Stochastic Spectral Methods for Parametric Uncertainty Propagation

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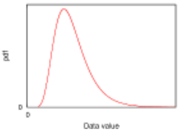
GdR Mascot-Num / working meeting

Model with uncertain parameters (data)

$$\mathcal{M}(U, D) = 0.$$

- U model solution (output), D model parameters (BCs, ICs, forcing, system properties, geometry,...).
- Consider the data uncertain and represented as a random quantity :
$$\mathcal{P} = (\Omega, \Sigma, \mathrm{d}\mu) \text{ and } D \to D(\omega).$$
- Since D is random the model solution is also random, $U \to U(\omega)$ and U and D are dependent random quantities :
$$\mathcal{M}(U(\omega), D()) = 0 \; a.s.$$

Data density



$$\mathcal{M}(U, D) = 0$$

Solution density

Parametrization of uncertainties

- Let $\xi := (\xi_1, \ldots, \xi_N) := \Xi \subset \mathbb{R}^N$ be a set of RVs defined on $(\Omega, \Sigma, d\mu)$ with known joint-density function p_ξ.
- Let $L^2(\Xi)$ be the space of 2nd order RV in ξ :

$$f : \xi \in \Xi \mapsto f(\xi) \in L^2(\Xi) \leftrightarrow \|f\|_{L^2(\Xi)} = \langle f, f \rangle^{1/2} < +\infty,$$

  where

$$\langle f, g \rangle := \int_\Xi f(\xi) g(\xi) p_\xi(\xi) d\xi.$$

- Assume available a representation of D using ξ :

$$D(\omega) \to D(\xi(\omega)).$$

- Iso-probabilistic transformation, KL expansion, identification, ...

Spectral representation of the solution

- Therefore,
$$\mathcal{M}(U(\xi), D(\xi)) = 0.$$
- So we have to compute U(ξ).
- We assume $U(\xi) \in L^2(\Xi) \otimes \mathcal{V}$. $\mathcal{V}$: deterministic space.
- $U(\xi)$ is sought as a functional expansion of the form
$$U(\xi) = \sum_{\alpha} U_\alpha \Psi_\alpha(\xi).$$
- $\{U_\alpha \in \mathcal{V}\}$ is the set of (determinsitic) expansion coefficients and $\{\Psi_\alpha\}$ is a set of functionals in ξ spanning $L^2(\Xi)$.

Generalized Polynomial Chaos Expansion

- A classical choice for $\{\Psi_\alpha\}$ is the COS of polynomials in ξ.
- We truncate the expansion to a polynomial degree No,

$$\mathcal{B} = \{\Psi_\alpha, deg(\Psi_\alpha) \leq \mathrm{No}\}, \quad \mathcal{S} = \text{span}\, \mathcal{B} \subset L^2(\Xi),$$

  where $\langle \Psi_\alpha, \Psi_\beta \rangle = \|\Psi_\alpha\|^2_{L^2(\Xi)} \delta_{\alpha,\beta}$.
- $U(\xi)$ is then approximated by $U^{\mathrm{P}}(\xi)$,

$$U(\xi) \approx U^{\mathrm{P}}(\xi) := \sum_{\alpha=1}^{\mathrm{P}} U_\alpha \Psi_\alpha(\xi) \in \mathcal{S} \otimes \mathcal{V}, \quad \mathrm{P} = \dim(\mathcal{S}).$$

- For a truncation at total degree No, we have $\mathrm{P} = (\mathrm{NNo})!/(\mathrm{N}!\mathrm{No}!)$: fast increase in the dimension of the expansion with N and No.

Generalized Polynomial Chaos Expansion

- Often, ξ_i are independent with "classical distributions" : leads to classical families of orthogonal polynomials.
- If the mapping $\boldsymbol{\xi} \in \Xi \mapsto U(\boldsymbol{\xi})$ is smooth we expect a fast convergence of the polynomial approximation.
- Other types of expansion functionals can be used, *e.g.* piecewise polynomial functionals, hierarchical orthogonal expansion (multi-wavelets), . . .
- Dimension of $\mathcal{S}$ is critical for computational efficiency : adaptive / progressive construction of $\mathcal{B}$.

Given $\mathcal{B}$, how to compute the expansion coefficients

$$U^{\mathrm{P}}(\boldsymbol{\xi}) := \sum_{\alpha=1}^{\mathrm{P}} U_\alpha \Psi_\alpha(\boldsymbol{\xi}).$$

Non intrusive methods

- Compute/estimate spectral coefficients *via* a set of deterministic model solutions
- Requires a deterministic solver only
- Overcome issues related to non-linearities.
- Suffers from the curse of dimensionnality

Non-intrusive methods

Basics

Use code as a black-box

- Compute/estimate spectral coefficients *via* a set of deterministic model solutions
- Requires a deterministic solver only

1. $\mathcal{S}_\Xi \equiv \{\xi^{(1)}, \dots, \xi^{(m)}\}$ sample set of ξ
2. Let $U^{(i)}$ be the solution of the deterministic problem
$$\mathcal{M}\left(U^{(i)}, D(\xi^{(i)})\right) = 0$$
3. $\mathcal{U}_S \equiv \{U^{(1)}, \dots, U^{(m)}\}$ sample set of model solutions
4. Estimate expansion coefficients U_α from this sample set.

- Complex models, reuse of determinsitic codes, planification, ...
- Error control and computational complexity (curse of dimensionality), ...

Least square fit "Regression"

- Best approximation is defined by minimizing a (weighted) sum of squares of residuals :

$$R^2(U_1,\dots,U_P) \equiv \sum_{i=1}^{m} w_i \left(U^{(i)} - \sum_{\alpha=1}^{P} U_\alpha \Psi_\alpha \left(\xi^{(i)}\right)\right)^2 .$$

Advantages/issues

- Convergence with number of regression points m
- Selection of the regression points and "regressors" Ψ_α
- Error estimate

Non intrusive spectral projection : NISP

Exploit the orthogonality of the basis :

$$\|\Psi_\alpha^2\|_{L^2(\Xi)}^2 U_\alpha = \langle U, \Psi_\alpha \rangle = \int_\Xi U(\xi)\Psi_\alpha(\xi)p(\xi)d\xi.$$

Computation of P N-dimensional integrals

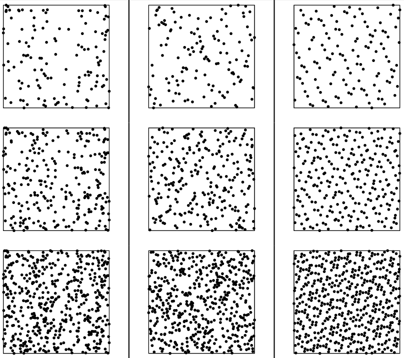
$$\langle U, \Psi_\alpha \rangle \approx \sum_{i=1}^{N_Q} w^{(i)} U\left(\xi^{(i)}\right) \Psi_\alpha\left(\xi^{(i)}\right).$$

Non intrusive projection — Random Quadratures

Approximate integrals from a (pseudo) random sample set $\mathcal{U}_S$:

$$\langle U, \Psi_\alpha \rangle \approx \frac{1}{m} \sum_{i=1}^{m} U^{(i)} \Psi_\alpha \left(\xi^{(i)} \right).$$

MC	LHS	QMC



- Convergence rate
- Error estimate
- Optimal sampling strategy
- Isotropy

Non intrusive projection — Deterministic Quadratures

Approximate integrals by N-dimensional quadratures :

$$\langle U, \Psi_\alpha \rangle \approx \sum_{i=1}^{N_Q} w^{(i)} U^{(i)} \Psi_\alpha \left(\xi^{(i)} \right).$$

Quadrature points $\xi^{(i)}$ and weights $w^{(i)}$ obtained by

- full tensorization of n points 1-D quadrature (*i.e.* Gauss) :

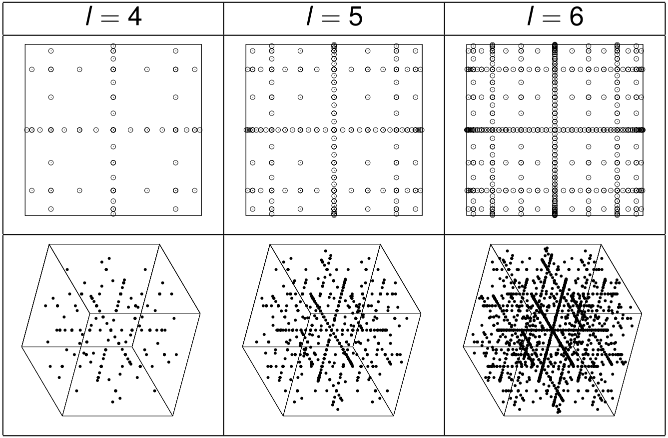
  $$\boxed{N_Q = n^{\mathrm{N}}}$$

- partial tensorization of nested 1-D quadrature formula (Féjer, Clenshaw-Curtis) :

  $$\boxed{N_Q << n^{\mathrm{N}}}$$

Non intrusive projection

Deterministic Quadratures



- Important development of sparse-grid methods
- Anisotropy and adaptivity
- Extension to collocation approach ($\mathbb{N}$-dimensional interpolation)
- Model selection, compressed-sensing, ...

Galerkin projection

Method of weighted residual

① Introduce truncated expansions in model equations

② Require residual to be $\perp$ to the stochastic subspace $\mathcal{S}$

$$\left\langle \mathcal{M}\left(\sum_{\alpha=1}^{\mathrm{P}} U_\alpha \Psi_\alpha(\xi), D(\xi)\right), \Psi_\beta(\xi)\right\rangle = 0 \quad \text{for } \beta = 1, \ldots, \mathrm{P}.$$

Set of P coupled problems.

Plus

- Virtues of Galerkin methods
- Often inherit properties of the deterministic model
- Can achieve better performance than NI

Minus

- Requires adaptation of deterministic solvers
- Treatment of non-linearities
- Full coupling penalize adaptive strategies.

Classical spectral expansions

$$U(\xi) \approx U^{\mathrm{P}}(\xi) = \sum_{\alpha=1}^{\mathrm{P}} U_\alpha \Psi_\alpha(\xi)$$

- a priori selection of $\mathcal{B}$
- Projection of $U(\xi)$ on $\mathcal{S}$
- Computational cost increases with $\mathrm{P} = \dim \mathcal{B}$
- Selection of $\mathcal{B}$ is critical
- Computation of the U_α are coupled
- Can require significant adaption of solvers (Galerkin).

Reduced basis approximation

Instead of

$$U(\xi) \approx U^{\mathrm{P}}(\xi) = \sum_{\alpha=1}^{\mathrm{P}} U_\alpha \Psi_\alpha(\xi),$$

one seeks for an expansion of the form

$$U(\xi) \approx U^{\mathrm{M}}(\xi) = \sum_{\alpha=1}^{\mathrm{M}<\mathrm{P}} U_\alpha \lambda_\alpha(\xi)$$

where the $\lambda_\alpha \in \mathcal{S}$ are not selected a priori.

Reduced basis approximation

$$U(\xi) \approx U^{\mathrm{M}}(\xi) = \sum_{\alpha=1}^{\mathrm{M}<\mathrm{P}} U_\alpha \lambda_\alpha(\xi)$$

where the $\lambda_\alpha \in \mathcal{S}$ are not selected a priori.

- The reduced basis expansion is defined as to minimize (in some sense) the Galerkin residual in $\mathcal{S}$
- In the case of a symmetric positive definite stochastic operator, optimality is defined with respect to the norm induced by the operator
- Algorithms have been proposed to approach the optimal decomposition
- Successive construction of the couples $(U_\alpha, \lambda_\alpha)$
- Iterative techniques (Power-iteration, Arnoldi, . . .) leads to weakly intrusive methods
- Convergence essentially function of the stochastic dimensionality of the solution.

PhD work of Julie Tryoen

with Alexandre Ern (Cermics, Univ. Paris-Est)
and Michael Ndjinga (CEA, Saclay)

Hyperbolic systems : deterministic case

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{f}(\boldsymbol{u}) = 0, \quad \boldsymbol{u}(\boldsymbol{x}, t=0) = \boldsymbol{u}^0(\boldsymbol{x}), \quad BCs$$

- $\boldsymbol{u} \in \mathcal{A}_{\boldsymbol{u}} \subset \mathbb{R}^m$ (conservative variables)
- $\boldsymbol{f} : \mathcal{A}_{\boldsymbol{u}} \mapsto \mathbb{R}^m$ (flux function)
- if $\nabla_{\boldsymbol{u}} \boldsymbol{f} \in \mathbb{R}^{m \times m}$ is $\mathbb{R}$-diagonalizable on $\mathcal{A}_{\boldsymbol{u}} \Longrightarrow$ hyperbolic
- $\boldsymbol{u}$ can develop shocks / discontinuities in finite time

Classical discretization (Finite Volume in 1-space dimension)

$$\frac{\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^n}{\Delta t} + \frac{\widetilde{\boldsymbol{f}}(\boldsymbol{u}_i^n, \boldsymbol{u}_{i+1}^n) - \widetilde{\boldsymbol{f}}(\boldsymbol{u}_{i-1}^n, \boldsymbol{u}_i^n)}{\Delta x} = 0$$

where $\boldsymbol{u}_i^n = \int_{\Delta x} \boldsymbol{u}(x, t_n) \mathrm{d}x$ and $\widetilde{\boldsymbol{f}}(,)$ is the numerical flux function (having had-hoc properties).

Uncertain hyperbolic problems :

- Uncertain initial & boundary conditions and parameters in $\boldsymbol{f}$
- Parametrization with $\boldsymbol{\xi}(\theta) = \{\xi_1(\theta), \ldots; \xi_{\mathrm{N}}(\theta)\}$ a set of N iid random variables with uniform distribution on $\Xi = [0,1]^{\mathrm{N}}$
- Stochastic Hyperbolic Problem

$$\frac{\partial \boldsymbol{U}(\boldsymbol{x},t,\xi)}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U};\xi) = 0, \quad \boldsymbol{U}(\boldsymbol{x},t=0,\xi) = \boldsymbol{U}^0(\boldsymbol{x},\xi) \quad (a.s.)$$

Hypotheses

1. $\boldsymbol{U}(\boldsymbol{x},t,\xi) \in \mathcal{A}_{\boldsymbol{U}}$ and $\nabla_{\boldsymbol{U}}\boldsymbol{F}(\boldsymbol{U};\xi)$ is $\mathbb{R}$-diagonalizable a.s.
2. all random quantities have finite variance.

The solution is sought in $\mathcal{S}^{\mathrm{P}} := \mathrm{span}\,\{\Psi_0,\ldots,\Psi_{\mathrm{P}}\} \subset L_2(\Xi)$, where Ψ_α are orthonormal polynomials in $\boldsymbol{\xi}$ with degree $\leq \mathrm{No}$: $\langle \Psi_\alpha, \Psi_\beta \rangle = \delta_{\alpha,\beta}$.

Galerkin problem :

- Since $\boldsymbol{U} \in L^2(\Xi)$ it has a convergent expansion :

$$\boldsymbol{U}(\boldsymbol{x}, t, \xi) = \sum_{\alpha} \boldsymbol{u}_\alpha(x, t)\Psi_\alpha(\xi)$$

- We denote $\boldsymbol{U}^{\mathrm{P}}$ the approximation of $\boldsymbol{U}$ in $\mathcal{S}^{\mathrm{P}}$
- Stochastic Galerkin projection of the hyperbolic problem : for $\alpha = 0, \ldots, \mathrm{P}$

$$\frac{\partial \boldsymbol{u}_\alpha(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot \boldsymbol{f}_\alpha(\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{\mathrm{P}}) = 0$$
$$\boldsymbol{f}_\alpha(\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{\mathrm{P}}) \equiv \left\langle \boldsymbol{F}(\boldsymbol{U}^{\mathrm{P}}; \xi), \Psi_\alpha \right\rangle$$
$$\boldsymbol{u}_\alpha(\boldsymbol{x}, t = 0) = \left\langle \boldsymbol{U}^0(\boldsymbol{x}), \Psi_\alpha \right\rangle$$

$(\mathrm{P}+1)$-coupled problems for the solution modes

Galerkin problem : (system form)

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{u}_0 \\ \vdots \\ \boldsymbol{u}_{\mathrm{P}} \end{pmatrix} + \boldsymbol{\nabla}\cdot\begin{pmatrix} \boldsymbol{f}_0(\boldsymbol{u}_0,\ldots,\boldsymbol{u}_{\mathrm{P}}) \\ \vdots \\ \boldsymbol{f}_{\mathrm{P}}(\boldsymbol{u}_0,\ldots,\boldsymbol{u}_{\mathrm{P}}) \end{pmatrix} = 0$$

$$\frac{\partial \mathcal{U}}{\partial t} + \boldsymbol{\nabla}\cdot\mathcal{F}(\mathcal{U}) = 0$$

- $\mathcal{U} \in \mathbb{R}^{m\times(\mathrm{P}+1)}$
- $\mathcal{F} : \mathbb{R}^{m\times(\mathrm{P}+1)} \mapsto \mathbb{R}^{m\times(\mathrm{P}+1)}$
- Is the Galerkin problem hyperbolic ?
- ($\nabla_{\mathcal{U}}\mathcal{F}$ $\mathbb{R}$-diagonalizable ?)
- What is the admissible domain $\mathcal{A}_{\mathcal{U}}$?

[J. Tryoen et al, JCP 2010, JCAM 2010]

Approximate Roe solver

$$\mathcal{U}_i^{n+1} = \mathcal{U}_i^n - \frac{\Delta t}{\Delta x}\left[\phi(\mathcal{U}_i^n, \mathcal{U}_{i+1}^n) - \phi(\mathcal{U}_{i-1}^n, \mathcal{U}_i^n)\right]$$

where the numerical flux Φ is chosen as

$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{1}{2}\left[\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)\right] - a\frac{\mathcal{U}_R - \mathcal{U}_L}{2}$$

where $a \in \mathbb{R}^{m(\mathrm{P}+1)\times m(\mathrm{P}+1)}$ is a non-negative upwind matrix

Theorem : It exists a Galerkin Roe state $\mathcal{U}_{L,R}^{\mathrm{Roe}}$ such that

$\nabla \mathcal{F}_{\mathcal{U}}(\mathcal{U}_{L,R}^{\mathrm{Roe}})$ is a Roe matrix for the Galerkin problem

i.e. has properties of consistency and conservativity through shocks.

We will take

$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{1}{2}\left[\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)\right] - \left|\nabla_{\mathcal{U}}\mathcal{F}(\mathcal{U}_{L,R}^{\mathrm{Roe}})\right|\frac{\mathcal{U}_R - \mathcal{U}_L}{2}$$

where $|A| = |LDR| = L\,|D|\,R$ for a $\mathbb{R}$-diagonalizable matrix

[J. Tryoen et al, JCP 2010]

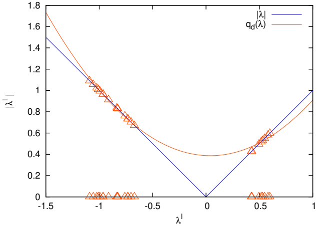
Fast approximation of the upwind matrix

To avoid the costly decomposition of the Roe matrix, we rely on a polynomial transform q_d :

- recall $q(LDR) = Lq(D)R$
- $|\nabla_{\mathcal{U}}\mathcal{F}| \approx q_d\left(\nabla_{\mathcal{U}}\mathcal{F}\right)$, where $q_d \in \mathbb{P}_d$ minimizes

$$J = \sum_{i,l}\left[q_d\left(\Lambda_i^l\right) - \left|\Lambda_i^l\right|\right]^2, \quad \Lambda_i^l \approx \Lambda^l\left(\boldsymbol{U}_{LR}^{\mathrm{Roe}}(\xi^{(i)})\right)$$

- In practice $d \leq 6$ is sufficient



Approximation polynomial q_d for $d = 2$ (left) and $d = 6$ (right).

Summary :

$$\mathcal{U}_i^{n+1} = \mathcal{U}_i^n - \frac{\Delta t}{\Delta x}\left[\phi(\mathcal{U}_i^n, \mathcal{U}_{i+1}^n) - \phi(\mathcal{U}_{i-1}^n, \mathcal{U}_i^n)\right]$$

where

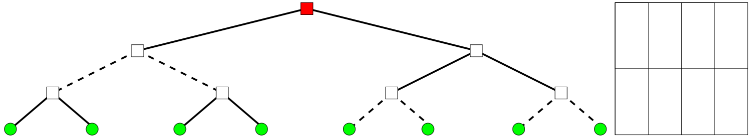
$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{1}{2}\left[\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)\right] - q_d\left(\nabla_{\mathcal{U}}\mathcal{F}(\mathcal{U}^{\mathrm{Roe}})\right)\frac{\mathcal{U}_R - \mathcal{U}_L}{2}$$

- Upwinding w.r.t. the actual waves in the Galerkin solution
- Applies conditionally to partially tensored stochastic basis
- May need Entropy corrector [J. Tryoen et al, JCAM 2010]
- Assume $\boldsymbol{U}(\xi)$ smooth and sufficient stochastic discretization
- But solutions are not smooth in general !

Call for piecewise polynomial approximations to allow for discontinuities at the stochastic level

Binary trees for piecewise polynomial space

$\mathcal{S} = \mathcal{S}(\mathtt{T})$



- Dyadic partitions of a node along a prescribe direction d : $\mathrm{p} \rightarrow (\mathrm{c}^-, \mathrm{c}^+)$
- Piecewise-polynomial with **fixed** order No on each leaf of $\mathtt{T}$.

1. Union of local modal basis : SE-basis

   [Deb et al, 2001], [Karniadakis et al]
   Uncoupled application of the Roe scheme over different leafs

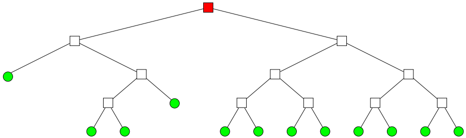
2. Hierarchical global basis over Ξ : MW-Basis

   [OLM et al, 2004]
   Hierarchical sequence of details, suited for adaptive scheme

Adaptivity

Singularity curves are localized in Ξ : **stochastic adaptivity**

- **Incomplete and anisotropic binary trees**



Operators for multi-resolution analysis :

- **Prediction operator** : define the solution in a stochastic space larger than the current one (add new leafs and L^2-injection).
- **Restriction operator** : define the solution in a stochastic space smaller one the current one (remove leafs and L^2-projection).
- Rely on recursive application of **elementary (directional) operators**, full exploitation of the tree structure.

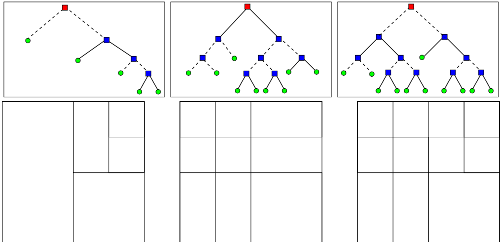
Adaptivity :

Singularity curves are localized in x and t

- **Each spatial cell** carries its own adapted stochastic discretization
- Flux computation,

$$\phi(\mathcal{U}_L,\mathcal{U}_R) = \frac{\mathcal{F}(\mathcal{U}_L)+\mathcal{F}(\mathcal{U}_R)}{2} - \left|a^{\mathrm{Roe}}(\mathcal{U}_L,\mathcal{U}_R)\right| \frac{\mathcal{U}_R-\mathcal{U}_L}{2},$$

  with $\mathcal{U}_R$ and $\mathcal{U}_L$ known on **different stochastic spaces**
- **Union operator** : given two stochastic spaces, construct the minimal stochastic space containing the two :



Adaptive Algorithm :

1. Loop over all interfaces of the spatial mesh :
   - Construct the union space of the left and right cells
   - **Enrich** this space
   - Predict left and right states of the interface
   - Evaluate the numerical flux (App. Roe scheme)
2. Loop over all cells of the spatial mesh :
   - Construct the union space of the cell's interfaces
   - Predict cell's fluxes on the union space
   - Compute fluxes difference and update cell's solution
   - Restrict cell's solution by **thresholding**
3. Repeat for the next time step

Two indicators needed : based on multiwavelet details of nodes.

- for Enrichment : anticipate emergence of new stochastic details,
- for Thresholding : remove unnecessary/negligible details.

Thresholding criterion :

Let us denote

- $\mathtt{T}$ a binary tree and $\mathcal{S}(\mathtt{T})$ the corresponding stochastic approximation space
- $\mathtt{n} \in \mathcal{N}(\mathtt{T})$ a node of the tree, and $\widehat{\mathcal{N}}(\mathtt{T})$ set set of nodes having children
- Nr the maximal depth allowed in a direction
- $\mathtt{T}_{[\mathrm{NNr}]}$ the maximal tree given Nr

We define for $U \in \mathcal{S}(\mathtt{T}_{[\mathrm{NNr}]})$ and $\eta > 0$ the subset of $\mathcal{N}(\mathtt{T}_{[\mathrm{NNr}]})$

$$\mathcal{D}(\eta) := \left\{ \mathtt{n} \in \widehat{\mathcal{N}}(\mathtt{T}_{[\mathrm{NrN}]}); \|\tilde{\boldsymbol{u}}^{\mathtt{n}}\|_{\ell^2} \leq 2^{-|\mathtt{n}|/2} \frac{\eta}{\sqrt{\mathrm{NNr}}} \right\},$$

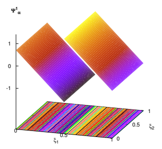
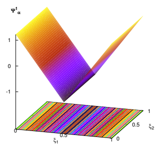
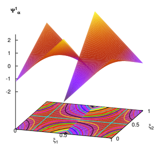
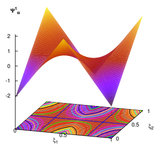
where $\tilde{\boldsymbol{u}}^{\mathtt{n}} := (\tilde{u}^{\mathtt{n}}_{\alpha})_{1 \leq \alpha \leq \mathrm{P}}$ are the MW coefficients of $\mathtt{n}$.

Coarsening strategy :

Two sisters (c^-, c^+) of a parent $p(c^-)$ are removed from the discretization if

$$\|\tilde{\boldsymbol{u}}^{p(c^-)}\|_{\ell^2} \leq 2^{-|n|/2} \frac{\eta}{\sqrt{\mathrm{NNr}}}$$

The criterion ensures that $\|U^{\mathrm{T}_{[\mathrm{NNr}]}} - U^{\mathrm{T}_{[\mathrm{NNr}]}\setminus\mathcal{D}}\| \leq \eta$.

$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
			

Mother wavelets $\tilde{\Psi}^{d}_{\alpha}$ for $N = 2$, $No = 1$ in direction $d = 1$.

Note : the coarsening is applied to the class of equivalent trees.

Enrichment strategy :

Enrichment is necessary to anticipate emergence of new-stochastic details.

- Isotropic enrichment is not an option for $\mathrm{N} > 2, 3$
- 1-D enrichment criterion : if U is (locally) smooth enough $\tilde{u}_{\alpha}^{\mathtt{n}}$ of a generic node $\mathtt{n}$ can be bounded as

$$|\tilde{u}_{\alpha}^{\mathtt{n}}| = \inf_{P \in \mathbb{P}_{\mathrm{No}}[\xi]} |\langle (U - P), \Psi_{\alpha}^{\mathtt{n}} \rangle| \leq C |S(\mathtt{n})|^{\mathrm{No}+1} \|U\|_{H^{\mathrm{No}+1}(S(\mathtt{n}))},$$

where $|S(\mathtt{n})| = 2^{-|\mathtt{n}|}$ is the volume of the node. Therefore $\|\tilde{\boldsymbol{u}}^{\mathtt{n}}\|_{\ell^2} \sim 2^{-(\mathrm{No}+1)} \|\tilde{\boldsymbol{u}}^{\mathtt{p}(\mathtt{n})}\|_{\ell^2}$ and a leaf $\mathtt{l}$ is refined if

$$\|\tilde{\boldsymbol{u}}^{\mathtt{p}(\mathtt{l})}\|_{\ell^2} \geq 2^{\mathrm{No}+1} 2^{-|\mathtt{l}|/2} \eta / \sqrt{\mathrm{Nr}} \quad \text{and} \quad |\mathtt{l}| < \mathrm{Nr}.$$

Enrichment strategy :

Extension of to the N-dimensional case :

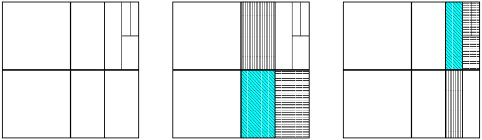
- Using the decay estimation

$$|\tilde{u}^{\mathrm{n}}_{\alpha}| = \inf_{P \in \mathbb{P}^{\mathrm{N}}_{\mathrm{No}}[\xi]} \left|\left\langle (U - P), \Psi^{\mathrm{n},d}_{\alpha} \right\rangle\right| \leq C \mathrm{diam}(S(\mathrm{n}))^{\mathrm{No}+1} \|U\|_{H^{\mathrm{No}+1}(S(\mathrm{n}))},$$

- a leaf $\mathtt{l}$ is partitioned in direction d if

$$\|\tilde{\boldsymbol{u}}^{\mathrm{p}^d(\mathtt{l})}\|_{\ell^2} \geq \frac{\mathrm{diam}(S(\mathrm{p}^d(\mathtt{l})))}{\mathrm{diam}(S(\mathtt{l}))}^{\mathrm{No}+1} 2^{-|\mathtt{l}|/2} \eta / \sqrt{\mathrm{NNr}} \quad \text{and} \quad |S(\mathtt{l})|_d > 2^{-\mathrm{Nr}}.$$

- the construction of the virtual sister and parent of $\mathtt{l}$ in arbitrary direction d



A sharper anisotropic criterion has been proposed using $1-D$ analysis functions in direction d [J.Tryoen, preprint 2012].

1. **Introduction**
   - Parametric uncertainty propagation
   - Spectral expansion
2. **Solution methods**
   - Non-Intrusive methods
   - Galerkin projection
   - Reduced basis approach
3. **Stochastic hyperbolic systems**
   - Galerkin solver
   - Stochastic adaptation
   - Adaptive scheme
   - Burgers equation
   - Traffic equation

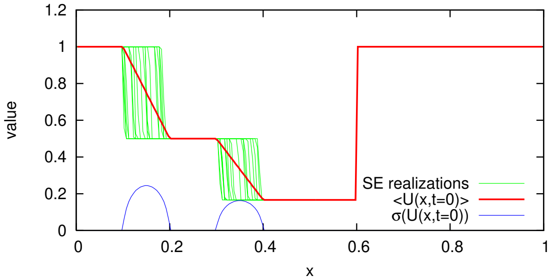
Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}$$

Uncertain initial condition $U^0(x, \xi)$:

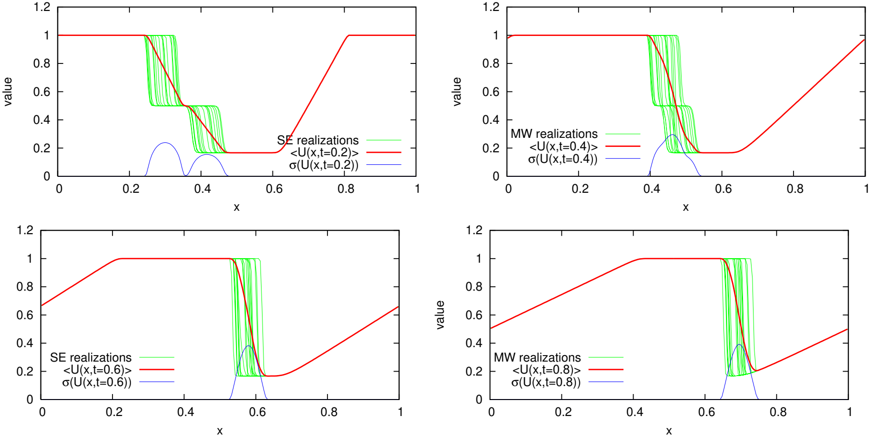
$$X_{1,2} = 0.1 + 0.1\xi_1, \quad X_{2,3} = 0.3 + 0.1\xi_2, \quad \xi_1, \xi_2 \sim \mathcal{U}[0, 1]$$

2 stochastic dimensions.



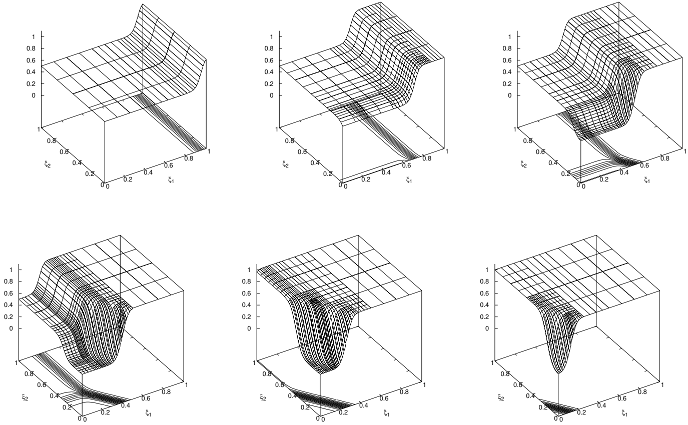
Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}$$



Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}$$



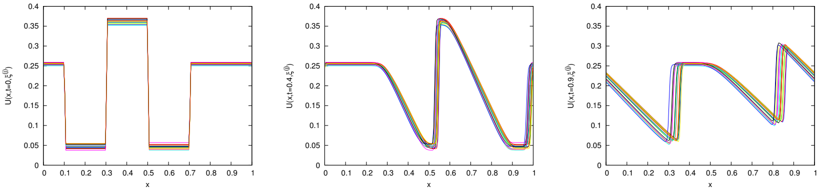
Trafic equation in periodic $[0,1]$-domain

$$F(U(\xi);\xi) = A(\xi)U(\xi)(1-U(\xi)) \quad \text{1-Periodic BC.}$$

- uncertain initial density of vehicles

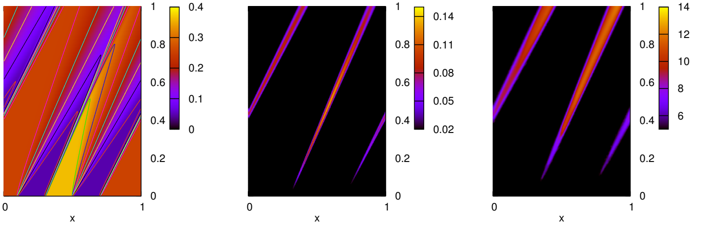
$$U^0(x,\xi) = 0.25 + 0.01\xi_1 - \mathbb{I}_{[0.1,0.3]}(x)(0.2+0.015\xi_2) + \mathbb{I}_{[0.3,0.5]}(x)(0.1+0.015\xi_3) - \mathbb{I}_{[0.5,0.7]}(x)(0.2+0.015\xi_4)$$

- uncertain characteristic velocity $A(\xi) = 1 + 0.1\xi_5$
- 5-dimensional problem $(\xi_1,\ldots,\xi_5) \sim U[0,1]^5$.

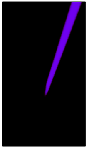
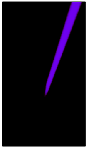
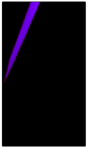
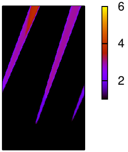


20 realizations of the initial condition (left) and solution at $t = 0.4$ (middle) and $t = 0.9$ (right) : 2 shocks and 2 rarefaction waves.

Space-time diagrams of the solution mean (left), standard deviation (center) and average depth of the leafs (right) :



Averaged number of partitions in each direction D_i and anisotropy factor ρ :

D_1	D_2	D_3	D_4	D_5	ρ
					

Hoeffding decomposition. Orthogonal hierarchical decomposition

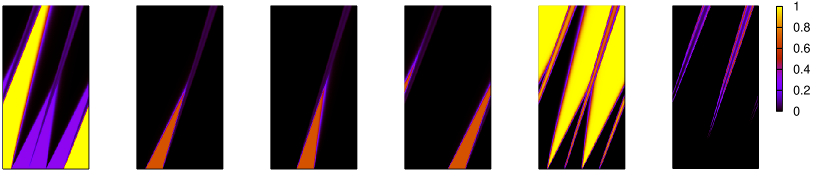
$$U(\xi_1,\ldots,\xi_N) = U_0 + \sum_{i_1=1}^{N} U_{i_1}(\xi_{i_1}) + \sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N} U_{i_1,i_2}(\xi_{i_1},\xi_{i_2}) + \ldots$$
$$+ U_{1,\ldots,N}(\xi_{i_1},\ldots,\xi_{i_N}),$$

Sobol ANOVA (analysis of the variance)

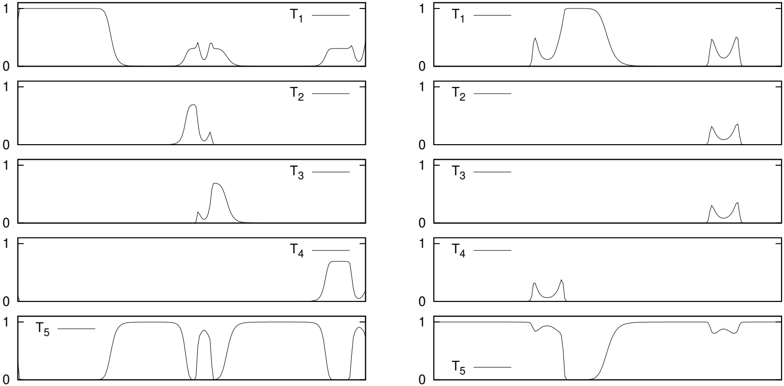
$$V(U) = \sum_{i_1=1}^{N} V_{i_1} + \sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N} V_{i_1,i_2} + \cdots + V_{1,\ldots,N},$$

- First order sensitivity indexes : $S_i = V_i/V$
- Total sensitivity indexes : $T_i = \sum_{\{u\}\ni\{i\}} V_{\{u\}}/V$

S_1	S_2	S_3	S_4	S_5	$1 - \sum_d S_d$

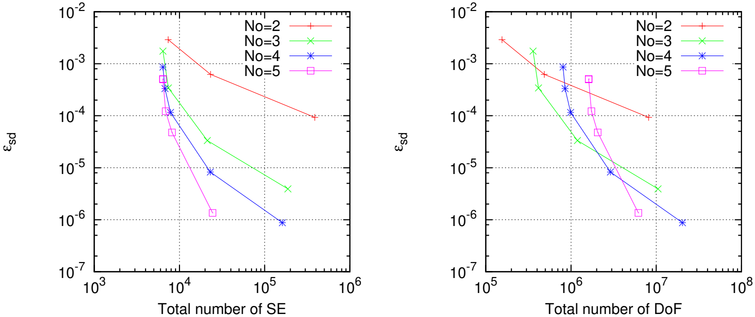


Space-time diagrams of the 1-st order sensitivity indexes S_i and contribution of higher order indexes.



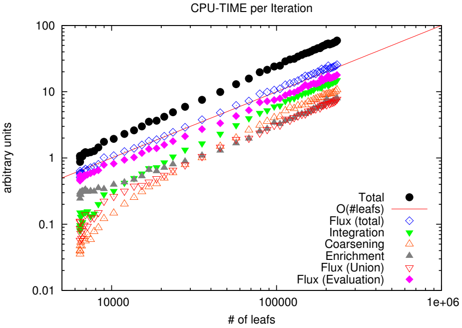
Total sensitivity indices as a function of $x \in [0, 1]$ at $t = 0.4$ (left) and $t = 0.9$ (right).

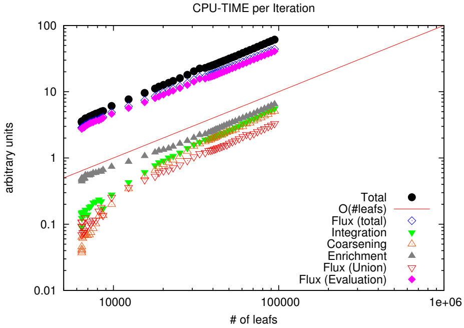
L^2-norm of stochastic error for different values of $\eta \in [10^{-2}, 10^{-5}]$ and polynomial degrees No



Left : error as a function of the total number of leafs in the final discretization ($t^n = 0.5$). Right : error as a function of the total number of degrees of freedom (number of leafs times the dimension of the local polynomial basis).

Traffic equation





Computational time (per time-iteration) as a function of the stochastic discretization (total number of leafs) ; left : $\mathrm{No} = 2$ and $\eta = 10^{-3}$; right : $\mathrm{No} = 3$ and $\eta = 10^{-4}$.

-Thank you for your attention-

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