

Asymptotic MLMC-FVM error (denoting $\|\mathbf{I}\|_V = K\big(\|\mathbf{U}_0\|_V + T\|\mathbf{S}\|_V\big)$):

$$C_1 \Delta x_L^s \|\mathbf{I}\|_{L^1(\Omega,H)} + C_2 \sum_{\ell=1}^{L} M_\ell^{-\frac{1}{2}} \Delta x_\ell^s \|\mathbf{I}\|_{L^2(\Omega,H)} + C_3 M_0^{-\frac{1}{2}} \|\mathbf{I}\|_{L^2(\Omega,E)}$$

Equilibrate MC and FVM errors:

$$M_\ell = M_L \cdot 2^{2(L-\ell)s}$$

$$\text{Error} \lesssim \mathbb{E}[\text{Work}]^{-s/(d+1)} \log(\mathbb{E}[\text{Work}])$$

Optimize[9] MC and FVM errors for M_ℓ:

$$M_\ell = M_L \cdot 2^{\frac{2}{3}(L-\ell)(s+d+1)}$$

$$\text{Error} \lesssim \mathbb{E}[\text{Work}]^{-s/(d+1)}$$

! Same complexity as a single FVM solve. Constants differ by $\sqrt{M_L}$.

To find M_L, equilibrate first error term with error terms in the sum:

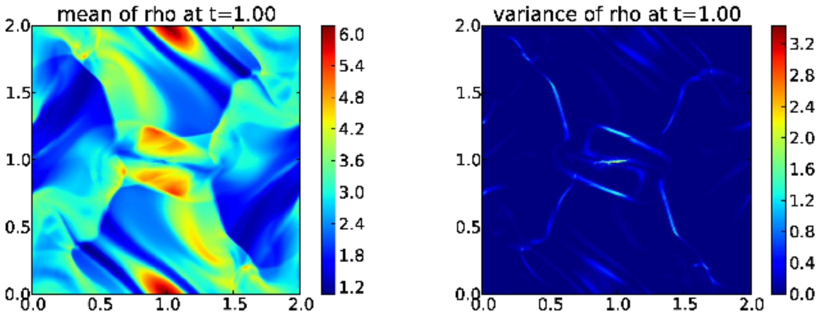
$$M_L = \left(\frac{C_2 \|\mathbf{I}\|_{L^2(\Omega,H)}}{C_1 \|\mathbf{I}\|_{L^1(\Omega,E)}}\right)^2 \approx \left(\frac{C_2}{C_1}\right)^2 = \begin{cases} 16 & \text{scalar } (C_1 = 1, C_2 = 4) \\ 4 & \text{linear systems } (C_1 = 1, C_2 = 2) \end{cases}$$

[9] Giles (Oper. Res., 2008); Pauli and Arbenz (2014)

Numerical experiments
and
error convergence

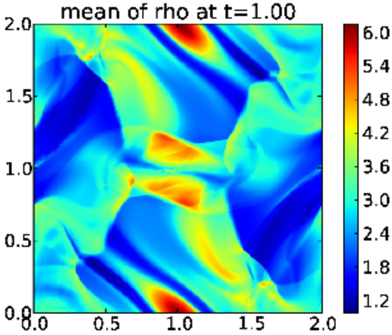
MHD: MLMC-FVM for Orszag-Tang vortex

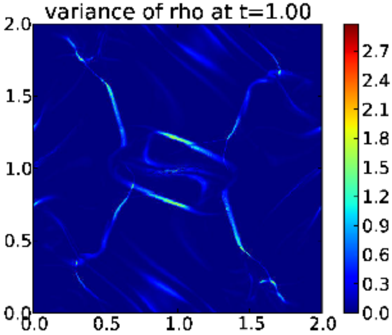
with uncertain initial magnetic field (2 sources of uncertainty)



MHD: MLMC-FVM for Orszag-Tang vortex

with uncertain initial magnetic field (2 sources of uncertainty)

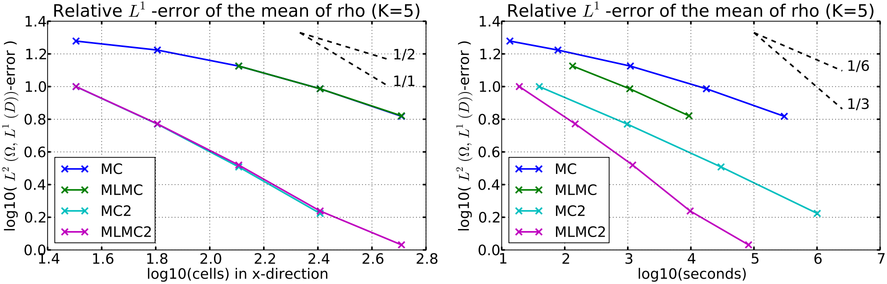




L	M_L	grid size	CFL	cores	runtime	efficiency
7	4	2048x2048	0.475	128	5:02:14	98.4%

MHD: Orszag-Tang vortex - convergence for mean

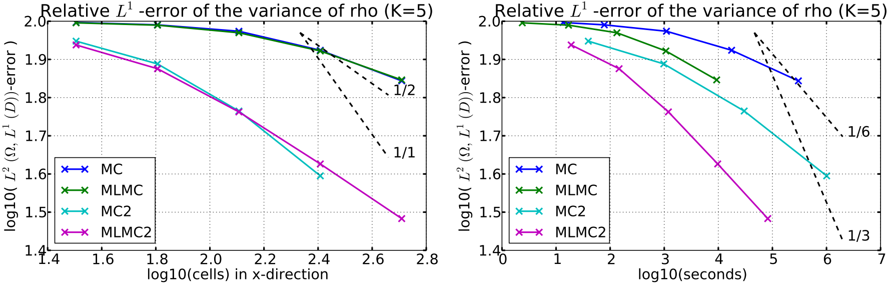
with 2 sources of uncertainty



Convergence rates coincide with the rigorous theory for SCL!

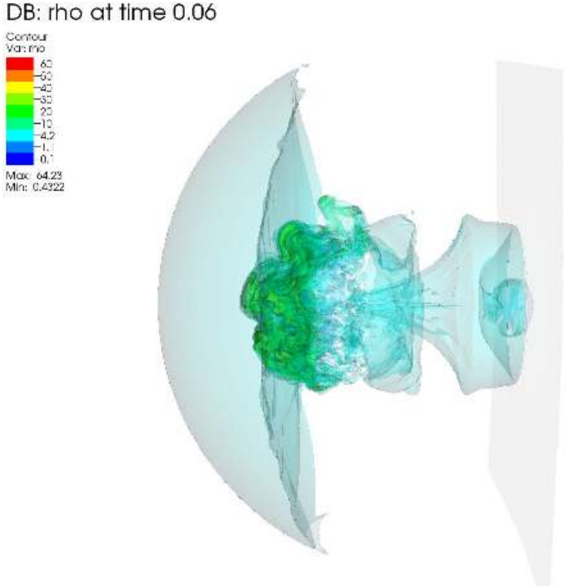
MHD: Orszag-Tang vortex - convergence for variance

with 2 sources of uncertainty



Euler: FVM for cloud shock - one sample

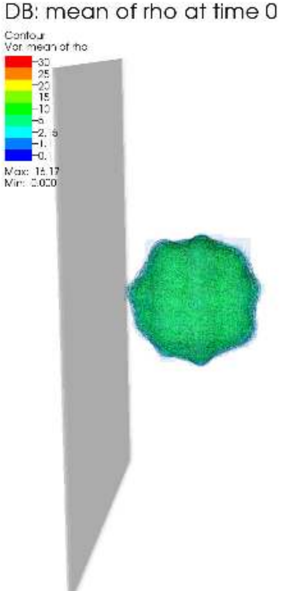
with uncertain shock location/magnitude and geometry of the cloud

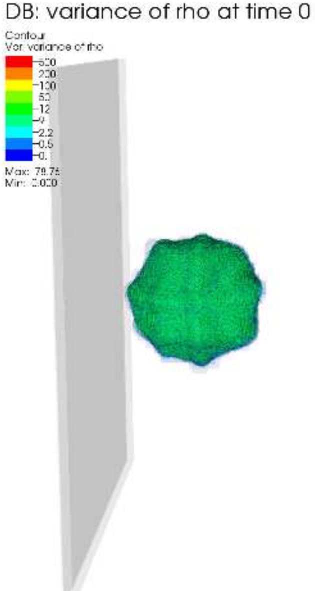


L	0
M_L	1
cells	1 Billion
CFL	0.475
cores	4096
runtime	4:29:44
eff.	95.7%

Euler: MLMC-FVM for cloud shock - mean and variance

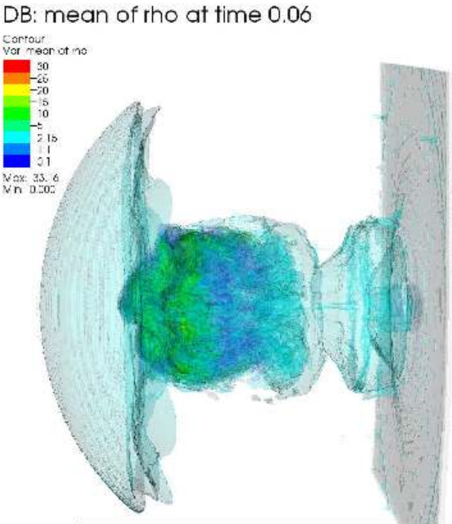
with uncertain shock location/magnitude and geometry of the cloud

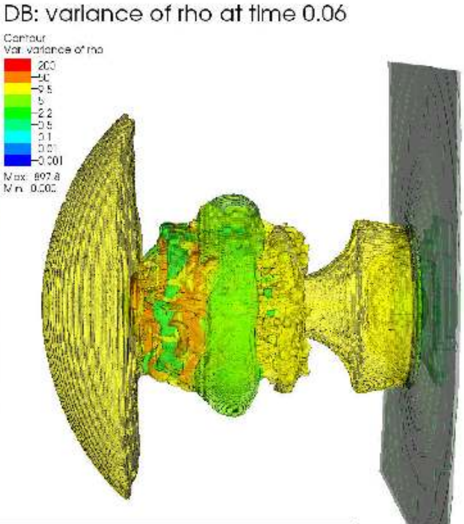




Euler: MLMC-FVM for cloud shock - mean and variance

with uncertain shock location/magnitude and geometry of the cloud





L	M_L	grid size	CFL	cores	runtime	efficiency
6	4	1024x1024x1024	0.475	21844	4:49:42	93.9%

MLMC algorithm is non-intrusive

$\downarrow$

Parallelization

MLMC: Distributions of random run-times across levels

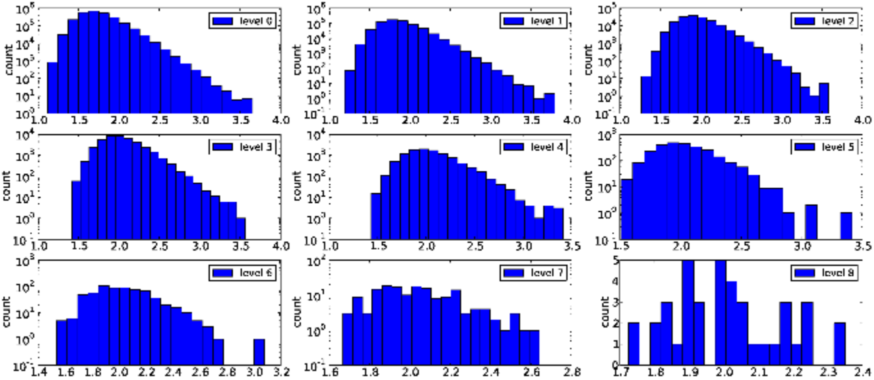
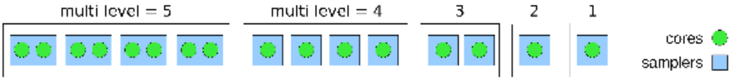


Figure : Distributions of random wave speeds for all resolution levels

Adaptive load balancing algorithm

Generalization of "greedy" algorithm for "workers" with non-uniform speed of execution

Setup: "Workers" $\mathcal{G}_m^s$ (blue) with "computing capacities" C_m (green):



Loads: (pre-FVM step, computed in parallel)

$$\text{Load}_\ell^i = \lambda_\ell^i \Delta x_\ell^{(d+1)} \sim \mathcal{O}(N_{\text{cells}} N_t), \qquad \ell = 0, \dots, L, \quad i = 1, \dots, M_\ell.$$

Recursive rule:

Assign the **largest** Load_ℓ^i to the worker $\mathcal{G}_m^s$ for which the total load is **minimized**.

Pseudocode

$\mathcal{L} = \{\text{Load}_\ell^i : \ell = 0, \dots, L, i = 1, \dots, M_\ell\}$

while $\mathcal{L} \neq \varnothing$ **do**

$\quad \text{Load}_\ell^i = \max \mathcal{L}$

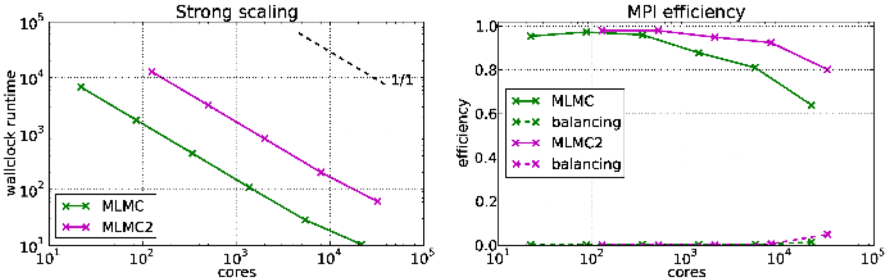
$\quad \mathcal{G}_m^s = \arg\min \sum \left\{ \text{Load}/C_m : \text{Load} \in \mathcal{G}_m^s \cup \text{Load}_\ell^i \right\}$

$\quad \mathcal{G}_m^s = \mathcal{G}_m^s \cup \text{Load}_\ell^i$

$\quad \mathcal{L} = \mathcal{L} \setminus \text{Load}_\ell^i$

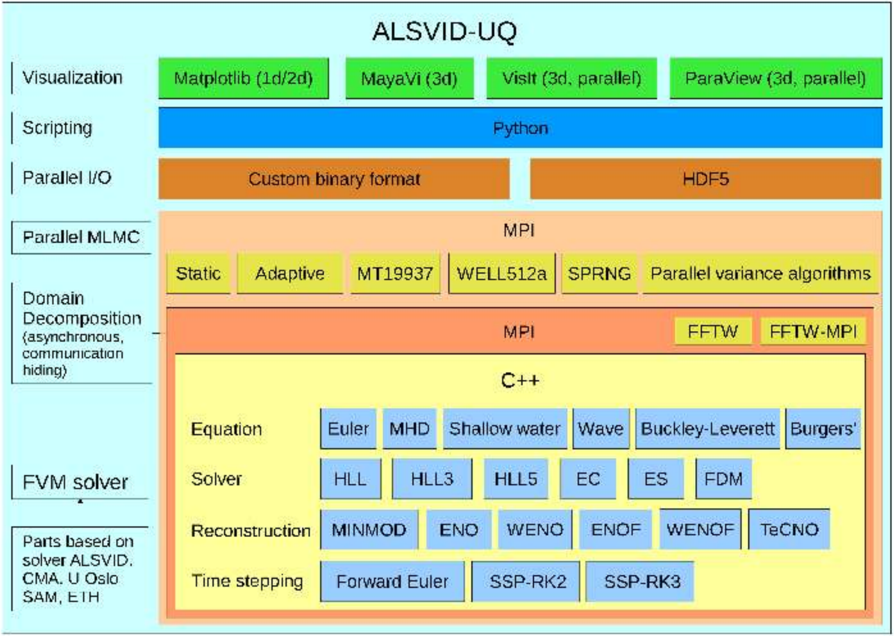
Linear (strong) scaling of adaptive load balancing

(with domain decomposition)



Strong and weak scaling up to 40 000 cores with high efficiency.
(Cray XE6, CSCS)

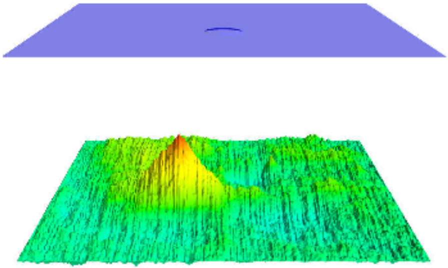
Parallel MLMC-FVM implementation: ALSVID-UQ



Do we ever need more than
10 or even 100 sources of uncertainty?

If yes, does MLMC-FVM still work?

Shallow water equations



Flows in rivers, lakes and oceans; atmospheric flows for weather prediction, etc.

$$\mathbf{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ -ghb_x(\omega) \\ -ghb_y(\omega) \end{bmatrix};$$

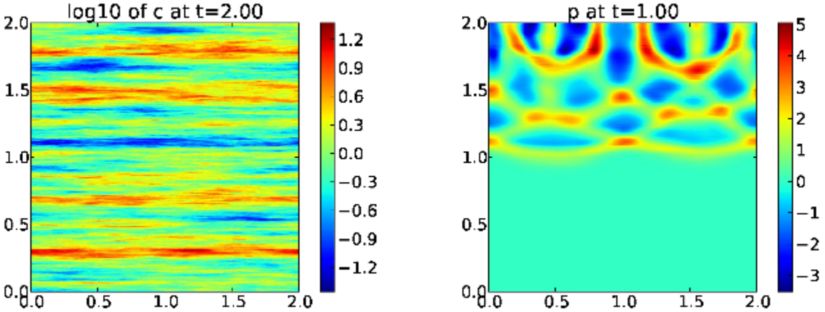
with bottom topography $b \in L^2(\Omega, W^{1,\infty}(\mathbf{D}))$.

$$\begin{cases} \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y = \mathbf{S}(\mathbf{U}, x, y, \omega), \\ \mathbf{U}(x, y, 0) = \mathbf{U}_0(x, y, \omega). \end{cases} \quad (x, y) \in \mathbf{D}, \quad t > 0, \quad \forall \omega \in \Omega.$$

Wave equation: log-normal material coefficient

One realization - 16 384 sources of uncertainty

$$p_{tt}(\mathbf{x}, t, \omega) - \nabla \cdot (c(\mathbf{x}, \omega) \nabla p(\mathbf{x}, t, \omega)) = 0$$



Coefficient $c(\mathbf{x}, \omega)$ is assumed to be **log-normal**, determined by its covariance

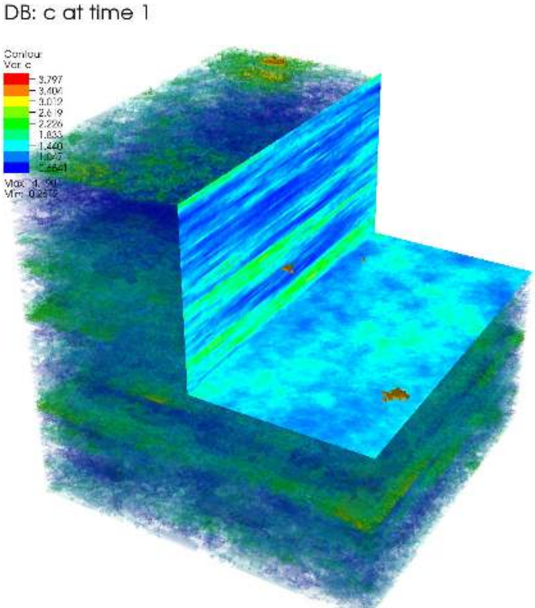
$$\operatorname{Cov}(\log c(\mathbf{x}, \cdot), \log c(\mathbf{y}, \cdot)) = k(\|\mathbf{x} - \mathbf{y}\|_\eta) = \sigma^2 \exp\left(-\sqrt{\sum_{r=1}^{d} \frac{|\mathbf{x}_r - \mathbf{y}_r|^2}{\eta_r^2}}\right)$$

where

- covariance kernel $k : \mathbb{R} \to \mathbb{R}_+$
- correlation lengths in each direction $\eta = \{\eta_1, \ldots, \eta_d\} \in \mathbb{R}_+^d$ (anisotropy)

Log-normal anisotropically correlated coefficient in 3d

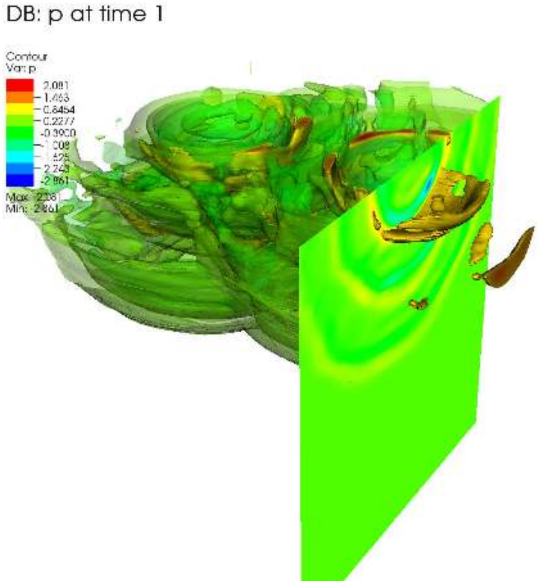
One realization - 2 million sources of uncertainty



L	0
M_L	1
grid size	1024^3
CFL	0.475
cores	4096
runtime	2:45:36
efficiency	99.9%

Wave equation with log-normal coefficient in 3d

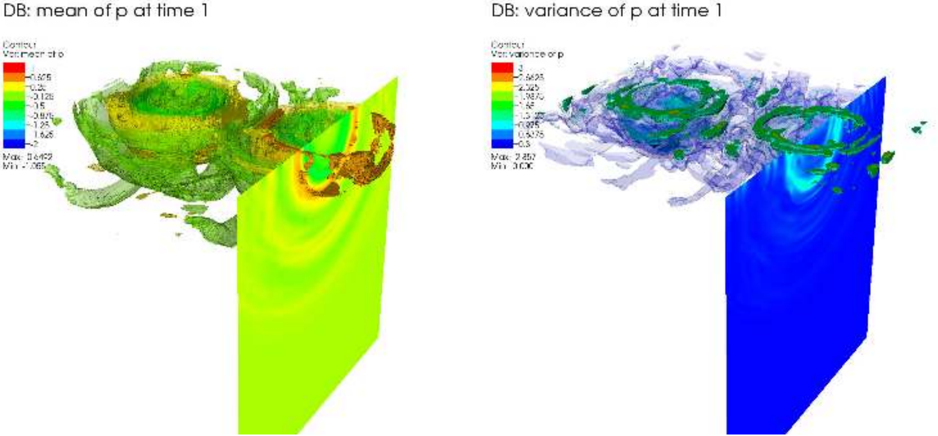
One realization, reflecting/periodic b.c.



L	0
M_L	1
grid size	1024^3
CFL	0.475
cores	4096
runtime	2:45:36
efficiency	99.9%

Wave equation with log-normal coefficient in 3d

MLMC-FVM, reflecting/periodic b.c., adaptive load balancing



L	M_L	grid size	CFL	cores	runtime	efficiency
6	8	1024x1024x1024	0.475	43680	2:48:50	98.1%

Figure : Solution of wave equation using MLMC-FVM and reflecting/periodic b.c.

Summary for MLMC-FVM method

- notion of random weak entropy solution is formulated
- the resulting stochastic hyperbolic system of CLs is shown to be well-posed

- applications: **Euler**, **MHD**, **shallow water**, **Buckley-Leverett**, **wave, etc.**
- flexible w.r.t. the origin of the uncertainty: $\mathbf{U}_0, \mathbf{S}, c, \mathbf{F}$

- optimal computational complexity (same as for deterministic systems)
- 2-3 orders of magnitude speed-up of MLMC-FVM vs. MC-FVM
- linear complexity w.r.t. stochastic dimension (unlike in gPC)

- low regularity requirements
- non-intrusive - deterministic FVM solvers can be reused
- easily parallelizable and scalable (tested up to 40 000 cores)

- algorithmic fault tolerant parallelization: [10]
  - lost samples (due to node failures) are dropped (NO checkpoint/restore)
  - MLMC-FVM error bound is still valid, in the sense of expected accuracy

[10] Pauli, Arbenz and Schwab (SAM Report No. 2012-24, PARCO 2013)

Joint work in progress with

- Siddhartha Mishra
  - SAM, ETH Zürich, Switzerland
- Christoph Schwab
  - SAM, ETH Zürich, Switzerland

- Other collaborators:
  - Florian Müller
  - Stefan Pauli
  - Svetlana Tokareva
  - Luc Grosheintz
  - Manuel Kohler
  - Franziska Weber

- Part of ETH interdisciplinary research grant
  - CH1-03 10-1

- Grant from the Swiss National Supercomputing Centre (CSCS)
  - Project ID S366

Publications (JŠ, S. Mishra, Ch. Schwab, A. Barth)

List available at: http://pub.sukys.lt

- **ALSVID-UQ:** http://www.sam.math.ethz.ch/alsvid-uq.
- *MLMC approximations of statistical solutions to the Navier-Stokes equation.* In review, 2014.
- *MLMC-FVM: uncertainty quantification in nonlinear systems of balance laws.* **Springer LNCSE (92)**, 225–294, 2013.
- *MLMC-FVM for stochastic linear hyperbolic systems.* **MCQMC 2012, Springer Proc. Math. Stat. (65)**, 649–666, 2013.
- *Adaptive load balancing for massively parallel multi-level Monte Carlo solvers.* **PPAM 2013** (to appear).
- *MLMC-FVM for shallow water equations with uncertain topography.* **SIAM J. Sci. Comput., 34(6)**, B761–B784, 2012.
- *MLMC-FVM for nonlinear systems of conservation laws in multi-dimensions.* **J. Comp. Phys., 231(8)**, 3365–3388, 2012.
- *Sparse tensor MLMC-FVM for conservation laws with random initial data.* **Math. Comp., 280**, 1979–2018, 2012.
- *Static load balancing for multi-level Monte Carlo finite volume solvers.* **PPAM 2011, Part I, LNCS 7203**, 245–254. Springer, Heidelberg 2012.