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Benchmark reliability problems – A selected list

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1 Example 1

Source: Bourinet (2018, Appendix A, Example 2)

Random inputs: $U_1, U_2 \sim \mathcal{N}(0,1)$, U_1 and U_2 independent.

Limit-state function:

$$G(\mathbf{u}) = \min_{k \in \{1,2\}} G_k(\mathbf{u}) \quad \text{for } \mathbf{u} = (u_1, u_2)^{\mathrm{T}} ,$$

where

$$G_1(\mathbf{u}) = (u_1 - \epsilon) + \beta_1 ,$$

$$G_2(\mathbf{u}) = \beta_1 \left(1 - \left[\frac{1}{2} \left(\frac{u_1 - \epsilon}{\beta_2} + \left| \frac{u_1 - \epsilon}{\beta_2} \right| \right) \right]^{\gamma} \right) ,$$

and where $\beta_1 = 6$, $\beta_2 = 4.5$, $\gamma = 30$, $\epsilon = 10^{-6}$.

Reference failure probability: $p_{f\,\mathrm{ref}} = \Phi(-(\beta_1 - \epsilon)) + \Phi(-(\beta_2 + \epsilon)) \approx \Phi(-\beta_2) = 3.40 \times 10^{-6}$.

2 Example 2

Source: Bjerager (1988), Engelund and Rackwitz (1993), and Bourinet (2016)

Random inputs: $U_1, \ldots, U_5 \sim \mathcal{N}(0,1)$, $U_1, \ldots, U_5$ independent.

Limit-state function:

$$G(\mathbf{u}) = \max_{k \in \{1,4\}} G_k(\mathbf{u}) \quad \text{for } \mathbf{u} = (u_1, \ldots, u_5)^{\mathrm{T}} ,$$

where

$$\begin{aligned} G_1(\mathbf{u}) &= 2.677 - u_1 - u_2 , \\ G_2(\mathbf{u}) &= 2.500 - u_2 - u_3 , \\ G_3(\mathbf{u}) &= 2.323 - u_3 - u_4 , \\ G_4(\mathbf{u}) &= 2.250 - u_4 - u_5 . \end{aligned}$$

Reference failure probability: $p_{f\,\mathrm{ref}} = \Phi_4(-\boldsymbol{\beta}, \mathbf{0}, \mathbf{R}) = 2.13 \times 10^{-4}$ where $\boldsymbol{\beta}$ is the vector of HL reliability indices, $\mathbf{R}$ is the correlation matrix between the four limit-state surfaces and Φ_4 is the 4-dimensional normal CDF.

3 Example 3

Source: De Stefano and Der Kiureghian (1990) and Bourinet et al. (2011)

Random inputs: The 8 random inputs are independent. The marginal PDFs are defined in Table 1.

Variable	m_p	m_s	k_p	k_s	ζ_p	ζ_s	F_s	S_0
Distribution				lognormal				
Mean	1.5	0.01	1	0.01	0.05	0.02	27.5	100
C.o.V.	0.1	0.1	0.2	0.2	0.4	0.5	0.1	0.1

Table 1 – Example 3 - Random inputs.

Limit-state function:

$$g(\mathbf{x}) = F_\mathrm{s} - 3k_\mathrm{s}\sqrt{\frac{\pi S_0}{4\zeta_\mathrm{s}\omega_\mathrm{s}^3}\left[\frac{\zeta_\mathrm{a}\zeta_\mathrm{s}}{\zeta_\mathrm{p}\zeta_\mathrm{s}\left(4\zeta_\mathrm{a}^2+\theta^2\right)+\gamma\zeta_\mathrm{a}^2}\frac{\left(\zeta_\mathrm{p}\omega_\mathrm{p}^3+\zeta_\mathrm{s}\omega_\mathrm{s}^3\right)\omega_\mathrm{p}}{4\zeta_\mathrm{a}\omega_\mathrm{a}^4}\right]}$$

where $\mathbf{x} = (m_\mathrm{p}, m_\mathrm{s}, k_\mathrm{p}, k_\mathrm{s}, \zeta_\mathrm{p}, \zeta_\mathrm{s}, F_\mathrm{s}, S_0)^\mathrm{T}$,
and where $\omega_\mathrm{p} = \sqrt{k_\mathrm{p}/m_\mathrm{p}}$, $\omega_\mathrm{s} = \sqrt{k_\mathrm{s}/m_\mathrm{s}}$, $\omega_\mathrm{a} = (\omega_\mathrm{p}+\omega_\mathrm{s})/2$, $\zeta_\mathrm{a} = (\zeta_\mathrm{p}+\zeta_\mathrm{s})/2$, $\gamma = m_\mathrm{s}/m_\mathrm{p}$ and $\theta = (\omega_\mathrm{p}-\omega_\mathrm{s})/\omega_\mathrm{a}$.

Reference failure probability: $p_{\mathrm{f\,ref}} = 3.78 \times 10^{-7}$.

4 Example 4

Source: Kouassi et al. (2016), Bourinet (2018, Appendix A, Example 1), Bourinet (2019)

Random inputs: The 11 random inputs are independent. The marginal PDFs are defined in Table 2.

variable X_i	mean μ_{X_i}	c.o.v. δ_{X_i}	distribution / support
$X_1 = L$ (m)	4.2	0.10	lognormal / $\mathbb{R}_{\geq 0}$
$X_2 = h$ (m)	0.02	0.10	lognormal / $\mathbb{R}_{\geq 0}$
$X_3 = d$ (m)	0.001	0.05	lognormal / $\mathbb{R}_{\geq 0}$
$X_4 = Z_L$ (Ω)	1000	0.20	lognormal / $\mathbb{R}_{\geq 0}$
$X_5 = Z_0$ (Ω)	50	0.05	lognormal / $\mathbb{R}_{\geq 0}$
$X_6 = a_e$ (V/m)	1	0.20	lognormal / $\mathbb{R}_{\geq 0}$
$X_7 = \theta_e$ (rad)	$\pi/4$	0.577	uniform / $[0, \pi/2]$
$X_8 = \theta_p$ (rad)	$\pi/4$	0.577	uniform / $[0, \pi/2]$
$X_9 = \phi_p$ (rad)	π	0.577	uniform / $[0, 2\pi[$
$X_{10} = f$ (MHz)	30	0.096	uniform / $[25, 35]$
$X_{11} = \alpha$ (-)	0.0010	0.289	uniform / $[0.0005, 0.0015]$

Table 2 – Example 4 - Random inputs.

Limit-state function:

$$g(\mathbf{x}) = I_{\text{cr}} - I(\mathbf{x}) ,$$

where $I_{\text{cr}} = 1.5 \times 10^{-4}$ A is a given current magnitude level to be not exceeded,
where

$$I(\mathbf{x}) = \left| \frac{2ha_e}{I_1} I_2 \left[I_3 \left(I_4 - I_5 \right) + I_6 \right] \right| \quad \text{for } \mathbf{x} = (L, h, d, Z_L, Z_0, a_e, \theta_e, \theta_p, \phi_p, f, \alpha) ,$$

and where:

$$\begin{aligned}
I_1 &= (Z_0 Z_C + Z_L Z_C) \cosh(\gamma L) + \left(Z_C^2 + Z_0 Z_L \right) \sinh(\gamma L) , \\
I_2 &= \frac{\sin\left(\beta h \cos\theta_p\right)}{\beta h \cos\theta_p} , \\
I_3 &= i\beta \cos\theta_p \left(-\sin\theta_e \cos\theta_p \sin\phi_p + \cos\theta_e \cos\phi_p \right) , \\
I_4 &= \frac{1}{2} (Z_C + Z_0) \frac{\exp\left[\left(\gamma + i\beta \sin\theta_p \sin\phi_p \right) L \right] - 1}{\gamma + i\beta \sin\theta_p \sin\phi_p} , \\
I_5 &= \frac{1}{2} (Z_C - Z_0) \frac{\exp\left[-\left(\gamma - i\beta \sin\theta_p \sin\phi_p \right) L \right] - 1}{\gamma - i\beta \sin\theta_p \sin\phi_p} , \\
I_6 &= \sin\theta_e \sin\theta_p \left[Z_C - (Z_C \cosh(\gamma L) + Z_0 \sinh(\gamma L)) \exp\left(i\beta L \sin\theta_p \sin\phi_p \right) \right] ,
\end{aligned}$$

in which $Z_C = 60\,\text{acosh}(2h/d)$, $\beta = 2\pi f / 3 \times 10^8$ and $\gamma = \alpha + i\beta$.
In these expressions, $i = \sqrt{-1}$ is the imaginary number and $|\cdot|$ denotes the modulus of a complex number.

Reference failure probability: $p_{f\,\text{ref}} = 2.24 \times 10^{-4}$.

5 Example 5

Source: Blatman and Sudret (2010) and Bourinet (2017)

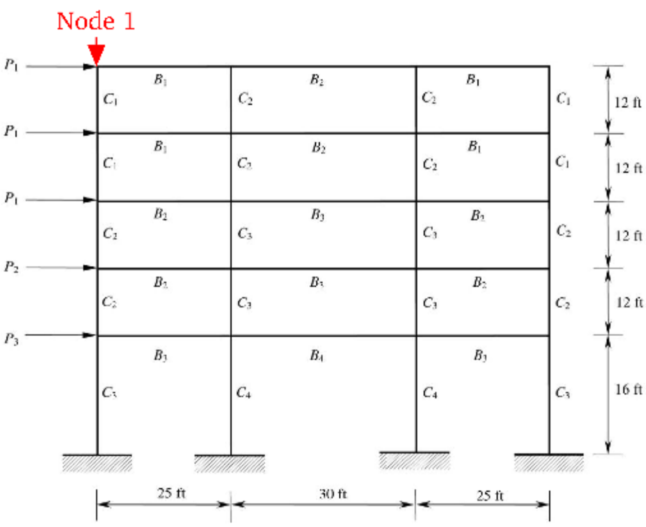

Figure 1 – Example 5. A three-span, five-story, linear elastic frame subjected to lateral loads

Random inputs: A three-span, five-story, linear elastic frame structure subjected to lateral loads is studied in this second example, see Figure 1. The problem has $n = 21$ random inputs: 3 applied lateral loads, 2 Young's moduli, 8 moments of inertia and 8 cross-sectional areas. This problem was initially proposed by Liu and Der Kiureghian (1986). It was later studied in several research works with differences in the input data. The problem settings here are those of Blatman and Sudret (2010), which specifically considers normal distributions left-truncated at zero for the Young's moduli E_{4-5}, the moments of inertia I_{6-13} and the cross-sectional areas A_{14-21}. The loads P_{1-3} are assumed independent and lognormally distributed by Blatman and Sudret (2010).

The frame element structural details and the statistical properties of each random input of $\mathbf{X}$ are given in Table 3 and 4 respectively. A linear correlation is considered between the following inputs:

- material properties: $\rho_{E_i E_j} = 0.9$ for $i, j = 4, 5$, $i \neq j$,
- moments of inertia: $\rho_{I_i I_j} = 0.13$ for $i, j = 6, \ldots, 13$, $i \neq j$,
- cross-sectional areas: $\rho_{A_i A_j} = 0.13$ for $i, j = 14, \ldots, 21$, $i \neq j$,
- moment of inertias and cross-sectional areas: $\rho_{A_i I_j} = \rho_{I_j A_i} = 0.13$ for $i = 6, \ldots, 13$, $j = 14, \ldots, 21$, except for properties of a single frame element for which we have $\rho_{A_{i+8} I_i} = \rho_{I_i A_{i+8}} = 0.95$ for $i = 6, \ldots, 13$.

Table 3 – Example 5. Frame element properties

Element	Young's modulus	Moment of inertia	Cross-sectional area
B_1	E_4	I_{10}	A_{18}
B_2	E_4	I_{11}	A_{19}
B_3	E_4	I_{12}	A_{20}
B_4	E_4	I_{13}	A_{21}
C_1	E_5	I_6	A_{14}
C_2	E_5	I_7	A_{15}
C_3	E_5	I_8	A_{16}
C_4	E_5	I_9	A_{17}

Table 4 – Example 5. Definition of marginal distributions of $\mathbf{X}$

X_i	Mean μ_i	Std dev. σ_i	X_i	Mean μ_i	Std dev. σ_i
P_1	30	9	E_4	454,000	40,000
P_2	20	8	E_5	497,000	40,000
P_3	16	6.40			
I_6	0.94	0.12	A_{14}	3.36	0.60
I_7	1.33	0.15	A_{15}	4.00	0.80
I_8	2.47	0.30	A_{16}	5.44	1.00
I_9	3.00	0.35	A_{17}	6.00	1.20
I_{10}	1.25	0.30	A_{18}	2.72	1.00
I_{11}	1.63	0.40	A_{19}	3.13	1.10
I_{12}	2.69	0.65	A_{20}	4.01	1.30
I_{13}	3.00	0.75	A_{21}	4.50	1.50

N.B.: P_i, E_i, I_i, and A_i are respectively in kip, kip/ft^2, ft^4 and ft^2.

Limit-state function: Failure is considered when the horizontal displacement u_1 at node 1 exceeds 0.07 cm $= 2.2966 \times 10^{-1}$ ft, see Figure 1. The LSF reads:

$$g(\mathbf{x}) = g(x_1, \ldots, x_{21}) = 0.07 - u_1(\mathbf{x}) .$$

Reference failure probability: $p_{\mathrm{f\,ref}} = 1.05 \times 10^{-4}$.

6 Example 6

Source: Bourinet (2018, Chap. 1)

Random inputs: $U_1, \ldots, U_{16} \sim \mathcal{N}(0,1)$, $U_1, \ldots, U_{16}$ independent.

Limit-state function:

$$G(\mathbf{u}) = \min_{k \in \{1,\ldots,4\}} G_k(\mathbf{u}) \quad \text{for } \mathbf{u} = (u_1, \ldots, u_{16})^{\mathrm{T}},$$

where

$$G_k(\mathbf{u}) = \beta_k - \boldsymbol{\alpha}_k^{\mathrm{T}} \mathbf{u}$$

for $k = 1, \ldots, 4$.
We take:

$$\beta_1 = 4.0118 \qquad \beta_2 = 4.0109 \qquad \beta_3 = 4.0108 \qquad \beta_4 = 4.0051\,,$$

and

$$\boldsymbol{\alpha}_1 = \begin{Bmatrix} 0.3040 \\ -0.4013 \\ 0.2131 \\ 0.3218 \\ -0.3253 \\ 0.2534 \\ 0.1582 \\ 0.1067 \\ 0.2065 \\ -0.3173 \\ 0.1614 \\ 0.2842 \\ -0.2397 \\ 0.2022 \\ 0.1601 \\ 0.1306 \end{Bmatrix} \quad \boldsymbol{\alpha}_2 = \begin{Bmatrix} 0.3061 \\ 0.4007 \\ 0.2131 \\ 0.3219 \\ 0.3253 \\ 0.2534 \\ -0.1582 \\ -0.1060 \\ 0.2064 \\ 0.3164 \\ 0.1614 \\ 0.2838 \\ 0.2394 \\ 0.2020 \\ -0.1600 \\ -0.1319 \end{Bmatrix} \quad \boldsymbol{\alpha}_3 = \begin{Bmatrix} 0.3038 \\ 0.4006 \\ -0.2133 \\ 0.3217 \\ -0.3262 \\ -0.2535 \\ -0.1582 \\ 0.1065 \\ 0.2065 \\ 0.3166 \\ -0.1615 \\ 0.2841 \\ -0.2396 \\ -0.2023 \\ -0.1603 \\ -0.1319 \end{Bmatrix} \quad \boldsymbol{\alpha}_4 = \begin{Bmatrix} 0.3004 \\ -0.4077 \\ -0.2181 \\ 0.3262 \\ 0.2787 \\ -0.2586 \\ 0.1605 \\ -0.1120 \\ 0.2102 \\ -0.3231 \\ -0.1649 \\ 0.2894 \\ 0.2445 \\ -0.2067 \\ 0.1632 \\ 0.1346 \end{Bmatrix}.$$

Reference failure probability: $p_{f\,\mathrm{ref}} = 1.21 \times 10^{-4}$.

7 Example 7

Source: Dubourg (2011, Chap. 6)

Random inputs: $U_1, \ldots, U_{93} \sim \mathcal{N}(0,1)$, $U_1, \ldots, U_{93}$ independent.

Limit-state function:

$$G(\mathbf{u}) = \min_{k \in \{1,\ldots,4\}} G_k(\mathbf{u}) \quad \text{for } \mathbf{u} = (u_1, \ldots, u_{93})^{\mathrm{T}},$$

where

$$G_k(\mathbf{u}) = \beta_k - \boldsymbol{\alpha}_k^{\mathrm{T}} \mathbf{u}$$

for $k = 1, \ldots, 4$, and

$$\beta_1 = 4.0118 \qquad \beta_2 = 4.0109 \qquad \beta_3 = 4.0108 \qquad \beta_4 = 4.0051\,,$$

Reference failure probability: $p_{f\,\mathrm{ref}} = 1.22 \times 10^{-4}$.

8 Example 8

Source: Schuëller and Pradlwarter (2007, Problem 3, Case 3.2.1)

Reference failure probability: $p_{\mathrm{f\,ref}} = 2.41 \times 10^{-4}$.

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Appendix: FERUM-based LSF evaluations

```
addpath('/xxx/xxx/FERUM4.1');

inputfile_ex1 % selected input file

[probdata,gfundata,analysisopt] = update_data(1,probdata,analysisopt,gfundata,femodel);
probdata.marg  = distribution_parameter(probdata.marg);
[ Ro, dRo_drho, dRo_dthetafi, dRo_dthetafj ] = mod_corr_solve(probdata.marg,probdata.correlation,0);
probdata.Ro = Ro;
[ Lo , ierr ] = my_chol(Ro);
probdata.Lo = Lo;
iLo = inv(Lo);
probdata.iLo = iLo;
nrv = size(probdata.marg,1);

nu = 10;
U = randn(nrv,nu); % selected realizations in the standard normal space
X = u_to_x(U,probdata);
[ G , dummy ] = gfun(1,X,'no ',probdata,analysisopt,gfundata,femodel,randomfield); % corresponding LSF evaluations
```