Roe Solver and Entropy Corrector for Hyperbolic Systems with Uncertain Coefficients

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Introduction

Parametric Uncertainty Quantification

- uncertainties in input quantities (model parameters, initial and boundary conditions)
- uncertain quantities parametrized by random variables with known distribution functions

Stochastic spectral methods

- decompose random quantities on suitable approximation bases (Ghanem and Spanos 91)
- Stochastic expansion of the solution :

$$U(x,t,\xi) \approx U^{\mathrm{P}}(x,t,\xi) = \sum_{\alpha=1}^{\mathrm{P}} u_\alpha(x,t)\Psi_\alpha(\xi).$$

$u_\alpha(x,t)$ stochastic modes of the solution.

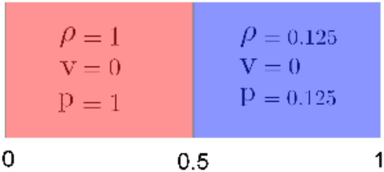
Different computational strategies

- probabilistic collocation : stochastic modes evaluated by polynomial interpolation
- non-intrusive projection : stochastic modes evaluated by numerical integration
- stochastic Galerkin : reformulated deterministic problem for the stochastic modes

Stochastic Galerkin methods :

- rely on a weak form of the problem
- well suited for mathematical analysis
- design of adaptive methods

Euler equations (Sod Shock Tube)



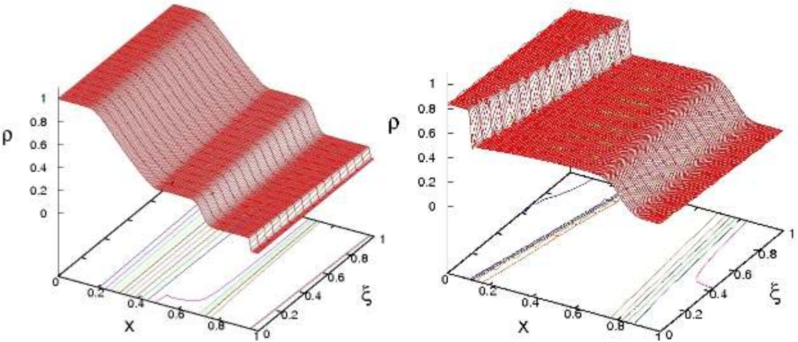
$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0,$$

$$U = (\rho, q, E)^T, \quad F(U) = (\rho v, \rho v^2 + p, v(E + p))^T,$$

$$v = \frac{q}{\rho}, \quad p = (\gamma - 1)\left(E - \frac{1}{2}\rho v^2\right).$$

$$\boxed{\gamma(\xi) = 1.4 + 0.2\,\xi, \quad \xi \sim U[0, 1].}$$

Euler equations (Sod Shock Tube)



Stochastic density $\rho(x, t, \xi)$ at $t = 0.25$ and $t = 3.25$.

Two main difficulties :

- solutions discontinuous in physical as well as in stochastic spaces
- nonlinearities in the flux functions

State of the art

Stochastic spectral methods applied to a large variety of engineering problems (elasticity, thermal science, fluid flows, chemical/biological systems,...) governed by elliptic, parabolic ,ODE or incompressible NS models.

Hyperbolic models

- non-intrusive approaches : multi-element probabilistic collocation methods (Lin et al. 08)
- pseudo-intrusive methods : stochastic modes of flux computed by quadrature methods (Ge et al. 08, Poette et al. 09)
- fully intrusive methods : scalar linear wave equation (Gottlieb and Xiu 08), mean flux upwinding (Lin et al. 06)

Objectives

- Develop fully intrusive stochastic Galerkin method
- Investigate hyperbolicity of the Galerkin system
- Design a Roe-type solver with entropy corrector
- Account for non-smooth solutions

Outline

Outline

Stochastic parametrization

$\xi = (\xi_1, \dots, \xi_N) \sim \mathcal{U}(\Xi = [0,1]^N) \to L^2(\Xi)$ corresponding space of the second-order random variables with the expectation operator $\langle H \rangle = \int_\Xi H(y)dy$.

Stochastic hyperbolic systems

We seek for $U(x,t,\xi) \in \mathbb{R}^m \otimes L^2(\Xi, p_\xi)$ solving a.s.

$$\begin{cases} \frac{\partial}{\partial t} U(x,t,\xi) + \frac{\partial}{\partial x} F(U(x,t,\xi);\xi) = 0, \\ U(t=0,x,\xi) = U_0(x,\xi). \end{cases}$$

$(x,t,\xi) \in \Omega \times [0,T] \times \Xi$,

$\nabla_U F \in \mathbb{R}^{m,m}$ stochastic Jacobian matrix $\mathbb{R}$-diagonalizable almost surely.

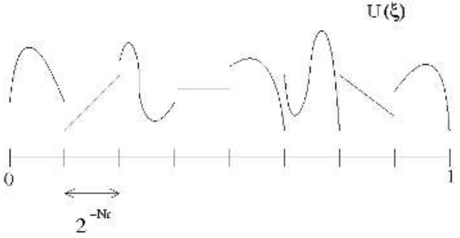
Stochastic discretization

We approximate $U(x,t,\xi)$ in the stochastic space of fully tensorized piecewise polynomial functions $\mathcal{S}^{\mathrm{No},\mathrm{Nr}}$:

- Nr : resolution level,
- No : expansion order.

Remark : Also possible to work with smaller stochastic approximation spaces, using for instance sparse tensorization.

Case $\mathrm{N} = 1$.
Exemple pour $\mathrm{Nr} = \mathrm{No} = 3$: $U(\xi) \in \mathcal{S}^{3,3}$.



Stochastic discretization

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- No : expansion order.

Case $\mathrm{N} = 1$.

$$\dim \mathcal{S}^{\mathrm{No},\mathrm{Nr}} = (\mathrm{No} + 1)2^{\mathrm{Nr}} =: \mathrm{P}_\pi \mathrm{P}_\sigma := \mathrm{P}.$$

Select the Stochastic Element (SE) orthonormal basis $\{\Psi_\alpha\}_{\alpha=1,\dots,\mathrm{P}}$, which corresponds to local (rescaled) Legendre polynomial bases , s.t.

$$span(\Psi_1, \dots, \Psi_\mathrm{P}) = \mathcal{S}^{\mathrm{No},\mathrm{Nr}}.$$

$\alpha = (\alpha_\sigma, \alpha_\pi)$, where α_σ refers to the stochastic element and α_π to the polynomial function in the stochastic element.

Stochastic discretization

General case $\mathrm{N} > 1$ obtained by full tensorization.

$$\dim \mathcal{S}^{\mathrm{No},\mathrm{Nr}} = (\mathrm{No}+1)^{\mathrm{N}} 2^{\mathrm{N}\mathrm{Nr}} =: \mathrm{P}_\pi \mathrm{P}_\sigma := \mathrm{P}.$$

Select the Stochastic Element (SE) orthonormal basis $\{\Psi_\alpha\}_{\alpha=1,\dots,\mathrm{P}}$, which corresponds to local fully tensorized (rescaled) Legendre polynomial bases , s.t.

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$\alpha = (\alpha_\sigma, \alpha_\pi)$, where α_σ refers to the stochastic element and α_π to the polynomial function in the stochastic element.

Stochastic expansion of the solution :

$$U(x,t,\xi) \approx U^{\mathrm{P}}(x,t,\xi) = \sum_{\alpha=1}^{\mathrm{P}} u_\alpha(x,t)\Psi_\alpha(\xi).$$

$u_\alpha(x,t) \in \mathbb{R}^m$ stochastic modes of the solution.

The Galerkin system

Galerkin projection of the original stochastic problem :

$$\begin{cases} \left\langle \Psi_\alpha \dfrac{\partial U^{\mathrm{P}}}{\partial t} \right\rangle + \left\langle \Psi_\alpha \dfrac{\partial F(U^{\mathrm{P}};\cdot)}{\partial x} \right\rangle = 0, & \forall \alpha = 1,\dots,\mathrm{P}, \\ \left\langle \Psi_\alpha U^{\mathrm{P}} \right\rangle (t=0) = \left\langle \Psi_\alpha U_0(x,\cdot) \right\rangle, & \forall \alpha = 1,\dots,\mathrm{P}. \end{cases}$$

We seek for $u(x,t)$ solving

$$\boxed{\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} f(u(x,t)) = 0, \quad u(x,t=0) = u^0(x),}$$

$$u(x,t) = (u_1(x,t),\dots,u_{\mathrm{P}}(x,t))^T \in \mathbb{R}^{m\mathrm{P}},$$

$$u^0(x) = (\left\langle \Psi_\alpha U^0 \right\rangle),$$

$$f(u(x,t)) = (f_1(u),\dots,f_{\mathrm{P}}(u))^T \in \mathbb{R}^{m\mathrm{P}},$$

$$f_\alpha(u) := \left\langle \Psi_\alpha F\left(U^{\mathrm{P}};\cdot\right)\right\rangle, \quad \alpha = 1,\dots,\mathrm{P}.$$

Hyperbolicity of the Galerkin system

$$(\nabla_u f(u))_{\alpha,\beta=1,\dots,\mathrm{P}} = \left\langle \nabla_U F(U^{\mathrm{P}};\cdot)\Psi_\alpha \Psi_\beta \right\rangle_{\alpha,\beta=1,\dots,\mathrm{P}}.$$

Is the Galerkin system hyperbolic ?

$\Leftrightarrow \nabla_u f(u) \in \mathbb{R}^{m\mathrm{P},m\mathrm{P}}$ $\mathbb{R}$-diagonalizable ?

Diagonal block structure owing to the decoupling of the problem over different stochastic elements :

$$\nabla_u f(u) = \begin{pmatrix} [\nabla_u f]^1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & [\nabla_u f]^{\alpha_\sigma} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & [\nabla_u f]^{\mathrm{P}_\sigma} \end{pmatrix}.$$

$\rightarrow$ It suffices to consider $\mathrm{P}_\sigma = 1$ and $\nabla_u f$ of size $m\mathrm{P}_\pi$.

Hyperbolicity of the Galerkin system

HYPERBOLICITY proven in two specific cases :

Theorem
HYPERBOLICITY, if the original stochastic system is a scalar conservation law.

Theorem
HYPERBOLICITY, if the stochastic Jacobian matrix $\nabla_U F(\cdot;\xi)$

- *is either symmetric (almost surely)*
- *or its eigenvectors are deterministic (independent of the uncertainty).*

Applications : Scalar wave equation with uncertain sound velocity. / Linear hyperbolic systems with uncertainty only on initial or boundary counditions.

Hyperbolicity of the Galerkin system

In the general case, we consider the approximate Galerkin Jacobian matrix $\overline{\nabla_u f}$ whose coefficients are obtained by approaching the coefficients of $\nabla_u f$ by a Gauss quadrature

$$\left(\overline{\nabla_u f}(u)\right)_{\alpha_\pi,\beta_\pi=1,\dots,\mathrm{P}_\pi} = \left(\sum_{\gamma=1}^{\mathrm{P}_\pi} \varpi_\gamma \nabla_U F(U^{\mathrm{P}}(\xi_\gamma);\xi_\gamma)\Psi_{\alpha_\pi}(\xi_\gamma)\Psi_{\beta_\pi}(\xi_\gamma)\right)_{\alpha_\pi,\beta_\pi=1,\dots,\mathrm{P}_\pi},$$

$\{\xi_\gamma\}_{\gamma=1,\dots,\mathrm{P}_\pi}$ set of the Gauss points, with associated weights $\{\varpi_\gamma\}_{\gamma=1,\dots,\mathrm{P}_\pi}$.

Hyperbolicity of the Galerkin system

Assume $\nabla_U F(U^{\mathrm{P}}(\xi);\xi) = L(\xi)\Lambda(\xi)R(\xi)$,

Theorem

$\overline{\nabla_u f(u)}$ *is* $\mathbb{R}$*-diagonalizable with eigenvalues* $\{\lambda'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$ *and right and left eigenvectors* $\{r'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$ *and* $\{l'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$ *given by*

$$\begin{cases} \{\lambda'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi} = \lambda'_{k\eta} = \Lambda^k(\xi_\eta), \\ \{r'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi} = \left(r'_{k\eta}\right)_\beta = \left(\varpi_\eta R^k(\xi_\eta)\Psi_\beta(\xi_\eta)\right)_\beta, \\ \{l'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi} = \left(l'_{k\eta}\right)_\beta = \left(\varpi_\eta L^k(\xi_\eta)\Psi_\beta(\xi_\eta)\right)_\beta, \end{cases}$$

where $k = 1,\dots,m, \eta = 1,\dots,\mathrm{P}_\pi, \beta = 1,\dots,\mathrm{P}_\pi$.

$\rightarrow$ $\{\lambda'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$, $\{r'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$, and $\{l'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$ approximations of the eigenvalues and eigenvectors of $\nabla_u f$.

Outline

Numerical scheme

Discretization of the Galerkin system using a FV method :

$$\boxed{u_i^{n+1} = u_i^n - \frac{\Delta^n t}{\Delta x}\left(\varphi(u_i^n, u_{i+1}^n) - \varphi(u_{i-1}^n, u_i^n)\right),}$$

- Δx (uniform) spatial step,
- $\Delta^n t$ time step,
- $\varphi(\cdot,\cdot)$ 1^{st} order numerical flux function :

$$\varphi(u_i^n, u_{i+1}^n) = \underbrace{\frac{f(u_i^n) + f(u_{i+1}^n)}{2}}_{\uparrow} - \underbrace{|a(u_i^n, u_{i+1}^n)|}_{\uparrow} \frac{u_{i+1}^n - u_i^n}{2}.$$

centered part of the flux — upwind matrix chosen as explained below

Roe linearized matrix

- Assume that the original stochastic problem possesses a Roe linearized matrix and a Roe state a.s.,

$$(U_L, U_R) \rightarrow A^{\mathrm{Roe}}(U_L, U_R) = \nabla_U F(U_{LR}^{\mathrm{Roe}}; \cdot).$$

- Given two states u_L and u_R of the Galerkin system,

$$\left.\begin{array}{l} u_L \rightarrow U_L^{\mathrm{P}} \\ u_R \rightarrow U_R^{\mathrm{P}} \end{array}\right\rangle \rightarrow U_{LR}^{\mathrm{Roe}},$$

$$\rightarrow a^{\mathrm{Roe}}(u_L, u_R) = \langle \nabla_U F(U_{LR}^{\mathrm{Roe}}; \cdot)\Psi_\alpha \Psi_\beta \rangle.$$

Theorem

a^{Roe} is a Roe linearized matrix for the Galerkin system.

Choice of upwinding :

$$\varphi(u_i^n, u_{i+1}^n) = \frac{f(u_i^n) + f(u_{i+1}^n)}{2} - |a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| \, \frac{u_{i+1}^n - u_i^n}{2}.$$

$\rightarrow$ Consistency of the numerical scheme

$\rightarrow$ Conservativity through shocks

Efficient approximation of the absolute value of a matrix

A deterministic $\mathbb{R}$-diagonalizable matrix of size N_A.
Known data : $\{\lambda_i\}_{i=1,\dots,\mathrm{N}_A}$ eigenvalues of A.

$$A = \sum_{i=1}^{\mathrm{N}_A} \lambda_i l_i \otimes r_i, \quad |A| = \sum_{i=1}^{\mathrm{N}_A} |\lambda_i| l_i \otimes r_i.$$

For a polynomial q

$$q(A) = \sum_{i=1}^{\mathrm{N}_A} q(\lambda_i) l_i \otimes r_i.$$

$\rightarrow$ Determination of a polynomial $q_{d,\{\lambda_i\}}$ with low degree $d \ll \mathrm{N}_A$ which minimizes the least-square error $\sum_{i=1}^{\mathrm{N}_A}(|\lambda_i| - q_{d,\{\lambda_i\}}(\lambda_i))^2$.

In fact, determination of $q_{d,\{\lambda'_i\}}(A)$ from $\{\lambda'_i\}_{i=1,\dots,\mathrm{N}_A}$ approximate eigenvalues of A.

$$\rightarrow \boxed{|A| \approx q_{d,\{\lambda'_i\}}(A).}$$

The upwind scheme

At each interface LR in physical space,
$u_{i,i+1}^{\mathrm{Roe}} := \left(\left\langle \Psi_\alpha U_{i,i+1}^{\mathrm{Roe}} \right\rangle \right)_{\alpha=1,\dots,\mathrm{P}}$, projected Roe state in $\mathcal{S}^{\mathrm{No,Nr}}$.

Parallelisation of the procedure on each stochastic element α_σ, $1 \leq \alpha_\sigma \leq \mathrm{P}_\sigma$,

→ Evaluate approximate eigenvalues $\{\lambda'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$

$$\{\lambda'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi} = spec(\overline{\nabla_u f}(u_{i,i+1}^{\mathrm{Roe}})).$$

→ Determine the local polynomial $q_{d,\{\lambda'_\gamma\}}$ fitting $\{\lambda'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$.

→ Approximate the absolute value of $a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)$

$$|a^{\mathrm{Roe}}(u_i^n, u_{i+1}^n)| \approx q_{d,\{\lambda'_\gamma\}}(\nabla_u f(u_{i,i+1}^{\mathrm{Roe}})).$$

The upwind scheme

$$u_i^{n+1} = u_i^n - \frac{\Delta^n t}{\Delta x} \left(\varphi(u_i^n, u_{i+1}^n) - \varphi(u_{i-1}^n, u_i^n) \right),$$

where the numerical flux $\varphi(u_i^n, u_{i+1}^n)$ is computed in this way

$$\varphi(u_i^n, u_{i+1}^n) = \frac{f(u_i^n) + f(u_{i+1}^n)}{2} - q_{d,\{\lambda'_\gamma\}}(\nabla_u f(u_{i,i+1}^{\mathrm{Roe}})) \frac{u_{i+1}^n - u_i^n}{2}.$$

CFL condition :

$$\frac{\Delta^n t^{\alpha_\sigma}}{\Delta x} = \frac{CFL}{\max\limits_{LR \in \mathcal{I}, \gamma=1,\dots,m\mathrm{P}_\pi} |\lambda'_\gamma|}, \quad \Delta^n t = \min_{1 \le \alpha_\sigma \le \mathrm{P}_\sigma} \Delta^n t^{\alpha_\sigma}.$$

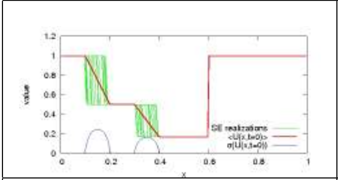
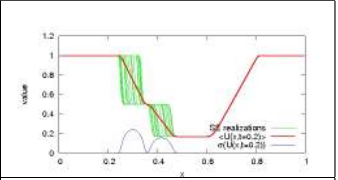
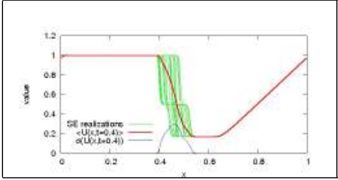
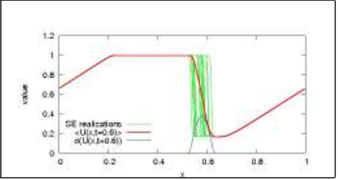
Outline

Periodic Burgers equation

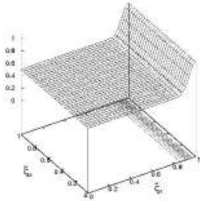
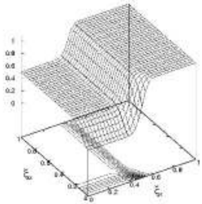
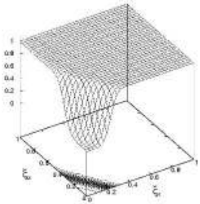
$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2},$$

initial random shock locations :

$$X_{1,2} = 0.1 + 0.1\xi_1, \quad X_{2,3} = 0.3 + 0.1\xi_2, \quad \xi_1, \xi_2 \sim \mathcal{U}[0,1].$$

$t = 0\ s$	$t = 0.2\ s$
	
$t = 0.4\ s$	$t = 0.6\ s$
	

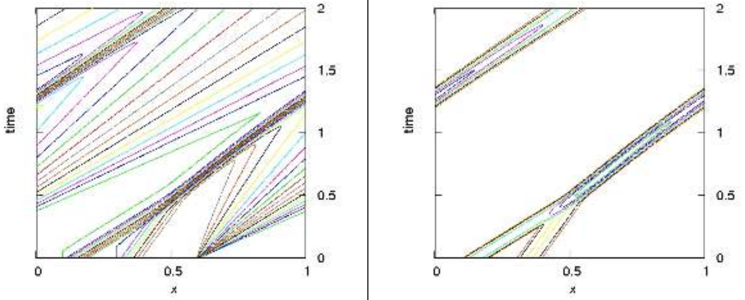
Periodic Burgers equation

$x = 0.35, t = 0.2\ s$	$x = 0.45, t = 0.4\ s$	$x = 0.55, t = 0.6\ s$
		

Stochastic solution $U(x, t, (\xi_1, \xi_2))$ at observation point $x_0(t) = 0.25 + 0.5t$ as a function of (ξ_1, ξ_2) and for different times. Computations with $\mathrm{No} = \mathrm{Nr} = 3$.

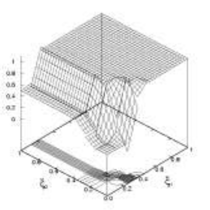
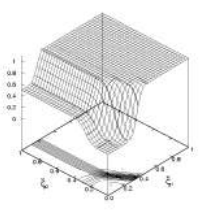
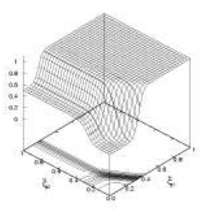
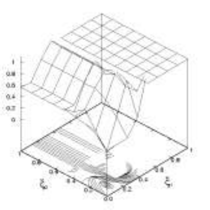
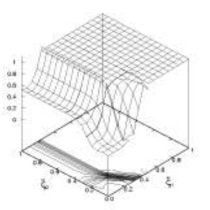
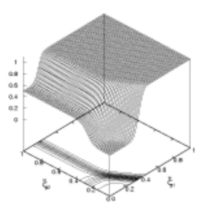
Periodic Burgers equation

Expectation	Standard Deviation



Space-time diagrams of the expectation $\langle U(x,t,\cdot)\rangle$ and standard deviation $\sigma(U(x,t,\cdot))$ of the stochastic solution. $\mathrm{No} = \mathrm{Nr} = 3$.

Burgers equation

$\mathrm{Nr} = 3, \mathrm{No} = 1$	$\mathrm{Nr} = 3, \mathrm{No} = 2$	$\mathrm{Nr} = 3, \mathrm{No} = 3$
		
$\mathrm{Nr} = 1, \mathrm{No} = 3$	$\mathrm{Nr} = 2, \mathrm{No} = 3$	$\mathrm{Nr} = 4, \mathrm{No} = 3$
		

Stochastic solution of the Burgers equation as a function of (ξ_1, ξ_2) at $x = 0.5$ and $t = 0.5$ for different Nr and No.

Euler equations (Sod Shock Tube)

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0,$$

$$U = (\rho, q, E)^T, \quad F(U) = (\rho v, \rho v^2 + p, v(E+p))^T,$$

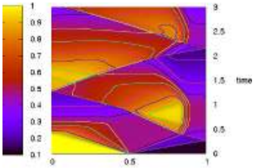
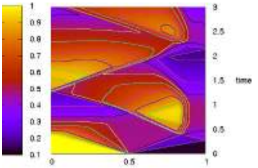
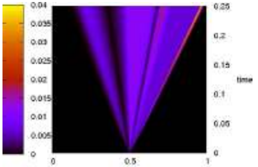
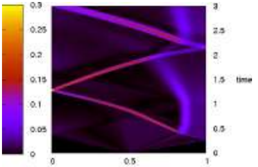
$$v = \frac{q}{\rho}, \quad p = (\gamma - 1)\left(E - \frac{1}{2}\rho v^2\right).$$

$$\boxed{\gamma(\xi) = 1.4 + 0.2\,\xi, \quad \xi \sim U[0,1].}$$

Computation of the Galerkin flux : using a pseudo-spectral approximation (Debusschere et al. 04)

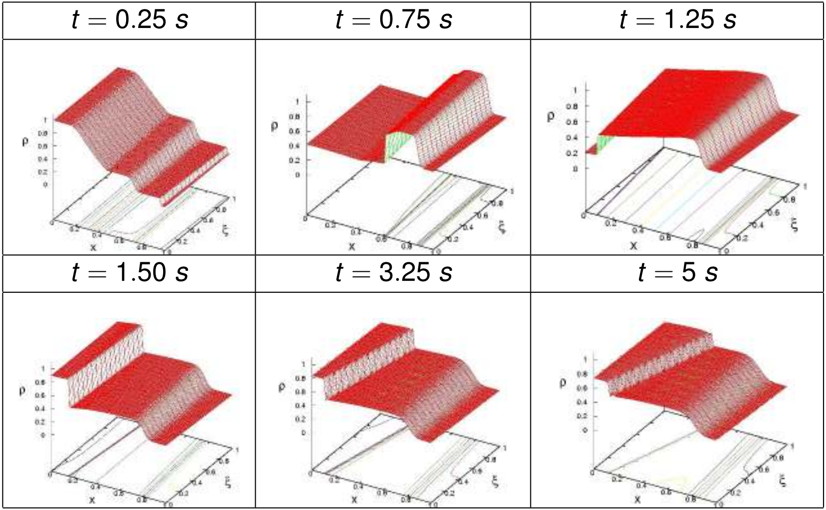
- $a \times b \approx a * b = \sum_{\alpha=0}^{\mathrm{P}} (a*b)_\alpha \Psi_\alpha,$
  $(a*b)_\alpha = \sum_{\beta,\delta=0}^{\mathrm{P}} a_\beta b_\delta \mathcal{M}_{\alpha\beta\delta}, \quad \mathcal{M}_{\alpha\beta\delta} = \langle \Psi_\alpha \Psi_\beta \Psi_\gamma \rangle$
- $1/a \approx a^{-*}$ obtained by solving $a * a^{-*} = 1$
- $p \approx (\gamma - 1) * (E - (q*q) * (1/\rho)/2)$
- $\sqrt{a} \approx a^{*/2}$ obtained by solving $(a^{*/2}) * (a^{*/2}) = a$

Euler equations

Deterministic problem ($\gamma = 1.5$)	Expected density
	
Standard deviation (early times)	**Standard deviation (longer times)**
	

Space-time diagrams of the deterministic density for $\gamma = 1.5$, the expected density, and the standard deviations in the density for early and longer times. $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$.

Euler equations



Stochastic density as a function of (x, ξ). $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$.

Euler equations

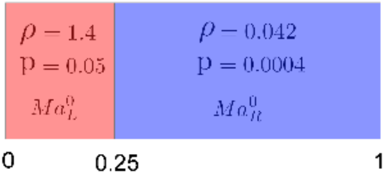
	$\mathrm{Nr}=2$		$\mathrm{Nr}=3$		$\mathrm{Nr}=4$	
	T_{CPU}	$\dim \mathcal{S}^{\mathrm{Nr},\mathrm{No}}$	T_{CPU}	$\dim \mathcal{S}^{\mathrm{Nr},\mathrm{No}}$	T_{CPU}	$\dim \mathcal{S}^{\mathrm{Nr},\mathrm{No}}$
$\mathrm{No}=0$	4.0	(4)	8.1	(8)	16.1	(16)
$\mathrm{No}=1$	6.9	(8)	13.9	(16)	27.8	(32)
$\mathrm{No}=2$	11.8	(12)	23.2	(24)	46.5	(48)
$\mathrm{No}=3$	17.1	(16)	34.1	(32)	68.1	(64)
$\mathrm{No}=4$	24.8	(20)	49.3	(40)	98.0	(80)

Normalized computational times T_{CPU} for different stochastic discretization parameters Nr and No. $\mathrm{Nc}=250$.

$\rightarrow$ computational costs scale as *dim* $S^{\mathrm{No},\mathrm{Nr}}$ at least for moderate No.

Entropy corrector

Euler equations (Sod Shock Tube)



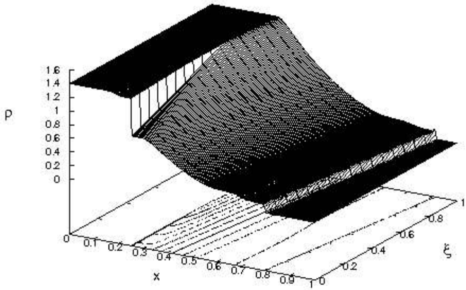
$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0,$$
$$U = (\rho, q, E)^T, \quad F(U) = (\rho v, \rho v^2 + p, v(E+p))^T,$$
$$v = \frac{q}{\rho}, \quad p = (\gamma - 1)\left(E - \frac{1}{2}\rho v^2\right).$$

$$Ma^0(\xi) = \begin{cases} 0.7 + 0.5\,\xi, \xi \in [0, 1/4], \\ 2.46 \times (0.7 + 0.5\,\xi), \xi \in]1/4, 1], \end{cases} \qquad \xi \sim U[0,1].$$

Entropy corrector

Euler equations (Sod Shock Tube)

Stochastic density $\rho(x, t, \xi)$ at $t = 1$ obtained without entropy corrector (using $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$) :



Entropy-violating shock ! → Need for an entropy corrector !

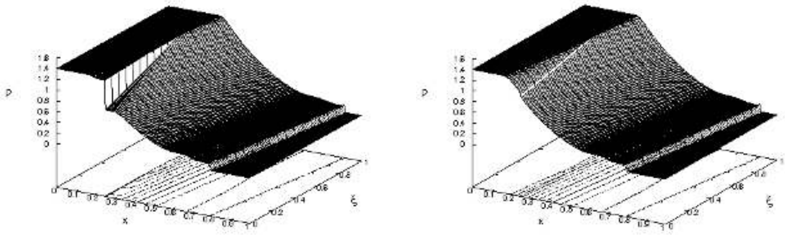
Entropy corrector

Non-parametrized entropy corrector proposed by Dubois and Mehlmann (96) for Roe solver in the deterministic case

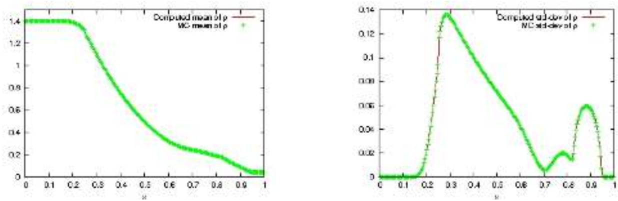
- Avoid entropy-violating shocks
- Nonlinear modification of the numerical flux in the vicinity of sonic points
- Detection of sonic expansion waves based on reconstruction of intermediate states for each couple of left and right states and test on sign of eigenvalues of the Roe linearized matrix

$\rightarrow$ Adaptation to the present context

Euler equations (Sod Shock Tube)



Stochastic density $\rho(x, t, \xi)$ at $t = 1$ obtained without (left) and with (right) the entropy corrector using $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$.



Comparaison of the mean and standard deviation of the numerical density at $t = 1$, computed with a Galerkin method (using $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$) and a MC method.

Some details

- Parallelisation of the procedure on each stochastic element α_σ, $1 \leq \alpha_\sigma \leq \mathrm{P}_\sigma$
- Compute the $m\mathrm{P}_\pi$ characteristic variables $\{\beta'_\gamma\}_{\gamma=1,\dots,m\mathrm{P}_\pi}$

$$u_L - u_R \approx \sum_{\gamma=1}^{m\mathrm{P}_\pi} \beta'_\gamma r'_\gamma(u_{LR}^{\mathrm{Roe}}).$$

- Reconstruct the $m\mathrm{P}_\pi$ intermediate states at each physical interface

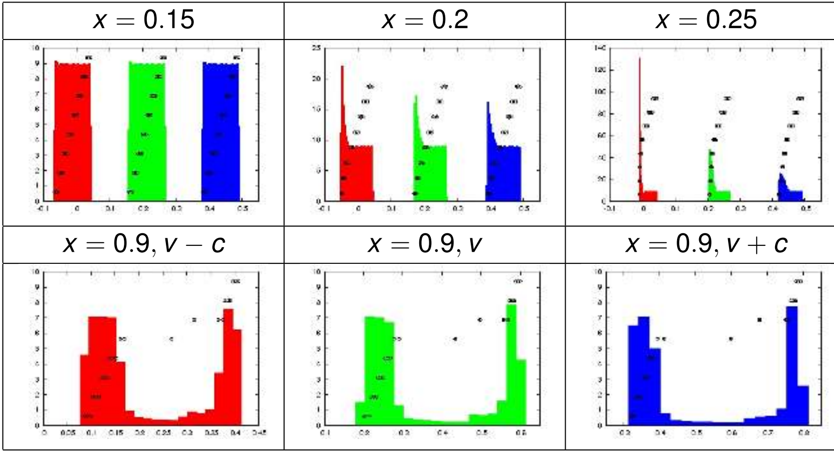
$$u'_\gamma = u'_{\gamma-1} + \beta'_\gamma r'_\gamma(u_{LR}^{\mathrm{Roe}}).$$

- Determine the set of sonic indices :

$$S' = \{\gamma, \lambda'_\gamma(u'_{\gamma-1}) < 0 < \lambda'_\gamma(u'_\gamma)\}.$$

- The indexing of $\{\lambda'_\gamma\}_\gamma$ and $\{r'_\gamma\}_\gamma$, $\gamma = 1, \dots, m\mathrm{P}_\pi$, provides a correspondence between approximate eigenvalues and eigenvectors and is central to determine S'.

Euler equations (Sod Shock Tube)



Approximate eigenvalues $(v_{LR}^{\mathrm{Roe},*} - c_{LR}^{\mathrm{Roe},*})(\xi_\eta)_{\eta=0,\dots,\mathrm{No}}$ (red), $v_{LR}^{\mathrm{Roe},*}(\xi_\eta)_{\eta=0,\dots,\mathrm{No}}$ (green), and $(v_{LR}^{\mathrm{Roe},*} + c_{LR}^{\mathrm{Roe},*})(\xi_\eta)_{\eta=0,\dots,\mathrm{No}}$ (blue) corresponding to each stochastic element together with their density functions. Computations at $t = 1$ with $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$.

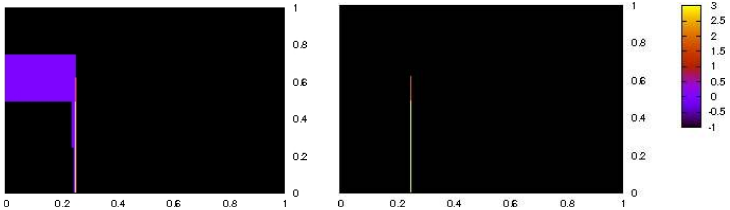
CPU improvements

- Only the eigenvalue $v - c$ can change its sign.
- Mean value averaged criterium $\rightarrow$ portions of (x, ξ) actually selected for the entropy correction : $]x_{L-1/2}, x_{R+1/2}[\times \alpha_\sigma$ such that $E^{\alpha_\sigma}[v_{LR}^{\mathrm{Roe},*} - c_{LR}^{\mathrm{Roe},*}]$ changes its sign at the physical interface LR.
- Use a numerical tolerance $ctol$ $(= Cv_{ref})$

$$E^{\alpha_\sigma}[(v_L^* - c_L^*)] - ctol < 0 < E^{\alpha_\sigma}[(v_R^* - c_R^*)] + ctol.$$

Euler equations (Sod Shock Tube) CPU improvements

Portion of the domain (x, ξ) selected for the entropy correction and card S'. Value=-1 if no correction. Value=0 if test. Card S' else. Computations with $ctol = 1e^{-2}$ (left) and $ctol = 1e^{-5}$ (right).



$\dim \mathcal{S}^{\mathrm{Nr,No}}$	$\mathrm{No}=1, \mathrm{Nr}=3$ 16		$\mathrm{No}=2, \mathrm{Nr}=3$ 24		$\mathrm{No}=3, \mathrm{Nr}=3$ 32	
	T_{CPU}	*factor*	T_{CPU}	*factor*	T_{CPU}	*factor*
$ctol = +\infty$	11.7	1.0e-0	16.1	1.0e-0	21.6	1.0e-0
$ctol = 1e^{-1}$	8.2	3.7e-1	11.8	3.7e-1	16.4	3.7e-1
$ctol = 1e^{-2}$	6.5	7.1e-2	9.8	7.1e-2	13.9	7.1e-2
$ctol = 1e^{-3}$	6.1	2.8e-3	9.3	2.8e-3	13.5	2.8e-3
$ctol = 0$	6	2.5e-3	9.2	2.5e-3	13.4	2.5e-3
ϵ_h	1.32e-3		7.17e-4		2.88e-4	

Conclusion

- fully intrusive multi-resolution scheme
- Roe-type solver with upwind matrices efficiently computed by an original and fast method
- accurate and robust method
- entropy correction in the presence of sonic points only requiring marginal costs
- yet, computational costs scale as *dim* $S^{\mathrm{No},\mathrm{Nr}}$ (at least for moderate No)
- savings in computational costs for problems with higher stochastic dimensions
  $\rightarrow$ adaptive stochastic mesh refinement

References :

Intrusive Projection Methods with Upwinding for Uncertain Nonlinear Hyperbolic Systems (submitted)
J. Tryoen, O. Le Maître, M. Ndjinga, A. Ern

Roe solver with Entropy Corrector for Uncertain Hyperbolic Systems (submitted)
J. Tryoen, O. Le Maître, M. Ndjinga, A. Ern

Case of random variables with non-uniform distribution functions

Stochastic parametrization

$\xi = (\xi_1, \dots, \xi_N)$ vector of random variables with known independent distribution functions.

Change of variables

$x(\xi) = (x_1(\xi_1), \dots, x_N(\xi_N)) = (p_1(\xi_1), \dots, p_N(\xi_N))$ with $(p_d(\xi_d))_{d=1,\dots,N}$ cumulative density functions

$$\rightarrow \boxed{x(\xi) \sim \mathcal{U}([0,1]^N).}$$

Expansion of a process

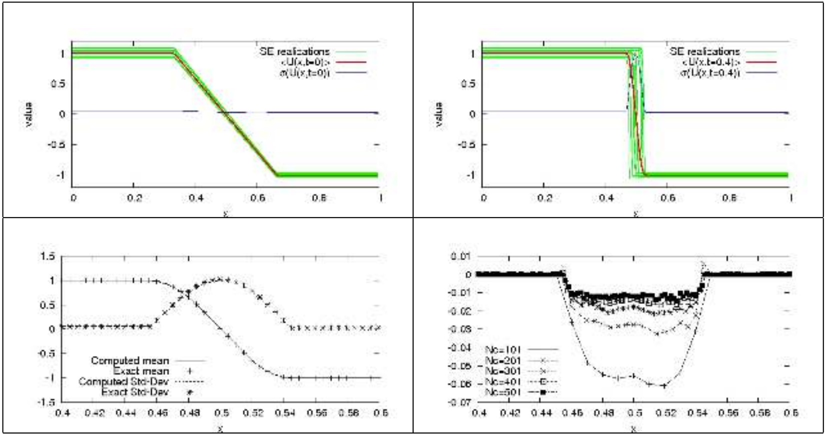
$$\boxed{H(\xi) = \tilde{H}(x(\xi)) = \sum_{\alpha=1}^{P} \tilde{H}_\alpha \Psi_\alpha(x(\xi)).}$$

Burgers equation

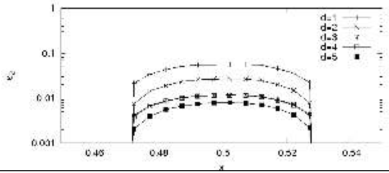
$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2},$$

$$U^+(\xi_1) = 1 + 0.1(2\xi_1 - 1), \qquad \xi_1 \sim \mathcal{U}[0,1],$$

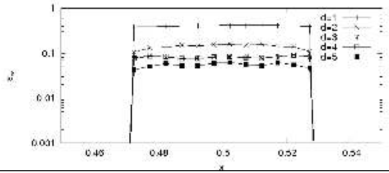
$$U^-(\xi_2) = -1 + 0.05(2\xi_2 - 1), \quad \xi_2 \sim \mathcal{U}[0,1].$$



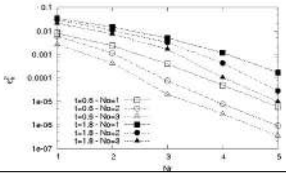
Error L^2 on the eigenvalues of $|\nabla_u f(u_{LR}^{\mathrm{Roe}})|$ at $t = 0.4$.



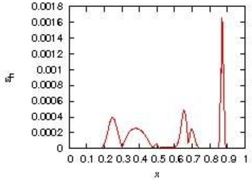
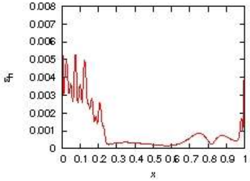
Error L^∞ on the eigenvalues of $|\nabla_u f(u_{LR}^{\mathrm{Roe}})|$ at $t = 0.4$.



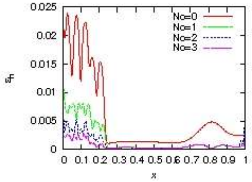
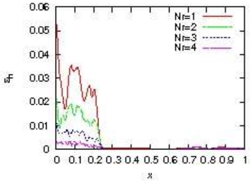
Stochastic error at $t = 0.6$ on the semi-discrete solution.



Euler equations

$t = 0.20$	$t = 6.5$
	

Stochastic error $\epsilon_h(x, t)$ for early (left) and longer (right) times.

$\mathrm{Nr} = 3$	$\mathrm{No} = 1$
	

Stochastic error $\epsilon_h(x, t = 6.5)$ for various No and Nr.
Computations with $\mathrm{Nc} = 250$.