Discrete least squares polynomial approximations for high dimensional uncertainty propagation

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Center for Advanced Modeling and Science

Outline

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UQ for deterministic PDE models

- Consider the parametric equation (typically a PDE)

$$\text{find } u : \quad \mathcal{F}(\mathbf{y}, u) = 0 \tag{1}$$

  where $\mathbf{y}$ is a vector of N parameters: $\mathbf{y} = (y_1, \ldots, y_N) \in \mathbb{R}^N$ ($N = \infty$ when dealing with distributed fields).

- Often in applications the parameters $\mathbf{y}$ are not perfectly known or are intrinsically variable. Examples are:
  - subsurface modeling: porous media flows; seismic waves; basin evolutions; ...
  - modeling of living tissues: mechanical response; growth models;
  - material science: properties of composite materials
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Assumption: $\forall \mathbf{y} \in \Gamma$ the problem admits a unique solution $u \in V$ in a suitable finite or infinite dimensional Hilbert space V. Moreover,

$$\forall \mathbf{y} \in \Gamma, \quad \exists C(y) > 0; \qquad \|u(\mathbf{y})\|_V \leq C(\mathbf{y})$$

- Then, equation (1) induces a map $u = u(\mathbf{y}) : \Gamma \to V$.
- if $\int_\Gamma C(\mathbf{y})^p \rho(\mathbf{y}) d\mathbf{y} < \infty$, then $u \in L^p_\rho(\Gamma, V)$.

Goals:

- Construct a reduced model $u_\Lambda(\mathbf{y}) \approx u(\mathbf{y})$
- Compute statistics of the solution

Expected value: $\bar{u} \approx \mathbb{E}[u_\Lambda]$

Variance: $Var[u] \approx \mathbb{E}[u_\Lambda^2] - \mathbb{E}[u_\Lambda]^2$

two points corr. (if u is a distributed field)

$$Cov_u(x_1, x_2) \approx \mathbb{E}[u_\Lambda(x_1)u_\Lambda(x_2)] - \mathbb{E}[u_\Lambda(x_1)]\mathbb{E}[u_\Lambda(x_2)]$$

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Example: Elliptic PDE with random coefficients

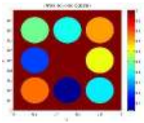
$$\begin{cases} -\operatorname{div}(a(\mathbf{y},x)\nabla u(\mathbf{y},x)) = f(x) & x \in D, \quad \mathbf{y} \in \Gamma, \\ u(\mathbf{y},x) = 0 & x \in \partial D, \quad \mathbf{y} \in \Gamma \end{cases}$$

with $a_{min}(\mathbf{y}) = \inf_{x \in D} a(\mathbf{y},x) > 0$ for all $\mathbf{y} \in \Gamma$ and $f \in L^2(D)$. Then

$$\forall \mathbf{y} \in \Gamma, \quad u(\mathbf{y}) \in V \equiv H_0^1(D), \quad \text{and} \quad \|u(\mathbf{y})\|_V \leq \frac{C_P}{a_{min}(\mathbf{y})} \|f\|_{L^2(D)}.$$

Inclusions problem

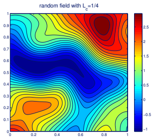
$\mathbf{y}$ describes the conductivitiy in each inclusion



$$a(\mathbf{y},x) = a_0 + \sum_{n=N}^{N} y_n \mathbb{1}_{D_n}(x)$$

Random fields problem

$a(\mathbf{y},x)$ is a random field, e.g. lognormal: $a(\mathbf{y},x) = e^{\gamma(\mathbf{y},x)}$ with γ expanded e.g. in Karhunen-Loève series



$$\gamma(\mathbf{y},x) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} y_n b_n(x), \quad y_n \sim N(0,1) \ i.i.d.$$

Outline

1. Introduction – parametric / stochastic equations
2. Stochastic polynomial approximation
3. Discrete least squares approx. using random evaluations
   - Convergence analysis
   - Numerical results
4. Adaptive algorithms
5. Conclusions

Stochastic multivariate polynomial approximation

- The parameter-to-solution map $u(\mathbf{y}) : \Gamma \to V$ is often smooth (even analytic for the elliptic diffusion model). It is therefore sound to approximate it by global multivariate polynomials.
- Let $\Lambda \subset \mathbb{N}^N$ be an index set of cardinality $|\Lambda| = M$, and consider the multivariate polynomial space

$$\mathbb{P}_\Lambda(\Gamma) = span\left\{\prod_{n=1}^N y_n^{p_n}, \quad \text{with } \mathbf{p} = (p_1, \ldots, p_N) \in \Lambda\right\}$$

  We seek an approximation $P_\Lambda u \in \mathbb{P}_\Lambda(\Gamma) \otimes V$.
- The optimal choice of Λ depends heavily on the problem at hand and the "structure" of the map $u(\mathbf{y})$.

Definition. An index set Λ is downward closed (or *lower set*) if

$$\mathbf{p} \in \Lambda \text{ and } \mathbf{q} \leq \mathbf{p} \quad \Longrightarrow \quad \mathbf{q} \in \Lambda$$

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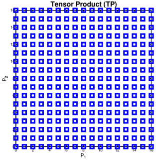
Common choices of polynomial spaces



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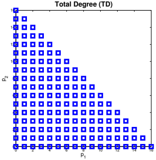
$\Lambda(w) = \{\mathbf{p} : \max_n p_n \leq w\}$



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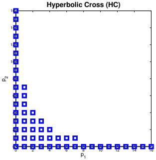

hyperbolic cross (HC)

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- Anisotropic versions are also possible.
- All these index sets are all downward closed.

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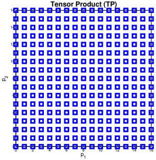

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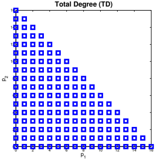

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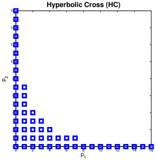

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Approximation using random evaluations

Goal: construct a polynomial approximation using random evaluations (Monte Carlo sampling):

1. Generate M random i.i.d. samples $\mathbf{y}^{(k)} \sim \rho(\mathbf{y})d\mathbf{y}$, $k = 1, \ldots, M$
2. Compute the corresponding solutions $u^{(k)} = u(\mathbf{y}^{(k)})$
3. Construct a suitable approximation $P_\Lambda^{M,\omega} u \in \mathbb{P}_\Lambda(\Gamma) \otimes V$

Notation

given two functions $u, v \in L^2_\rho(\Gamma; V)$

- Continuous inner product: $\mathbb{E}[(u,v)_V] = \int_\Gamma (u(\mathbf{y}), v(\mathbf{y}))_V \rho(\mathbf{y}) d\mathbf{y}$
- Continuous norm: $\|v\|^2_{L^2_\rho(\Gamma;V)} = \mathbb{E}[(v,v)_V]$.
- Discrete inner product: $\mathbb{E}_M[(u,v)_V] = \frac{1}{M}\sum_{i=1}^M (u(\mathbf{y}^{(i)}), v(\mathbf{y}^{(i)}))_V$
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Let $\{\psi_\mathbf{p}\}_{\mathbf{p}\in\Lambda}$ be an orthonormal basis of $\mathbb{P}_\Lambda$ w.r.t the weight ρ.
Then, the best approximation of u in $\mathbb{P}_\Lambda(\Gamma)\otimes V$ (exact L^2 projection) is

$$P_\Lambda u = \underset{v\in\mathbb{P}_\Lambda(\Gamma)\otimes V}{\operatorname{argmin}} \mathbb{E}[\|u-v\|^2_V] = \sum_{\mathbf{p}\in\Lambda} \mathbb{E}[u\psi_\mathbf{p}]\psi_\mathbf{p}$$

How to compute an approx. projection using the random sample?
Replace the exact expectation $\mathbb{E}[\cdot]$ with the sample average $E_M[\cdot]$.

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First idea (bad): discrete projection

We construct an approximation as

$$P_\Lambda^{M,\omega} u = \sum_{\mathbf{p}\in\Lambda} \mathbb{E}_M[u\psi_\mathbf{p}]\psi_\mathbf{p} = \sum_{\mathbf{p}\in\Lambda} \left(\frac{1}{M} \sum_{i=1}^{M} u(\mathbf{y}^{(i)})\psi_\mathbf{p}(\mathbf{y}^{(i)}) \right) \psi_\mathbf{p}$$

Error analysis:

$$\|u - P_\Lambda^{M,\omega} u\|^2_{L^2_\rho(\Gamma;V)} = \underbrace{\sum_{\mathbf{q}\notin\Lambda} \|\mathbb{E}[u\psi_\mathbf{q}]\|^2_V}_{L^2 \text{ projection error}} + \sum_{\mathbf{p}\in\Lambda} \underbrace{\|\mathbb{E}[u\psi_\mathbf{p}] - \mathbb{E}_M[u\psi_\mathbf{p}]\|^2_V}_{\text{Monte Carlo error}\sim O(M^{-1})}$$

from which, setting $K(\Lambda) = \sup_{\mathbf{y}\in\Gamma} \left(\sum_{\mathbf{p}\in\Lambda} |\psi_\mathbf{p}(\mathbf{y})|^2 \right)$, one can deduce

$$\mathbb{E}^\omega \|u - P_\Lambda^{M,\omega} u\|^2_{L^2_\rho(\Gamma;V)} \leq \inf_{v\in\mathbb{P}_\Lambda(\Gamma)\otimes V} \|u - v\|^2_{L^2_\rho(\Gamma;V)} + \frac{K(\Lambda)}{M} \|u\|^2_{L^2_\rho(\Gamma;V)}$$

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Second idea (good): Discrete least squares approximation

(see e.g. [Hosder-Walters et al. 2010, Blatman-Sudret 2008, Burkardt-Eldred 2009, Eldred 2011, Yan-Guo-Xiu 2012, Cohen-Davenport-Leviatan 2013, Migliorati etal 2011-2014])

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Two relevant questions

- What is the accuracy of the random discrete least square approximation?
- For a given set Λ, how many samples should one use?

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Algebraic formulation

- Let $V_h \subset V$ be a finite dimensional subspace (e.g. finite elements) and $\{\phi_j\}_{j=1}^{N_h}$ a basis; mass matrix $\mathcal{M}_{ij} = (\phi_j, \phi_i)_V$.
- Let $\{\psi_{\mathbf{p}}\}_{\mathbf{p}\in\Lambda}$ be an orthonormal basis of $\mathbb{P}_\Lambda$ w.r.t the weight ρ; define the design matrix $D_{i\mathbf{p}} = \frac{1}{\sqrt{M}}\psi_{\mathbf{p}}(\mathbf{y}^{(i)})$.

Then $P_\Lambda^{M,\omega} u(x,\mathbf{y}) = \sum_{\mathbf{p}\in\Lambda}\sum_{j=1}^{N_h} c_{\mathbf{p}j}\phi_j(x)\psi_{\mathbf{p}}(\mathbf{y})$ and the tensor $C = \{c_{\mathbf{p}j}\}$ satisfies the normal equations

$$(D^T D \otimes \mathcal{M})C = (D^T \otimes \mathcal{M})U$$

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Since the matrix $\mathcal{M}$ is invertible, the previous problem decouples in N_h least square problems, one for each spatial dof.

$$D^T D C_{:,j} = D^T U_{:,j}$$

Algebraic formulation

- Let $V_h \subset V$ be a finite dimensional subspace (e.g. finite elements) and $\{\phi_j\}_{j=1}^{N_h}$ a basis; mass matrix $\mathcal{M}_{ij} = (\phi_j, \phi_i)_V$.
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Error analysis

The error analysis goes through the *equivalence of norms* on the polynomial space. Remind:

- continuous norm: $\|v\|^2_{L^2_\rho(\Gamma;V)} = \int_\Gamma \|v(\mathbf{y})\|^2_V \rho(\mathbf{y}) d\mathbf{y}$
- discrete norm: $\|v\|^2_{M,V} = \frac{1}{M}\sum_{i=1}^M \|v(\mathbf{y}^{(i)})\|^2_V$

Define the random variable

$$\delta := \sup_{v\in\mathbb{P}_\Lambda(\Gamma)\otimes V} \left| \frac{\|v\|^2_{M,V}}{\|v\|^2_{L^2_\rho(\Gamma,V)}} - 1 \right|.$$

Whenever $\delta < 1$, we have norm equivalence

$$(1-\delta)\|v\|^2_{L^2_\rho(\Gamma,V)} \leq \|v\|^2_{M,V} \leq (1+\delta)\|v\|^2_{L^2_\rho(\Gamma,V)}, \quad \forall v \in \mathbb{P}_\Lambda(\Gamma)\otimes V$$

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Theorem [Migliorati-Nobile-von Schwerin-Tempone '11]

1. $\delta \to 0$ almost surely when $M \to \infty$
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A general result

[Cohen-Davenport-Leviatan '12], [Chkifa-Cohen-Migliorati-Nobile-Tempone '13]
Let $\{\psi_{\mathbf{p}}\}$ be any orthonormal basis of $L^2_\rho(\Gamma)$ and define

$$K(\Lambda) := \sup_{\mathbf{y}\in\Gamma}\left(\sum_{\mathbf{p}\in\Lambda}|\psi_{\mathbf{p}}(\mathbf{y})|^2\right) = \sup_{v\in\mathbb{P}_\Lambda}\frac{\|v\|^2_{L^\infty(\Gamma)}}{\|v\|^2_{L^2_\rho(\Gamma)}}$$

Theorem [Cohen-Davenport-Leviatan '13]

For any $\gamma > 0$, and $0 < \delta < 1$, and $\beta_\delta = \delta + (1-\delta)\log(1-\delta)$, if

$$\frac{M}{\log M} \geq \frac{1+\gamma}{\beta_\delta}K(\Lambda), \tag{2}$$

Then $P\left((1-\delta)\|v\|^2_{L^2_\rho(\Gamma;V)} \leq \|v\|^2_{M,V} \leq (1+\delta)\|v\|^2_{L^2_\rho(\Gamma;V)}\right) \geq 1 - 2M^{-\gamma}$

The result is based on properties of random matrices.

Implications

Convergence in probability: with probability greater than $1 - 2M^{-\gamma}$

$$\|u - P_\Lambda^{M,\omega} u\|_{L^2_\rho(\Gamma;V)} \leq \left(1 + \sqrt{\frac{1}{1-\delta}}\right) \inf_{v \in \mathbb{P}_\Lambda \otimes V} \|u - v\|_{L^\infty(\Gamma,V)}$$

Convergence in expectation: assume $\|u\|_{L^\infty(\Gamma,V)} \leq \tau$ and define the truncation operator

$$T_\tau : V \to V, \qquad T_\tau(v) = \begin{cases} v & \text{if } \|v\|_V \leq \tau \\ \frac{\tau}{\|v\|_V} v, & \text{if } \|v\|_V > \tau \end{cases}$$

Then $\quad \mathbb{E}^\omega(\|u - T_\tau P_\Lambda^{M,\omega} u\|^2_{L^2_\rho(\Gamma;V)}) \leq C\|u - P_\Lambda u\|^2_{L^2_\rho(\Gamma,V)} + 8\tau^2 M^{-\gamma}$

Stability of discrete least squares: $cond(D^T D) \leq \frac{1+\delta}{1-\delta}$.

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Uniform random vector in $[-1,1]^N$

Let $y_1, \dots, y_N$ be i.i.d. uniform random variables in $[-1,1]$.

Theorem [Chkifa-Cohen-Migliorati-Nobile-Tempone '13]

For any N and any downward closed set $\Lambda \subset \mathbb{N}^N$ it holds

$$K(\Lambda) \leq (\#\Lambda)^2.$$

Therefore, the discrete L^2 projection is stable and optimally convergent under the condition

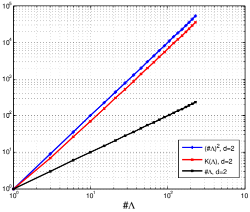
$$\frac{M}{\log M} \geq \frac{1+\gamma}{\beta_\delta}(\#\Lambda)^2$$

The result uses expansion on Legendre polynomials for which

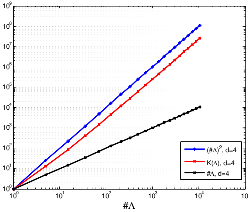
$$|\psi_{\mathbf{p}}(\mathbf{y})| \leq \prod_{n=1}^{N} \sqrt{2p_n+1}, \quad \forall \mathbf{y} \in [-1,1]^N.$$

Uniform random vector in $[-1,1]^N$

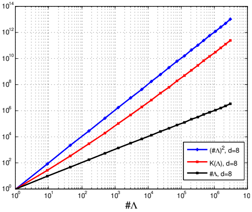
- For specific sets Λ the condition can be improved.
- For instance for the Total Degree polynomial space of degree w the bound $K(\Lambda) \leq (\#\Lambda)^2$ is very conservative



dimension $N = 2$



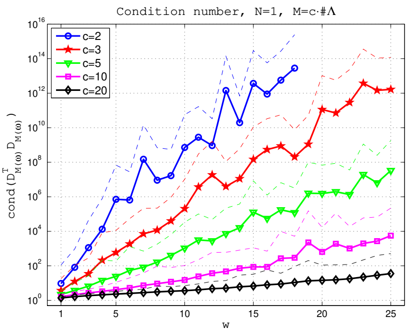
dimension $N = 4$



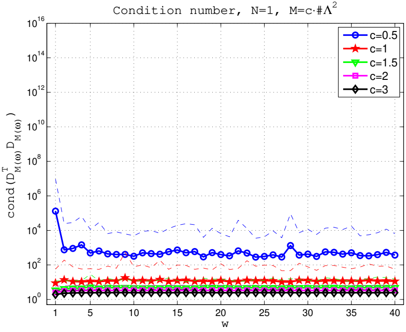
dimension $N = 8$

Some numerical examples – 1D function

Condition number of $D^T D$



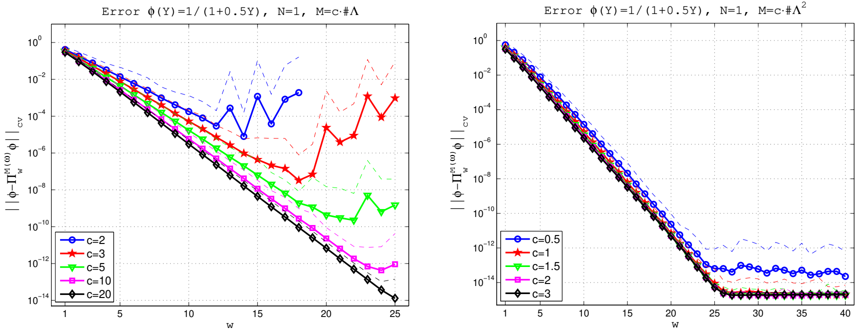
$M = c \cdot \#\Lambda$



$M = c \cdot (\#\Lambda)^2$

Some numerical examples – 1D function

Approximation of the meromorphic function $\phi(y) = \frac{1}{1+0.5y}$



$M = c \cdot \#\Lambda$ $\qquad$ $M = c \cdot (\#\Lambda)^2$

error with respect to polynomial degree.

Some numerical examples – 1D function

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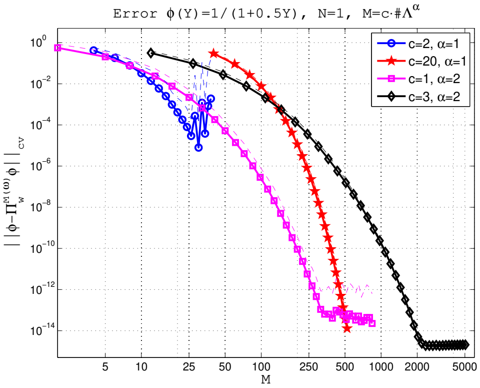
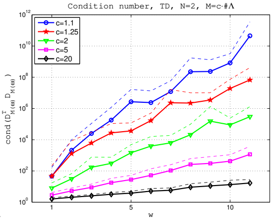


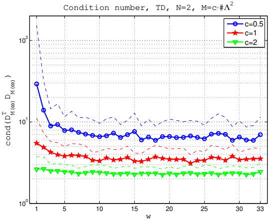
error with respect to total number of sampling points.

Some numerical examples

Condition number of $D^T D$ – multiD – Total Degree poly. space

$M = c \cdot \#\Lambda$





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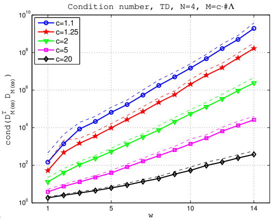
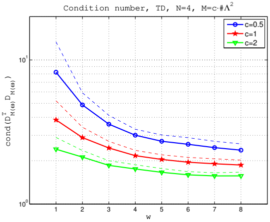




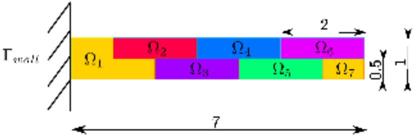
$M = c \cdot (\#\Lambda)^2$

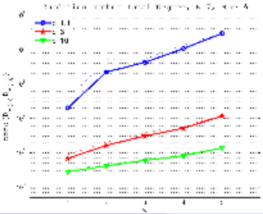
Cantilever beam

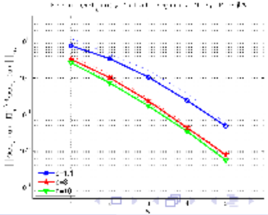
- linear elasticity equations
- Young modulus uncertain in each brick:

$$E_i = e^{7+Y_i}, \quad \text{in } \Omega_i,$$
$$Y_i \sim \mathcal{U}([-1,1]), \text{ iid}$$

- Uncertainty analysis on maximum vertical displacement.



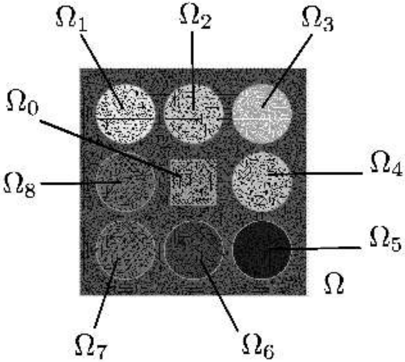


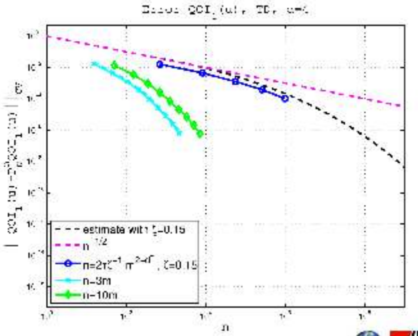


Elliptic PDE with random inclusions

The following bound has been derived in
[Chkifa-Cohen-Migliorati-Nobile-Tempone 13]

$$\mathbb{E}(\|u - T_\tau \circ P_\Lambda^{M,\omega} u\|^2_{L^2_\rho(\Gamma, H^1_0(D))}) \leq c_1 e^{-c_2 N M^{\frac{1}{1+2N}}}$$





4 random variables

Improvements on the quadratic relation

Improvements can be obtained by sampling from a different distribution $\hat{\rho}$.
Let us consider the weighted least squares approx.

$$P_\Lambda^M u = \underset{v \in \mathbb{P}_\Lambda(\Gamma) \otimes V}{\operatorname{argmin}} \frac{1}{M} \sum_{k=1}^{M} \frac{\rho(\mathbf{y}^{(k)})}{\hat{\rho}(\mathbf{y}^{(k)})} \| u^{(k)} - v(\mathbf{y}^{(k)}) \|_V^2$$

where the sample $\{\mathbf{y}^{(k)}\}_k$ is drawn from the distribution $\hat{\rho}(\mathbf{y})d\mathbf{y}$.

- $\rho(\mathbf{y}) = \hat{\rho}(\mathbf{y}) =$ Chebyshev distribution in $[-1,1]^N$, then the relation $M \propto \min\{2^N \# \Lambda, (\# \Lambda)^{\frac{\log(3)}{\log(2)}}\}$ is enough to guarantee optimal convergence [Chkifa-Cohen-Migliorati-N.-Tempone '13]
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Improvements on the quadratic relation

Improvements can be obtained by sampling from a different distribution $\hat{\rho}$. Let us consider the weighted least squares approx.

$$P_\Lambda^M u = \underset{v \in \mathbb{P}_\Lambda(\Gamma)\otimes V}{\operatorname{argmin}} \frac{1}{M} \sum_{k=1}^{M} \frac{\rho(\mathbf{y}^{(k)})}{\hat{\rho}(\mathbf{y}^{(k)})} \| u^{(k)} - v(\mathbf{y}^{(k)}) \|_V^2$$

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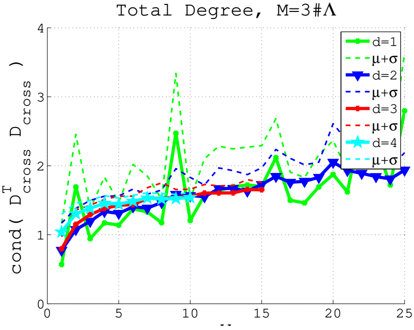
Numerical example with Chebyshev preconditioning

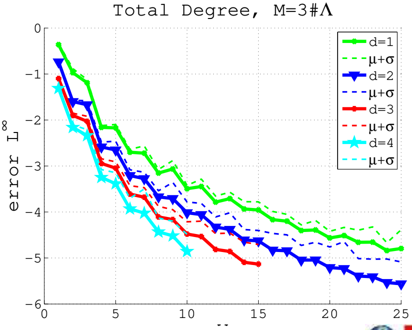
Expansion in Legendre polynomials ($\rho(\mathbf{y})$=uniform) and samples from Chebyshev distribution ($\hat{\rho}(\mathbf{y})$=Chebyshev)

condition number $cond(D^T D)$



$M = 3 \cdot \#\Lambda$

error for $u(y) = \left(1 + \frac{0.7}{2N}\sum_{n=1}^{N} y_n\right)^{-1}$



$M = 3 \cdot \#\Lambda$

Outline

Adaptive construction of polynomial spaces

$\{\Lambda_k\}_{k\geq 0}$ sequence of downward closed multi-index sets, with $\Lambda_0 = \{\mathbf{0}\}$.
The sequence is adaptively computed by means of greedy algorithms based on the random discrete L^2 projection.

Definitions:

- Margin $\mathcal{M}(\Lambda)$ associated to a multi-index set Λ:

$$\mathcal{M}(\Lambda) = \{\mathbf{p} \ : \ \mathbf{p} \notin \Lambda \text{ and } \exists j > 0 \ : \ \mathbf{p} - \mathbf{e}_j \in \Lambda\}$$

- Reduced margin $\mathcal{R}(\Lambda)$ associated to a multi-index set Λ:

$$\mathcal{R}(\Lambda) = \{\mathbf{p} \ : \ \mathbf{p} \notin \Lambda \text{ and } \forall j = 1, \dots, d \ : \ p_j \neq 0 \Rightarrow \mathbf{p} - \mathbf{e}_j \in \Lambda\}$$

set Λ and its Margin   set Λ and its Reduced margin

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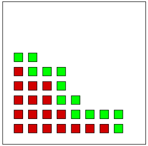
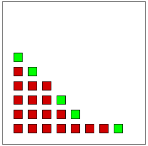


set Λ and its Margin



set Λ and its Reduced margin

The Dörfler marking

Idea proposed by W. Dörfler in 1996 for Adaptive Finite Elements.

Given a multi-index set Λ, a subset $R \subseteq \mathcal{R}(\Lambda)$, a (continuous) function $e : R \to \mathbb{R}$ and a parameter $\theta \in (0, 1]$, we define a procedure

$$\text{Dörfler} = \text{Dörfler}(R, e, \theta)$$

that computes a set $F \subseteq R \subseteq \mathcal{R}(\Lambda)$ of minimal cardinality such that

$$\sum_{\mathbf{p} \in F} e(\mathbf{p})^2 \geq \theta \sum_{\mathbf{p} \in R} e(\mathbf{p})^2.$$

In practice, for any $\mathbf{p} \in R$, the error indicator $e(\mathbf{p})$ will be either an estimator of the coefficient $c_{\mathbf{p}}$ of the function u expanded over the Legendre basis or the projected residual on the $\mathbf{p}$-th Legendre basis function.

This corresponds to choose a fraction θ of the energy associated with the (estimates of the) coefficients in the set R.

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Orthogonal Matching Pursuit with Dörfler marking

Algorithm 1 Orthogonal Matching Pursuit with Dörfler marking

Set $r_0 = u(\mathbf{y})$, $u_0 \equiv 0$ and $\Lambda_0 = \{\mathbf{0}\}$,
for $k = 1, \dots, k_{max}$ **do**
$F_1 = \text{Dörfler}(\mathcal{R}(\Lambda_{k-1}), \{|(r_{k-1}, \psi_{\mathbf{p}})_{M,V}|\}_{\mathbf{p}}, \theta_1)$
$\widetilde{\Lambda}_k = \Lambda_{k-1} \cup F_1$
$u_k = \text{argmin}_{v \in \mathbb{P}_{\widetilde{\Lambda}_k}} \|u - v\|_{M,V}, \qquad u_k = \sum_{\mathbf{p} \in \tilde{\Lambda}_k} c_{\mathbf{p}}^{(k)} \psi_{\mathbf{p}}$
$F_2 = \text{Dörfler}(F_1, \{c_{\mathbf{p}}^{(k)}\}_{\mathbf{p}}, \theta_2)$
$\Lambda_k = \Lambda_{k-1} \cup F_2$
$r_k = u - u_k|_{\Lambda_k}$
end for

$\theta_1 \in (0,1)$ and $\theta_2 = 1$: Dörfler marking only with the correlations.

$\theta_1 = 1$ and $\theta_2 \in (0,1)$: Dörfler marking only with the random discrete L^2 projection on $\Lambda_{k-1} \cup \mathcal{R}(\Lambda_k)$.

Some remarks and open issues

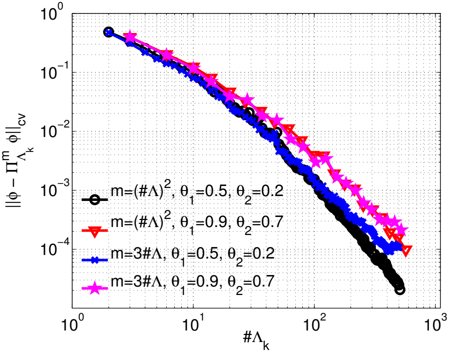
- The first Dörfler marking performs a screening of the reduced margin, to avoid an L^2 discrete minimization over a too large polynomial space.
- At each step the correlations $\{|(r_{k-1}, \psi_{\mathbf{p}})_{M,V}| \ : \ \mathbf{p} \in \mathcal{R}(\Lambda_k)\}$ are mutually uncoupled and cheap to compute, but might provide only a rough estimate of the coefficients (depending on the choice of M_k).
- The second Dörfler marking performs a selection based on the more accurate estimates of the coefficients coming from the L^2 projection.
- At each step the adaptive algorithm remains stable and accurate by choosing $M_k \propto (\#\Lambda_k)^2$ (consequence of the theory in the first part).
- The adaptive algorithm generates a sequence $\{\Lambda_k\}_{k\geq 0}$ of only quasi best N-term sets.
- Rate of convergence? Choice of θ_1, θ_2? What if $M_k \propto \#\Lambda_k$?

A numerical test

Approximation of a meromorphic function (16-variables)

$$\phi(\mathbf{y}) = \frac{1}{1 - \boldsymbol{\gamma} \cdot \mathbf{y}}, \quad \mathbf{y} \sim \mathcal{U}([-1,1]^{16})$$

$$\gamma = 0.3 * (1, 5 \cdot 10^{-1}, 10^{-1}, 5 \cdot 10^{-2}, \ldots, 5 \cdot 10^{-8})$$



Outline

1. Introduction – parametric / stochastic equations
2. Stochastic polynomial approximation
3. Discrete least squares approx. using random evaluations
   - Convergence analysis
   - Numerical results
4. Adaptive algorithms
5. Conclusions

Conclusions

- We have derived conditions under which the random discrete least squares approximation is stable and optimally convergent.
- The condition $M \geq C(\#\Lambda)^2$ for uniform random variables holds in any dimension and for any "shape" of the polynomial space, opening the possibility for adaptive algorithms.
- The condition $M \sim (\#\Lambda)^2$ seems to be too stringent in high dimension and a linear scaling is often enough, making this technique more attractive for high dimensional problems.
- Still open questions on preconditioned least squares or unbounded random variables.
- We have proposed an adaptive algorithm based on a double Dörfler marking that performs very well. The analysis is still ongoing.

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Thank you for your attention!

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