Optimization with probabilistic constraints of complex systems
Application to the design of an offshore wind turbine

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28-30 April, 2021

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Problem presentation: the stationary case

- Harmonic oscillator (spring/mass system):
  - x_{d_1} (mass), x_{d_2} (spring stiffness), x_p (damping coefficient);
  - Displacement of the object is solution of the oscillator equation:

$$x_{d_1}\mathcal{Y}''(t) + x_p\mathcal{Y}'(t) + x_{d_2}\mathcal{Y}(t) = \eta(t), t \in [0, T].$$

- Introduction of uncertainties:
  - uncertainties on x_{d_1}, x_{d_2} represented by a r.v. $X_d = (X_{d_1}, X_{d_2}), \mathbb{E}[X_d] = d = (d_1, d_2)$ (design variables);
  - uncertainties on x_p represented by a r.v. X_p;
  - uncertainties on the force: $\eta(t)$ **stationary Gaussian** random process (with **zero mean** and **known spectral density**).

For fixed x_d, x_p, the displacement of the object is modelled by the **stochastic process** denoted $t \to \mathcal{Y}(x_d, x_p; t)$.

Mathematical formulation

Time-variant Reliability-Based Design Optimization (t-RBDO)

$$\min_{d \in \Omega_d} cost(d) \quad \text{such that}$$

$$\mathbb{P}_{X_d, X_p, X_{r_1}, \eta}\left(\max_{t \in [0,T]} \mathcal{Y}'(X_d, X_p; t) > X_{r_1}\right) < p_s$$

$$\mathbb{P}_{X_d, X_p, X_{r_2}, \eta}\left(\max_{t \in [0,T]} \mathcal{Y}''(X_d, X_p; t) > X_{r_2}\right) < p_s$$

$$\mathbb{P}_{X_d, X_p, X_{r_3}, \eta}\left(\int_0^T \left(\left|\mathcal{Y}''(X_d, X_p; t)\right| - \rho\right)^+ dt > X_{r_3}\right) < p_s$$

with $(x)^+ = \max\{0, x\}$. All the sources of uncertainties are **independent**.

Difficulty

Estimate the failure probabilities at each iteration of the optimization algorithm. Especially when p_s is small (rare event).

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Main reformulation idea

$$\mathbb{P}_{X_d,X_p,X_{r_1},\eta}\left(\max_{t\in[0,T]}\mathcal{Y}'\left(X_d,X_p;t\right)>X_{r_1}\right)$$
$$=\mathbb{E}_{X_d,X_p,X_{r_1}}\left[\mathbb{P}_\eta\left(\max_{t\in[0,T]}\mathcal{Y}'\left(X_d,X_p;t\right)>X_{r_1}\right)\right]$$

Use a limit theorem to approximate $\mathbb{P}_\eta\left(\max_{t\in[0,T]}\mathcal{Y}'\left(x_d,x_p;t\right)>x_{r_1}\right)$ by a quantity that only depends on x_d, x_p, x_{r_1}.

$$\mathbb{P}_{X_d,X_p,X_{r_2},\eta}\left(\max_{t\in[0,T]}\mathcal{Y}''\left(X_d,X_p;t\right)>X_{r_2}\right)$$
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Properties of the velocity and acceleration processes

At $X_d = x_d, X_p = x_p$ fixed:

- $\mathcal{Y}(x_d, x_p; .)$ output of **linear filter**:

$$\mathcal{Y}(x_d, x_p; t) = h_{\mathcal{Y}}(x_d, x_p; .) * \eta(t);$$

- η zero-mean Gaussian, stationary with known spectral density.

$\Rightarrow \mathcal{Y}'(x_d, x_p; .)$ and $\mathcal{Y}''(x_d, x_p; .)$ are also **Gaussian** and **stationary** with **zero mean**. Their spectral moment of order n is computable from the **spectral density** of η, K_η, and the **transfer function** $FT(h_{\mathcal{Y}})$:

$$m_{\mathcal{Y}',n}(x_d, x_p) = \int \omega^{n+2} \left| FT(h_{\mathcal{Y}})(x_d, x_p; \omega) \right|^2 K_\eta(\omega) d\omega;$$

$$m_{\mathcal{Y}'',n}(x_d, x_p) = \int \omega^{n+4} \left| FT(h_{\mathcal{Y}})(x_d, x_p; \omega) \right|^2 K_\eta(\omega) d\omega.$$

Extreme value theory

Theorem (1)

Let ξ be zero-mean stationary Gaussian process with spectral moment of order n denoted $m_{\xi,n}$. Then:

$$\mathbb{P}\left(a_T\left(\max_{t\in[0,T]}\frac{\xi(t)}{\sqrt{m_{\xi,0}}}-a_T\right)\leqslant x\right)\to\exp(-e^{-x})\quad \textit{as}\quad T\to\infty$$

with $a_T=\sqrt{2\log\left(\frac{T}{2\pi}\sqrt{\frac{m_{\xi,2}}{m_{\xi,0}}}\right)}$.

Thus, for T large enough:

$$\mathbb{P}\left(\max_{[0,T]}\xi(t)\geqslant x\right)\simeq 1-\exp\left(-e^{a_T^2-\frac{a_T x}{\sqrt{m_{\xi,0}}}}\right)=F_\epsilon\left(e^{a_T^2-\frac{a_T x}{\sqrt{m_{\xi,0}}}}\right)$$

with $F_\epsilon(x)=1-\exp(-x)$.

[1] Leadbetter et al, *Extremes and Related Properties of Random Sequences and Processes*, Chapter 8, 1983.

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Reformulation of extreme-based constraints

At fixed x_d, x_p, $\mathcal{Y}'(x_d, x_p; .)$ is a zero-mean stationary Gaussian process.
$\Rightarrow$ for T large enough, we have:

$$\mathbb{P}_\eta \left(\max_{[0,T]} \mathcal{Y}'(x_d, x_p; t) > x_{r_1} \right) \simeq F_\epsilon \left(e^{a_T(x_d,x_p)^2 - \frac{a_T(x_d,x_p) x_{r_1}}{\sqrt{m_{\mathcal{Y}',0}(x_d,x_p)}}} \right)$$

where $a_T(x_d, x_p)$ depends on T and the spectral moments of order 0 and 2 of $\mathcal{Y}'(x_d, x_p; .)$. The initial failure probability becomes:

$$\mathbb{P}_{X_d,X_p,X_{r_1},\eta} \left(\max_{t\in[0,T]} \mathcal{Y}'(X_d, X_p; t) > X_{r_1} \right) \simeq \mathbb{E}_{X_d,X_p,X_{r_1}} \left[F_\epsilon \left(e^{a_T(X_d,X_p)^2 - \frac{a_T(X_d,X_p) X_{r_1}}{\sqrt{m_{\mathcal{Y}',0}(X_d,X_p)}}} \right) \right]$$

The same reasoning applies to the second constraint on $\mathcal{Y}''$.

Reformulation of integral-based constraint

For fixed x_d, x_p fixed, $(\mathcal{Y}''(x_d, x_p; .) - \rho)^+ = \mathcal{F}(x_d, x_p; .)$ is ergodic:

$$\frac{1}{T}\int_0^T \mathcal{F}(x_d, x_p; t)\, dt \xrightarrow{\mathbb{P}} \mathbb{E}_{\mathcal{F}}[\mathcal{F}(x_d, x_p; 0)] \quad \text{as} \quad T \to \infty.$$

Hence, for T large enough:

$$\mathbb{P}_\eta\left(\int_0^T \mathcal{F}(x_d, x_p; t)\, dt > x_{r_3}\right) \simeq \mathbb{1}_{T\mathbb{E}_{\mathcal{F}}[\mathcal{F}(x_d, x_p; 0)] > x_{r_3}}.$$

We only need $m_{\mathcal{Y}'',0}(x_d, x_p)$ to compute $\mathbb{E}_{\mathcal{F}}[\mathcal{F}(x_d, x_p; 0)]$. The failure probability becomes:

$$\mathbb{P}_{X_d, X_p, X_{r_3}, \eta}\left(\int_0^T \mathcal{F}(X_d, X_p; t)\, dt > X_{p_3}\right) \simeq \mathbb{E}_{X_d, X_p}\left[F_{r_3}\left(T\mathbb{E}_{\mathcal{F}}[\mathcal{F}(X_d, X_p; 0)]\right)\right]$$

with F_{r_3} the cumulative distribution function of X_{r_3}.

Reformulated problem

Reliability-Based Design Optimization (RBDO)

$$\min_{d \in \Omega_d} cost(d) \quad \text{such that}$$

$$\mathbb{E}_{X_d, X_p, X_{r_1}}\left[F_\epsilon\left(e^{a_T(X_d,X_p)^2 - \frac{a_T(X_d,X_p)X_{r_1}}{\sigma_{\mathcal{Y}'}(X_d,X_p)}}\right)\right] < p_s$$

$$\mathbb{E}_{X_d, X_p, X_{r_2}}\left[F_\epsilon\left(e^{b_T(X_d,X_p)^2 - \frac{b_T(X_d,X_p)X_{r_2}}{\sigma_{\mathcal{Y}'}(X_d,X_p)}}\right)\right] < p_s$$

$$\mathbb{E}_{X_d, X_p}\left[F_{r_3}\left(T\mathbb{E}_{\mathcal{F}}\left[\mathcal{F}\left(X_d, X_p; 0\right)\right]\right)\right] < p_s$$

Remark

Time-independent RBDO: easier to solve.

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Definition of a piece-wise stationary process

- The period $[0,T]$ is decomposed into n_T intervals $I_i = \left[(i-1)\Delta T, i\Delta T\right], i = 1, \ldots, n_T$.
- for fixed x_d, x_p the process $\mathcal{Y}\left(x_d, x_p; .\right)$ is defined as:

$$\mathcal{Y}(x_d, x_p; t) = \sum_{i=1}^{n_T} \mathcal{Y}_i\left(x_d, x_p, S_i; t\right) \mathbb{1}_{I_i}(t)$$

  where $\mathcal{Y}_i\left(x_d, x_p, S_i; t\right)$ is solution of the oscillator equation with external force $\eta(S_i, t)$.
- $S_1, \ldots, S_{n_T}$ are i.i.d. discrete random variables such that $\mathbb{P}\left(S_1 = s^j\right) = p^j$ for $j = 1, \ldots, 7$.
- $\eta(s^j, .)$ is a zero-mean stationary Gaussian process with spectral density depending on s^j.

Reformulated problem

Reliability-Based Design Optimization (RBDO)

$$\min_{d \in \Omega_d} \; cost(d) \quad \text{such that}$$

$$\mathbb{E}_{X_d, X_p, X_{r_1}} \left[F_\epsilon \left(\sum_{j=1}^{7} e^{a_{T_p^j}\left(X_d, X_p, s^j\right)^2 - \frac{a_{T_p^j}\left(X_d, X_p, s^j\right) X_{r_1}}{\sqrt{m_{\mathcal{Y}',0}\left(X_d, X_p, s^j\right)}}} \right) \right] < p_s$$

$$\mathbb{E}_{X_d, X_p, X_{r_2}} \left[F_\epsilon \left(\sum_{j=1}^{7} e^{b_{T_p^j}\left(X_d, X_p, s^j\right)^2 - \frac{b_{T_p^j}\left(X_d, X_p, s^j\right) X_{r_2}}{\sqrt{m_{\mathcal{Y}'',0}\left(X_d, X_p, s^j\right)}}} \right) \right] < p_s$$

$$\mathbb{E}_{X_d, X_p} \left[F_{r_3} \left(T \sum_{j=1}^{7} p^j \mathbb{E}_{\mathcal{F}_1} \left[\mathcal{F}_1 \left(X_d, X_p, s^j; 0 \right) \right] \right) \right] < p_s$$

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Metamodeling strategy

First constraint:

$$p_1(d) = \mathbb{E}_{X_d,X_p,X_{r_1}}\left[F_\epsilon\left(\sum_{j=1}^{7} e^{a_{Tp^j}\left(X_d,X_p,s^j\right)^2-\frac{a_{Tp^j}\left(X_d,X_p,s^j\right)X_{r_1}}{\sqrt{m_{\mathcal{Y}',0}\left(X_d,X_p,s^j\right)}}}\right)\right]$$
$$= \mathbb{E}_{X_d,X_p,X_{r_1}}\left[F_\epsilon\left(\sum_{j=1}^{7} e^{M\left(X_d,X_p,X_{r_1},s^j\right)}\right)\right]$$

From a space-filling design of experiments (DoE) of the **augmented space**, calibration of a metamodel by **Kriging**:

$$\widetilde{M}(x_d, x_p, x_{r_1}, s^j) \sim \mathcal{N}\left(\mu(x_d, x_p, x_{r_1}, s^j), \sigma(x_d, x_p, x_{r_1}, s^j)^2\right)$$

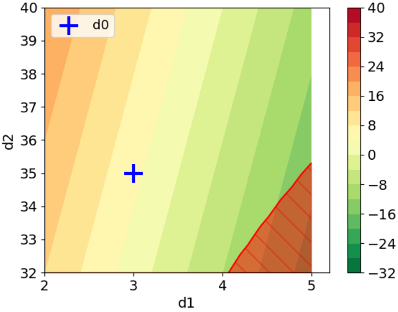
- Idea of AK-ECO : succession of cycles composed of a local enrichment of the metamodels and an optimization resolution.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0, DoE DoE^0 and metamodel $\widetilde{M}^0$, cycle $k = 1$.

Optimization cycle k: $d^{k-1}, \widetilde{M}^{k-1}, DoE^{k-1}$

- **Step 1:** local enrichment of the metamodel:
  - Step 1.a: accuracy criterion
  - Step 1.b: selection of x_{enr}. Update DoE^{k-1} and recalibration of $\widetilde{M}^{k-1}$.
- $DoE^{k-1} \rightarrow DoE^k$ and $\widetilde{M}^{k-1} \rightarrow \widetilde{M}^k$.
- **Step 2:** solve the reformulated RBDO starting from d^{k-1}.
  - optimization algorithm chosen by user + constraints evaluated with Monte Carlo replacing M by $\widetilde{M}^k \rightarrow d^k$.
- **Stopping condition:**
  - If $||d^k - d^{k-1}|| < \epsilon$ OR $|cost(d^k) - cost(d^{k-1})| < \epsilon$: AK-ECO ends and $d^{min} = d^k$.
  - Else: $k = k + 1$ and go back to step 1.



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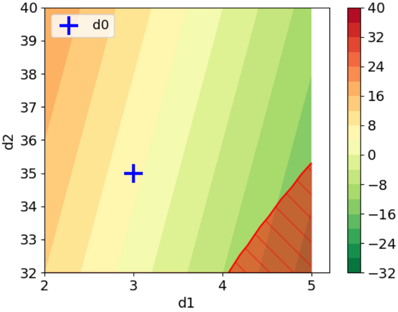

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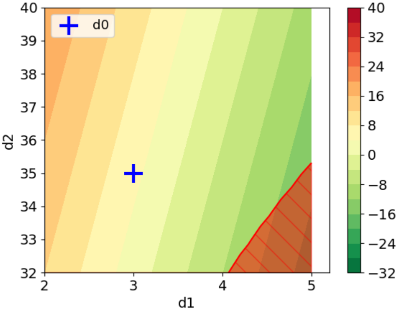

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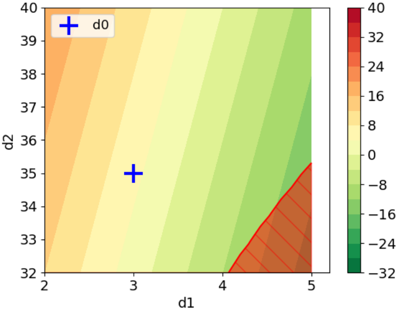

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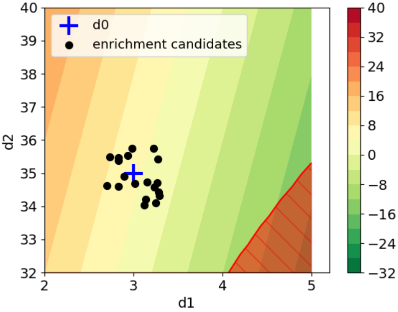

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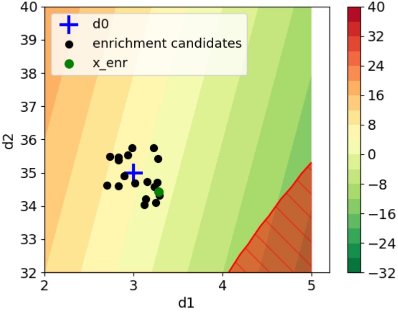

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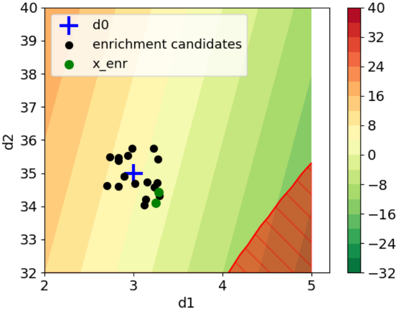

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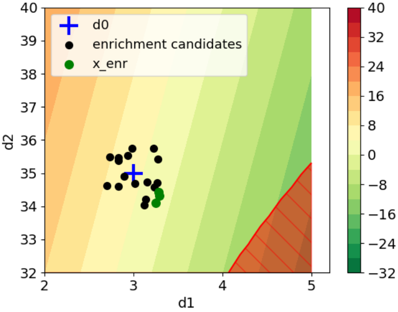

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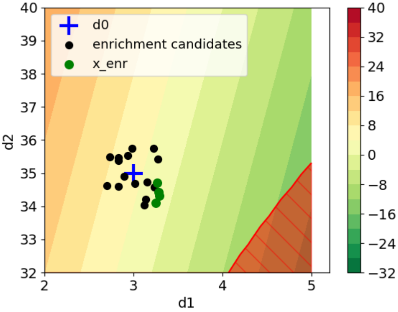

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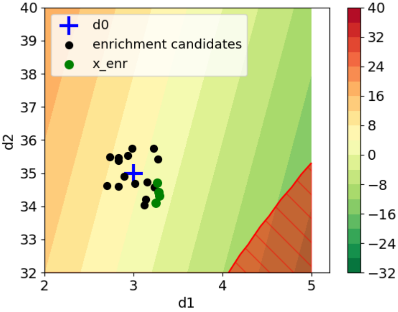

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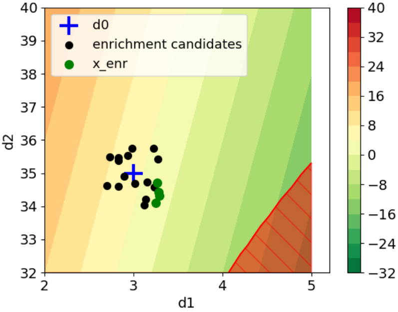

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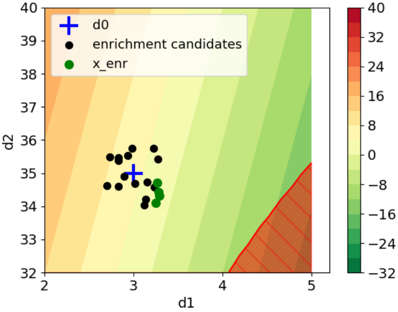

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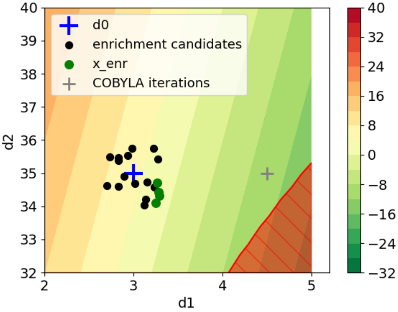

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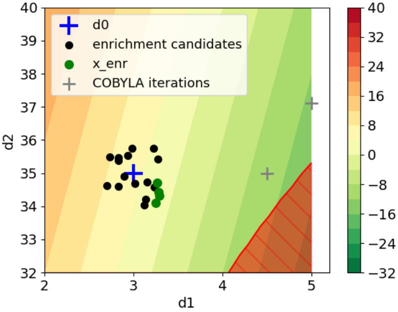

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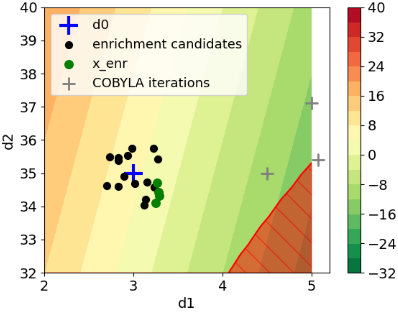

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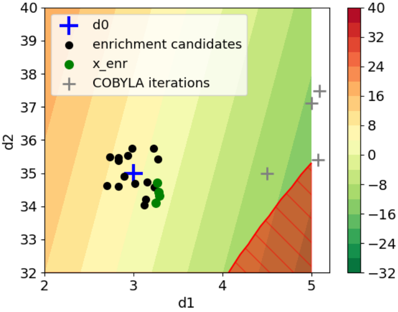

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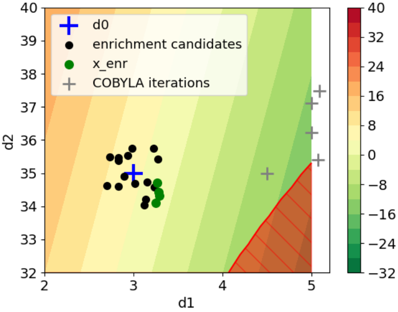

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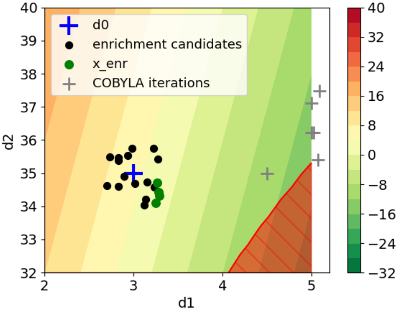

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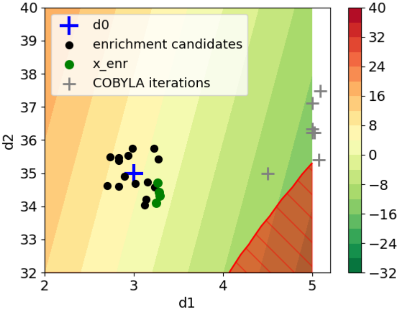

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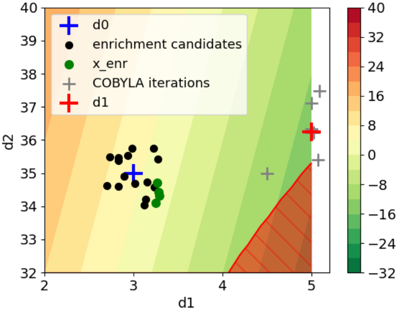

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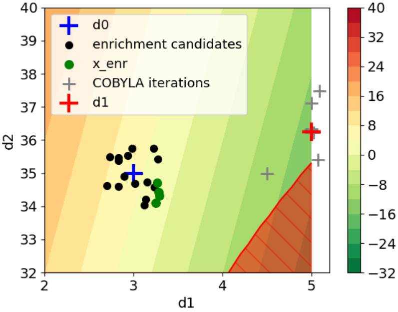

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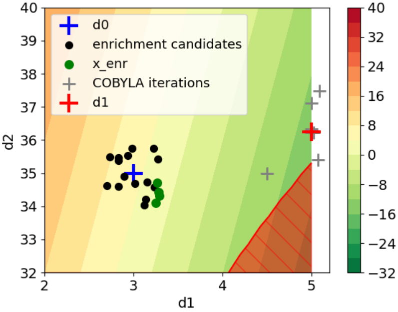

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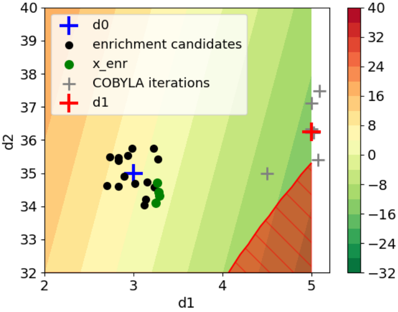

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Comparaison of AK-ECO with reference methods in RBDO

Results of different methods starting from $d^0 = (3, 35)$:

	MC	RIA[2]	PMA[2]	SORA[3]	Stieng[4]	AK-ECO
d^{min}	(5.0, 35.78)	(5.0, 35.07)	(5.0, 35.04)	(5.0, 36.06)	(4.98, 37.82)	(5.0, 35.67)
$cost(d^{min})$	-14.22	-14.93	-14.96	-13.94	-11.89	-14.33
$p_1^{MC}\left(d^{min}\right)$	0.6×10^{-4}	0.7×10^{-4}	0.8×10^{-4}	0.5×10^{-4}	0.1×10^{-4}	0.5×10^{-4}
$p_2^{MC}\left(d^{min}\right)$	1.0×10^{-4}	1.0×10^{-4}	1.4×10^{-4}	0.7×10^{-4}	0.3×10^{-4}	1.0×10^{-4}
$p_3^{MC}\left(d^{min}\right)$	0.1×10^{-4}	0.4×10^{-4}	0.4×10^{-4}	0.1×10^{-4}	0.02×10^{-4}	0.2×10^{-4}
n_{call}	3.36×10^6	431256	29393	16072	32242	150

Table: Results of AK-ECO and the comparison methods

- AK-ECO finds a reliable optimum with few calls to the expensive code.

[2] J. Tu, K.K. Choi and Y. H. Park. *A New Study on Reliability-Based Design Optimization*, Journal of Mechanical Design, 1999.

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Conclusion

- **Optimization problem** with **probabilistic constraints** dependent on **random variables** and a **piecewise-stationary stochastic Gaussian process**: t-RBDO.
- Methodology in two parts:
  - Reformulation of the constraints using **limit theorems**;
  - Resolution of the reformulated problem with a **new adaptive kriging** strategy **AK-ECO**.
- Application of the methodology to the harmonic oscillator problem.
- Not presented work:
  - Quantification of the approximation errors made during the reformulation steps;
  - Global enrichment of the metamodels before AK-ECO;
  - RBDO-oriented GSA;
  - Application of the methodology to the FOWT problem [5].

[5] A.Cousin, J. Garnier, M. Guiton, M. Munoz Zuniga. *Optimization with probabilistic constraints of complex systems. Application to the design of an offshore wind turbine.* ECCOMAS, 2021

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Surrogate-assisted reliability-based design optimization: a survey and a new general framework
Structural and Multidisciplinary Optimization, 60(5): 2157-2176, 2019.

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General formulation

$$\min_{d \in \Omega_d} cost(d) \quad \text{such that}$$

$$\mathbb{P}_{X_d, X_p, X_{r_{\mathrm{I}}}, \mathcal{Y}} \left(\min_{[0,T]} \mathcal{Y}(X_d, X_p; t) > X_{r_{\mathrm{I}}} \right) < p_s$$

$$\mathbb{P}_{X_d, X_p, X_{r_{\mathrm{II}}}, \mathcal{Y}} \left(\int_0^T f\left(\mathcal{Y}\left(X_d, X_p; t\right)\right) dt > X_{r_{\mathrm{II}}} \right) < p_s.$$

Graphs of the PW stationary oscillator problem

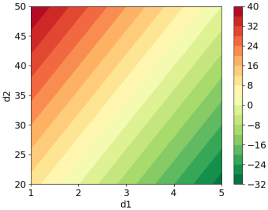

Figure: Level sets of the cost function

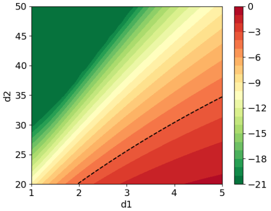

Figure: Level sets of the log of p_1

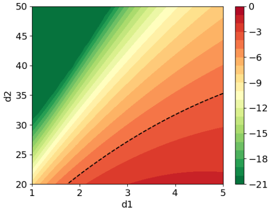

Figure: Level sets of the log of p_2

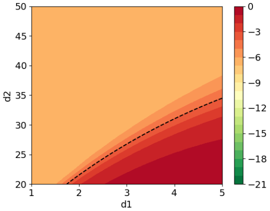

Figure: Level sets of the log of p_3

Reformulated constraints: probability version

Reformulated failure probabilities as expectations:

$$\mathbb{E}_{X_d,X_p,X_{r_{\mathrm{I}}}}\left[F_\epsilon\left(\sum_{j=1}^{n_s} e^{M_{\mathrm{I}}\left(X_d,X_p,X_{r_{\mathrm{I}}},s^j\right)}\right)\right] < p_s \tag{1}$$

$$\mathbb{E}_{X_d,X_p}\left[F_{r_{\mathrm{II}}}\left(\sum_{j=1}^{n_s} Tp^j\left(M_{\mathrm{II}}\left(X_d,X_p,s^j\right)\right)\right)\right] < p_s \tag{2}$$

with $F_\epsilon(x) = 1 - \exp(-x)$.
Reformulated failure probabilities as probabilities:

$$\mathbb{P}_{X_d,X_p,X_{r_{\mathrm{I}}},X_\epsilon}\left(X_\epsilon - \sum_{j=1}^{n_s} \exp\left(M_{\mathrm{I}}\left(X_d,X_p,X_{r_1},s^j\right)\right) < 0\right) < p_s, \tag{3}$$

$$\mathbb{P}_{X_d,X_p,X_{r_{\mathrm{II}}}}\left(X_{r_{\mathrm{II}}} - \sum_{j=1}^{n_s} Tp^j\left(M_{\mathrm{II}}\left(X_d,X_p,s^j\right)\right) < 0\right) < p_s \tag{4}$$

with X_ϵ r.v with exponential distribution of parameter 1.

AK-ECO Step 1.a: accuracy criterion

$$p_1^{k-1}\left(d^{k-1}\right) = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} F_\epsilon\left(\sum_{j=1}^{7} \exp\left(\mu^{k-1}\left(x_{d^{k-1}}^i, x_p^i, x_{r_1}^i, s^j\right)\right)\right) \quad (5)$$

$$p_{\mathrm{I},-}^{k-1}(d) = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} F_\epsilon\left(\sum_{j=1}^{n_s} \exp\left(\mu_{\mathrm{I}}^{k-1}\left(x_d^i, x_p^i, x_{r_{\mathrm{I}}}^i, s^j\right) - 2\sigma_{\mathrm{I}}^{k-1}\left(x_d^i, x_p^i, x_{r_{\mathrm{I}}}^i, s^j\right)\right)\right) \quad (6)$$

$$p_{\mathrm{I},+}^{k-1}(d) = \frac{1}{n_{MC}} \sum_{i=1}^{n_{MC}} F_\epsilon\left(\sum_{j=1}^{n_s} \exp\left(\mu_{\mathrm{I}}^{k-1}\left(x_d^i, x_p^i, x_{r_{\mathrm{I}}}^i, s^j\right) + 2\sigma_{\mathrm{I}}^{k-1}\left(x_d^i, x_p^i, x_{r_{\mathrm{I}}}^i, s^j\right)\right)\right) \quad (7)$$

Accuracy criteria for the first constraint:

$$\frac{\left|p_{\mathrm{I}}^{k-1}(d^{k-1}) - p_s\right|}{p_{\mathrm{I},+}^{k-1}(d^{k-1}) - p_{\mathrm{I},-}^{k-1}(d^{k-1})} > 1.$$

AK-ECO Step 1.b: enrichment criterion

$$p_1^{k-1}\left(d^{k-1}\right) = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} F_\epsilon \left(\sum_{j=1}^{7} e^{\mu^{k-1}\left(x^i_{d^{k-1}}, x^i_p, x^i_{r_1}, s^j\right)} \right)$$

$$\max_{i\in\{1,\dots,N_{MC}\}, j\in\{1,\dots,7\}} \mathcal{C}_{aug}(x^i_{d^{k-1}}, x^i_p, x^i_{r_1}, s^j) \quad \text{with} \quad \mathcal{C}_{aug}(x^i_d, x^i_p, x^i_{r_1}, s^j)$$

$$= \Bigg[F_\epsilon \left(\sum_{j'\neq j} e^{\mu^{k-1}\left(x^i_d, x^i_p, x^i_{r_1}, s^{j'}\right)} + e^{\left(\mu^{k-1}\left(x^i_d, x^i_p, x^i_{r_1}, s^j\right) + 2\sigma^{k-1}\left(x^i_d, x^i_p, x^i_{r_1}, s^j\right)\right)} \right)$$
$$- F_\epsilon \left(\sum_{j'\neq j} e^{\mu^{k-1}\left(x^i_d, x^i_p, x^i_{r_1}, s^{j'}\right)} + e^{\left(\mu^{k-1}\left(x^i_d, x^i_p, x^i_{r_1}, s^j\right) - 2\sigma^{k-1}\left(x^i_d, x^i_p, x^i_{r_1}, s^j\right)\right)} \right) \Bigg]$$
$$\times f_{X_d, X_p, X_{r_1}}(x^i_d, x^i_p, x^i_{r_1})$$

$F_\epsilon(x) = 1 - \exp(-x)$ and $f_{X_d,X_p,X_{r_1}}$ the pdf of (X_d, X_p, X_{r_1})