Uncertainty quantification for systems of conservation laws in high dimension random space

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ABSTRACT

We are interested in application of UQ (Uncertainty Quantification) with Polynomial Chaos (PC) technics for complex compressible flows such as the ones encountered in ICF (Inertial Confinement Fusion); the general frame is uncertain systems of conservation laws:

$\partial_t u(x,t,\xi) + \partial_x f(u(x,t,\xi)) = 0$ with $u(x,t,\xi) \in \mathbb{R}^n,$ where ξ parametrizes a random vector.

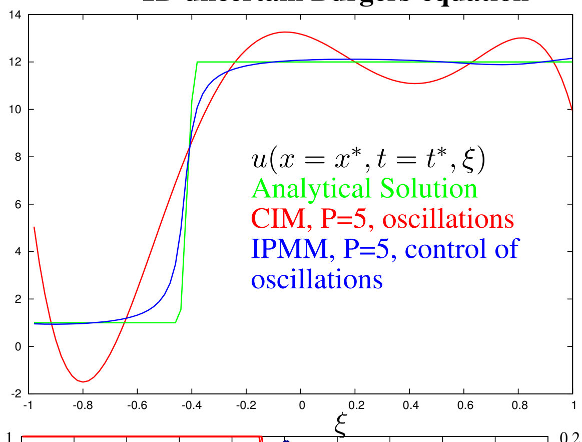
A basic PDE model is, for example, compressible Euler equations. PC methods were first introduced by Ghanem and Spanos [GS91] and are based on the Homogeneous Chaos Theory of Wiener [Wie38]. They appeared to be a good alternative to statistical methods (as Monte Carlo simulations and its modifications) for UQ as these latters can become too expensive due to the high number of samples required and time consuming codes. PC methods were successfully used for accurately solving many problems (incompressible flows [MK04], reacting flows and detonation [LEJS07]...) . However, classical approaches fail to approximate the solution in the case of "complex" flows implying for example discontinuities with respect to the random variable (see [PDL09, Cho74, MK04]). Conservation laws, known to generate shocks, can give birth to those kinds of difficulties. Several directions have been investigated in order to treat the Gibbs phenomenon due to polynomial order truncation as the use of Haar wavelets, adaptative methods as ME-GPC or ENO/WENO-like reconstruction in the random space [MK04, WK06, Abg07]. All these methods rely on a discretization of the random space: in the case of a moving discontinuity, with adaptative methods, the number of random subdomains can quickly become important (each time step needs a new refinement in the random space). Besides, they need interface tracking technics which are simple for one dimensional problems but are known to become quite tricky in higher dimensions.

To overcome these issues, we have developped a new method [PDL09](IPMM for Intrusive Polynomial Moment Method) which is based on a theoretical parallel between Classical PC (CIM for Classical Intrusive Method) and Theory of Moments (TM) ([MR98, CLL94]). We introduce a new variable v, the so-called *entropy variable* defined for a system of conservation laws through an entropy-entropy flux pair (s, g). In our approach, this variable becomes the main variable in the polynomial expansion $v(x,t,\xi) = \nabla_u s(u(x,t,\xi)) \approx \sum_{i=0}^{P} v_i(x,t)\phi_i(\xi)$ (where $(\phi_i)_{i\in\mathbb{N}}$ is the orthogonal polynomial basis). Several properties can then be proved for the new system as hyperbolicity (well-posedness), minoration and majoration of eigenvalues (control of the CFL condition), minimization of entropy and preservation of the invariant domain.

The talk will aim at the presentation of new results about the stability of IPMM, in the case of high dimension random space (≥ 3).

The next figures shows different results obtained with our method: the first one illustrates, on the inviscid Burgers'equation, how IPMM enables to constrain the oscillations in the random space to a certain domain dictated by the entropy expression. The second one presents a Sod shock tube (Euler system) with initially uncertain interface position in 1D space coordinates and the last one shows the same test-case in 2D space coordinates, in a convergent geometry. One has to figure out that on these test-cases, CIM fails for the presented polynomial order P: indeed, the Gibbs phenomenon occuring in the vicinity of the contact discontinuity makes the mass density ($\xi \longrightarrow \rho(x_{interface}, t = 0, \xi)$) become negative for certain values of ξ (see [PDL09]) and the computation crashes.

1D uncertain Burgers'equation

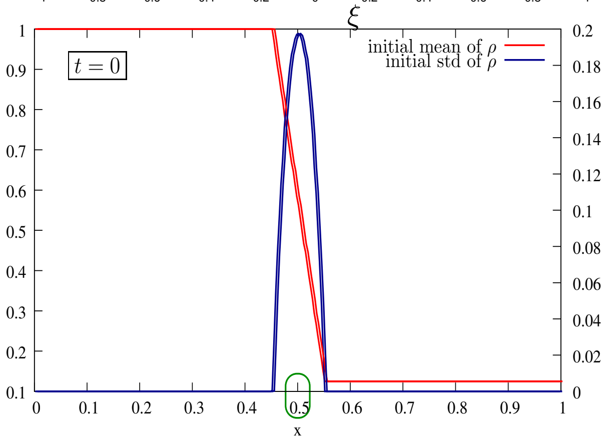


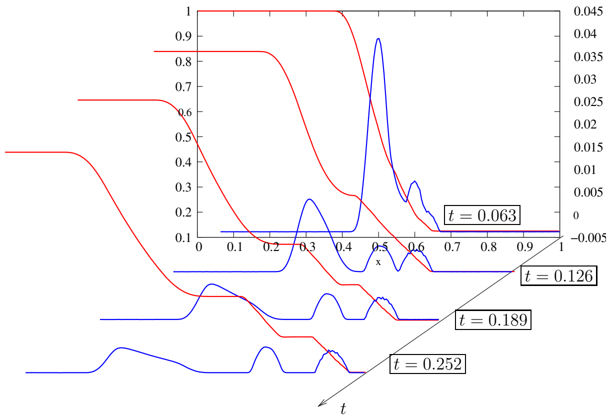
The new IPMM system for Burgers'equation:

$$\partial_t \int u\Big(\sum_{i=0}^{P} v_i\phi_i\Big)\begin{pmatrix}\phi_0\\ \ldots\\ \phi_P\end{pmatrix} \mathrm{d}w$$

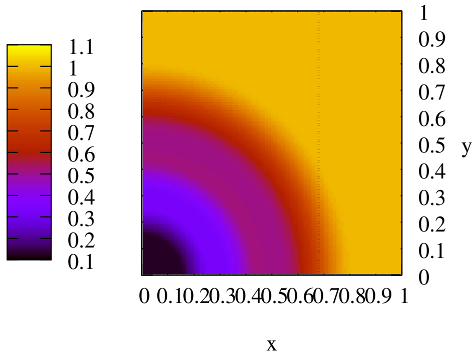
$$+\partial_x \int \frac{u^2\Big(\sum_{i=0}^{P} v_i\phi_i\Big)}{2}\begin{pmatrix}\phi_0\\ \ldots\\ \phi_P\end{pmatrix} \mathrm{d}w \quad = \quad 0$$

where $\mathrm{d}w$ is the probability measure.
The entropy: $s(u) = -\ln(u - u_-) - \ln(u_+ - u)$, with $(u_-, u_+) = (0.5, 12.5)$.

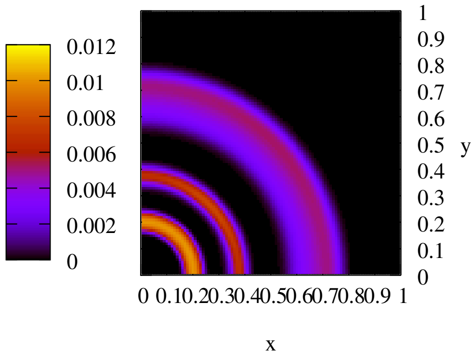




Sod shock tube, t=0.14, mean of ρ



Sod shock tube, t=0.14, std of ρ



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