Goal-oriented low-rank approximations for high dimensional stochastic problems

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Abstract : We propose a minimal residual method for the solution of high dimensional equations using low-rank tensor formats. The measure of the residual is such that the resulting approximation is quasi-optimal with respect to a specified distance to the solution. This distance is chosen such that the optimality of the approximation is achieved with respect to some quantity of interest that can expressed as a linear function of the solution. The resulting method can be seen as a quasi-optimal goal-oriented model reduction method.

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Context : Uncertainty Quantification

Random parameters $\xi = (\xi_1, \dots, \xi_d)$	$\longrightarrow$	Model $\mathcal{A}_\xi u(\cdot,\xi) = \mathcal{B}_\xi$	$\longrightarrow$	Solution $u(x,\xi)$

$$u \in V \otimes L^2(\Xi, P_\xi)$$

- **Linear Quantity of interest :** $\ell : V \to \mathbb{R}$ is a linear determinist extractor :
- Expectation : $Lu = \mathbb{E}(\ell(u))$
- Conditional expectation : $Lu = \mathbb{E}(\ell(u) \mid \xi_\mu)$

$\Rightarrow$ Sobol indices (sensitivity analysis) : $S_\mu(\ell(u)) = \frac{\mathbb{V}(\ Lu\ )}{\mathbb{V}(\ell(u))}$

1–Discretisation & low rank approximation

- **Stochastic Galerkin approximation [4] :**

$$X = V_n \otimes S_p \quad \subset \quad V \otimes L^2(\Xi, P_\xi)$$

- V_n finite element space of dimension n,
- S_p polynomial chaos space of dimension p.
- Galerkin approximation on X defined by an algebraic linear system

$$Au = b$$

× Dimensionality problem : $\dim(X) = np$

- **Low rank tensor format [5, 6] :**

$$\mathcal{M}_r = \left\{ \sum_{k=1}^{r} v^k \otimes w^k \;\middle|\; v^k \in V_n,\ w^k \in S_p \right\} \subset X$$

✓ $\dim(\mathcal{M}_r) = r(n+p) \ll np$ for r small

- **Minimal residual formulations :**

$$\min_{v \in \mathcal{M}_r} \|Au - b\|_*$$

Usual norms do not take into account the quantity of interest

2–Goal-oriented norm

Let $L : X \to Z$ be a linear quantity of interest :

$$\|u - v\|_{X_\alpha}^2 = \|u - v\|_X^2 + \alpha \|Lu - Lv\|_Z^2$$

- Riesz maps :

$$\|\cdot\|_X^2 = \langle R_X \cdot, \cdot \rangle$$
$$\|\cdot\|_Z^2 = \langle R_Z \cdot, \cdot \rangle$$
$$\|\cdot\|_{X_\alpha}^2 = \langle \underbrace{(R_X + \alpha L^* R_Z L)}_{R_{X_\alpha}} \cdot, \cdot \rangle$$

- Interesting property : for any $\varepsilon > 0$

$$\|u - v\|_{X_\alpha} \leq \varepsilon \sqrt{\frac{1 + \alpha \|L\|^2}{\|L\|^2}} = e_\alpha \quad \Longrightarrow \quad \|Lu - Lv\|_Z \leq \varepsilon$$

$\Rightarrow$ Control of the quantity of interest.

How to minimize $\|u - v\|_{X_\alpha}$ since u is unknown ?

3–Ideal Minimal Residual formulation [2, 3]

$$\|u - v\|_{X_\alpha} = \|R_{Y_\alpha}^{-1}(b - Av)\|_{Y_\alpha}$$

- Riesz map associated with the residual norm : $R_{Y_\alpha} = A R_{X_\alpha}^{-1} A^*$

× Inverse of R_{Y_α} not computable in practice.

$\Rightarrow$ Use a perturbated formulation : find v such that

$$\|\Lambda^\delta R_{Y_\alpha}^{-1}(b - Av)\|_{Y_\alpha} \leq e_\alpha$$

with Λ^δ a mapping providing an approximation with relative precision $\delta \in]0, 1]$:

$$\forall z, \quad \|\Lambda^\delta(z) - z\|_{Y_\alpha} \leq \delta \|z\|_{Y_\alpha} \quad (\delta\text{-proximality})$$

4– Algorithm [1]

Find $\{y^k\}_{k>0}$ and $\{u^k\}_{k>0}$ such that :

1. Residual approximation : y^{k+1} such that

$$y^{k+1} = \Lambda^\delta\left(R_{Y_\alpha}^{-1}(b - Au^k)\right)$$

2. Low rank recompression : u^{k+1} such that

$$\left\| \left(u^k + R_{X_\alpha}^{-1} A^* y^{k+1}\right) - u^{k+1} \right\|_{X_\alpha} \leq e_\alpha$$

Residual approximation :

✓ Adaptive low-rank approximation

× Heuristic estimation of the δ-proximality

× R_{Y_α} is ill-conditionned when $\alpha \gg 1$

$$R_{Y_\alpha}^{-1} = (A R_X^{-1} A^*)^{-1} + \underbrace{\alpha (LA^{-*})^*(LA^{-*})}_{\text{needs preconditioning strategy}}$$

Convergence result : $\limsup_k \|Lu - Lu^k\|_Z \leq \frac{\varepsilon}{1 - \delta}$

5– Numerical results

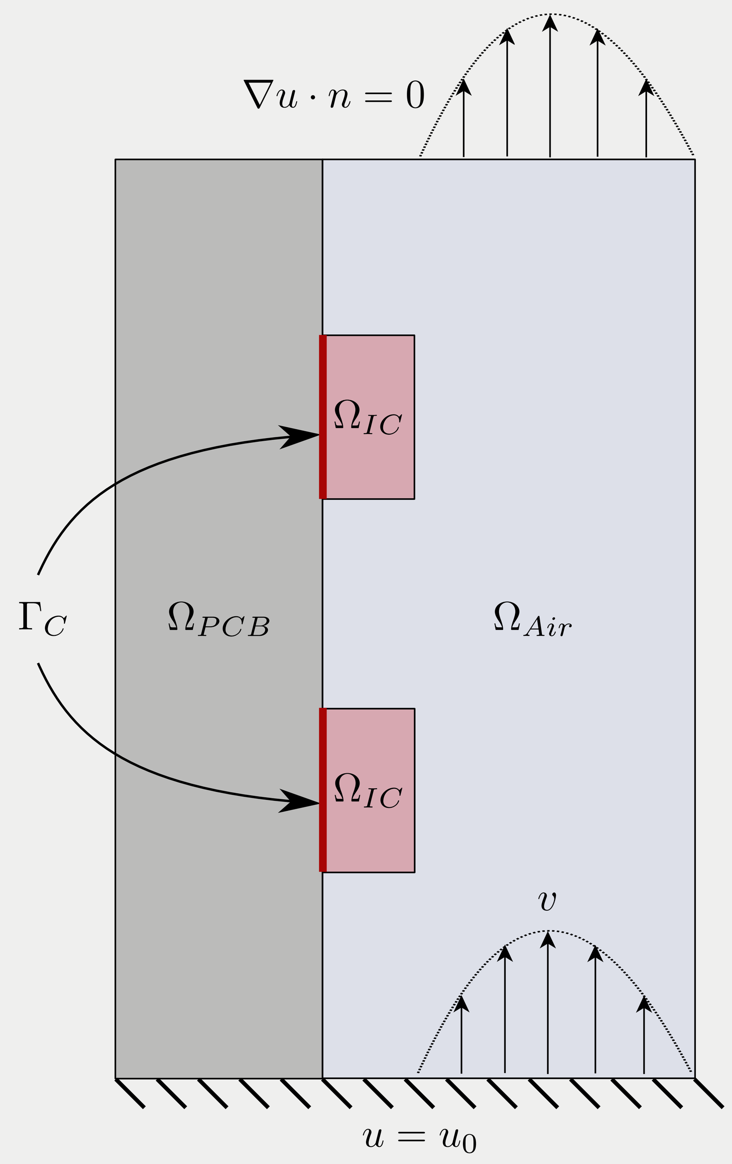
Cooling of electronic components (benchmark OPUS : http ://www.opus-project.fr)

Find u such that :

$$-\nabla \cdot \kappa \nabla u + D v \cdot \nabla u = f$$

Sources of uncertainty :

- diffusion coefficient : $\kappa_{IC} \sim \log \mathcal{U}\left(0.2, 2\right)$
- thermal contact conductance : $r \sim \log \mathcal{U}\left(0.1, 100\right)$
- advection field amplitude : $D \sim \log \mathcal{U}\left(5.10^{-4}, 10^{-2}\right)$



Quantity of interest :

$$Lu = \mathbb{E}\left(\int_{\Omega_{IC}} u \, d\Omega \;\middle|\; \kappa_{IC} \right)$$

Results :

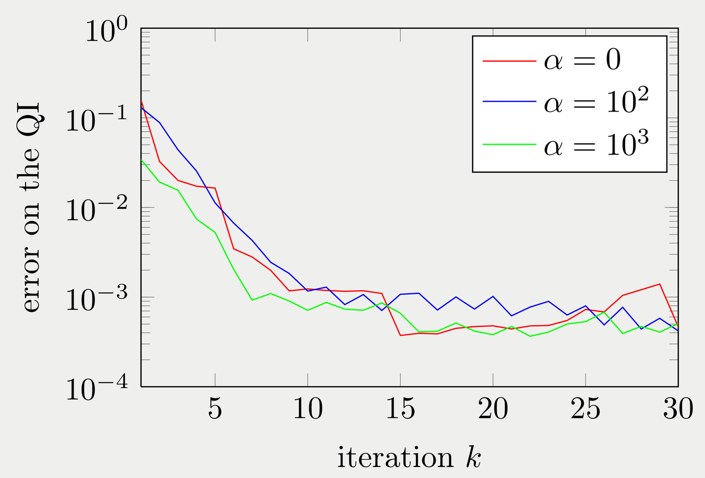


FIGURE: Evolution of the error on the quantity of interest during iteration process ($\delta = 0.5, \epsilon = 10^{-3}$)

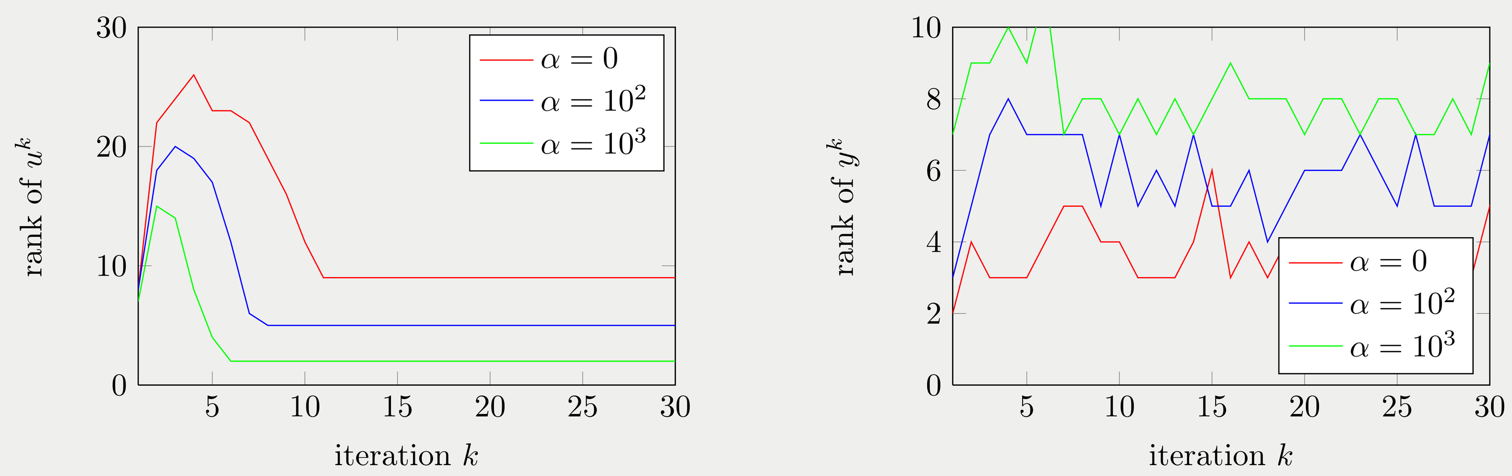


FIGURE: Evolution of the rank of u^k and y^k during iteration process ($\delta = 0.5, \epsilon = 10^{-3}$)

α \ ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
0	3	5	9	15	21
10^2	2	4	5	10	17
10^3	1	1	2	8	14
10^4	1	1	1	3	7
10^5	1	1	1	1	4
CMR	5	9	14	20	36

FIGURE: Final rank of the approximation u^k

- Achieve the requiered precision of the quantity of interest with respect to optimal rank,
- α larger implies a reduction of the rank.
- Comparison with the canonical minimal residual formulation (CMR) :

$$\min_{v \in \mathcal{M}_r} \|Av - b\|_2$$

Future work

- Control of the precision δ in the approximation of the residual.
- Non linear quantity of interest.

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