

Optimization with probabilistic constraints of complex systems

Application to the design of an offshore wind turbine

Alexis Cousin^{1,2} & Josselin Garnier² & Martin Guiton¹ & Miguel
Munoz Zuniga¹

¹ IFP Énergies Nouvelles

² CMAP, École Polytechnique

28-30 April, 2021



Table of contents

- 1 Presentation of the harmonic oscillator problem
- 2 Problem reformulation
- 3 From stationary to piecewise stationary
- 4 Solving the reformulated problem with AK-ECO
- 5 Conclusion

Table of contents

- 1** Presentation of the harmonic oscillator problem
- 2 Problem reformulation
- 3 From stationary to piecewise stationary
- 4 Solving the reformulated problem with AK-ECO
- 5 Conclusion

Problem presentation: the stationary case

Harmonic oscillator (spring/mass system):

X_{d_1} (mass), X_{d_2} (spring stiffness), X_p (damping coefficient);

Displacement of the object is solution of the oscillator equation:

$$X_{d_1} Y^2 p t q \quad X_p Y^1 p t q \quad X_{d_2} Y p t q \quad p t q, t \in \mathbb{R}, T_s.$$

Introduction of uncertainties:

uncertainties on X_{d_1}, X_{d_2} represented by a r.v.

$X_d = p X_{d_1}, X_{d_2} q, E r X_d s = d = p d_1, d_2 q$ (design variables);

uncertainties on X_p represented by a r.v. X_p ;

uncertainties on the force: $p t q$ **stationary Gaussian** random process (with **zero mean** and **known spectral density**).

For fixed X_d, X_p , the displacement of the object is modelled by the **stochastic process** denoted $t \tilde{N} Y p X_d, X_p; t q$.

Mathematical formulation

Time-variant Reliability-Based Design Optimization (t-RBDO)

$$\min_{d \in \Omega_d} \text{cost}(d) \quad \text{such that}$$

$$P_{X_d, X_p, X_{r_1}, \eta} \max_{t \in [0, T_s]} \gamma^1 p(X_d, X_p; t) \leq X_{r_1} \quad \rho_s$$

$$P_{X_d, X_p, X_{r_2}, \eta} \max_{t \in [0, T_s]} \gamma^2 p(X_d, X_p; t) \leq X_{r_2} \quad \rho_s$$

$$P_{X_d, X_p, X_{r_3}, \eta} \int_0^T \gamma^2 p(X_d, X_p; t) dt \leq X_{r_3} \quad \rho_s$$

with $p(x) = \max_{t \in [0, T_s]} x$. All the sources of uncertainties are **independent**.

Difficulty

Estimate the failure probabilities at each iteration of the optimization algorithm. Especially when ρ_s is small (rare event).

Table of contents

- 1 Presentation of the harmonic oscillator problem
- 2 Problem reformulation**
- 3 From stationary to piecewise stationary
- 4 Solving the reformulated problem with AK-ECO
- 5 Conclusion

Main reformulation idea

$$P_{X_d, X_p, X_{r_1}, \eta} \max_{t \in [0, T]} \gamma^1 p(X_d, X_p; tq) \mid X_{r_1}$$

$$E_{X_d, X_p, X_{r_1}} P_\eta \max_{t \in [0, T]} \gamma^1 p(X_d, X_p; tq) \mid X_{r_1}$$

Use a limit theorem to approximate $P_\eta \max_{t \in [0, T]} \gamma^1 p(X_d, X_p; tq) \mid X_{r_1}$ by a quantity that only depends on X_d, X_p, X_{r_1} .

$$P_{X_d, X_p, X_{r_2}} \max_{t \in [0, T]} \gamma^2 p(X_d, X_p; tq) \mid X_{r_2}$$

$$E_{X_d, X_p, X_{r_2}} P \max_{t \in [0, T]} \gamma^2 p(X_d, X_p; tq) \mid X_{r_2}$$

$$P_{X_d, X_p, X_{r_3}} \int_0^T \gamma^2 p(X_d, X_p; tq) \rho dt \mid X_{r_3}$$

$$E_{X_d, X_p, X_{r_3}} P \int_0^T \gamma^2 p(X_d, X_p; tq) \rho dt \mid X_{r_3}$$

Properties of the velocity and acceleration processes

At $X_d = x_d; X_p = x_p$ we have:

$Y = y(x_d; x_p; \omega)$ output of linear filter :

$$Y = y(x_d; x_p; \omega) = h_Y(x_d; x_p; \omega) p(\omega)$$

zero-mean Gaussian, stationary with known spectral density.

$\tilde{Y}^1(x_d; x_p; \omega)$ and $\tilde{Y}^2(x_d; x_p; \omega)$ are also Gaussian and stationary with zero mean. Their spectral moment of order m is computable from the spectral density of p , K , and the transfer function $FT(p, Y, \omega)$

$$\begin{aligned} \text{»} \\ m_{Y^1; n}(x_d; x_p, \omega) &= \int \int \dots \int |FT(p, Y, \omega)|^n K(p, \omega) \dots \omega^n \end{aligned}$$

$$\begin{aligned} \text{»} \\ m_{Y^2; n}(x_d; x_p, \omega) &= \int \int \dots \int |FT(p, Y, \omega)|^n K(p, \omega) \dots \omega^{2n} \end{aligned}$$

Extreme value theory

Theorem (1)

Let $\{X_t\}_{t \in \mathbb{R}}$ be zero-mean stationary Gaussian process with spectral moment of order n denoted $m_{;n}$. Then:

$$P(a_T \max_{t \in [0; T]} X_t \leq x) \sim \exp(-x^2) \text{ as } T \rightarrow \infty$$

$$\text{with } a_T = \frac{c}{2 \log \frac{T}{2}} \frac{b}{\frac{m_{;2}}{m_{;0}}}$$

Thus, for T large enough:

$$P(\max_{t \in [0; T]} X_t \leq x) \sim \exp(-x^2) \text{ as } T \rightarrow \infty$$

$$\text{with } F(x) = \exp(-x^2)$$

[1] Leadbetter et al, Extremes and Related Properties of Random Sequences and Processes, Chapter 8, 1983.

Extreme value theory

Theorem (1)

Let $\{X_t\}_{t \in \mathbb{R}}$ be zero-mean stationary Gaussian process with spectral moment of order n denoted $m_{;n}$. Then:

$$P \left(a_T \max_{t \in [0; T]} X_t \leq x \right) \sim \exp \left(- \frac{x^2}{2 a_T^2} \right) \quad \text{as } T \rightarrow \infty$$

$$\text{with } a_T = \sqrt{\frac{2 \log T}{m_{;2}}}.$$

Thus, for T large enough:

$$P \left(\max_{t \in [0; T]} X_t \leq x \right) \sim \exp \left(- \frac{x^2}{2 a_T^2} \right) \quad \text{with } F(x) = \exp \left(- \frac{x^2}{2 a_T^2} \right)$$

$$\text{with } F(x) = \exp \left(- \frac{x^2}{2 a_T^2} \right)$$

[1] Leadbetter et al, Extremes and Related Properties of Random Sequences and Processes, Chapter 8, 1983.

Reformulation of extreme-based constraints

At fixed $x_d; x_p$, $Y^1_{p|x_d; x_p}; :q$ is a zero-mean stationary Gaussian process.
 For T large enough, we have:

$$P_{r_0; T_S} \max_{r_0; T_S} Y^1_{p|x_d; x_p}; tq | X_{r_1} \approx F \left[e^{-a_T p|x_d; x_p|q^2} \frac{\mathbb{E}_{Y^1_{p|x_d; x_p}; :q} [Y^1_{p|x_d; x_p}; tq | X_{r_1}]}{m_{Y^1_{p|x_d; x_p}; :q}} \right]$$

where $a_T p|x_d; x_p|q$ depends on T and the spectral moments of order 0 and 2 of $Y^1_{p|x_d; x_p}; :q$. The initial failure probability becomes:

$$P_{X_d; X_p; X_{r_1}} \max_{t \in [0; T_S]} Y^1_{p|X_d; X_p}; tq | X_{r_1} \approx F \left[e^{-a_T p|X_d; X_p|q^2} \frac{\mathbb{E}_{Y^1_{p|X_d; X_p}; :q} [Y^1_{p|X_d; X_p}; tq | X_{r_1}]}{m_{Y^1_{p|X_d; X_p}; :q}} \right]$$

The same reasoning applies to the second constraint $Y^2_{p|x_d; x_p}; :q$

Reformulation of integral-based constraint

For fixed $x_d; x_p$ fixed, $\mathbb{P}^{Y^2; 0; x_d; x_p; :q}$ is ergodic:

$$\frac{1}{T} \int_0^T F(x_d; x_p; tq) dt \xrightarrow{\text{a.s.}} \mathbb{E}_F \int_0^1 F(x_d; x_p; 0) ds \text{ as } T \rightarrow \infty :$$

Hence, for T large enough:

$$\mathbb{P} \left(\int_0^T F(x_d; x_p; tq) dt \leq X_{r_3} \right) \approx \mathbb{1}_{T \mathbb{E}_F \int_0^1 F(x_d; x_p; 0) ds \leq X_{r_3}} :$$

We only need $\mathbb{P}^{Y^2; 0; x_d; x_p; :q}$ to compute $\mathbb{E}_F \int_0^1 F(x_d; x_p; 0) ds$. The failure probability becomes:

$$\mathbb{P}_{X_d; X_p; X_{r_3}} \left(\int_0^T F(x_d; x_p; tq) dt \leq X_{r_3} \right) = \mathbb{E}_{X_d; X_p} \mathbb{P}_{r_3} \left(T \mathbb{E}_F \int_0^1 F(x_d; x_p; 0) ds \leq X_{r_3} \right)$$

with F_{r_3} the cumulative distribution function of X_{r_3} .

Reformulated problem

Reliability-Based Design Optimization (RBDO)

$\min_{\mathbf{p}, \mathbf{d}} \text{cost}(\mathbf{p}, \mathbf{d})$ such that

$$E_{X_d; X_p; X_{r_1}} F e^{a_T \mathbf{p}^T X_d; X_p \mathbf{q}^2} \frac{a_T \mathbf{p}^T X_d; X_p \mathbf{q}^2 r_1}{\gamma \sqrt{1 + \mathbf{p}^T X_d; X_p \mathbf{q}^2}} \leq \rho_s$$

$$E_{X_d; X_p; X_{r_2}} F e^{b_T \mathbf{p}^T X_d; X_p \mathbf{q}^2} \frac{b_T \mathbf{p}^T X_d; X_p \mathbf{q}^2 r_2}{\gamma \sqrt{1 + \mathbf{p}^T X_d; X_p \mathbf{q}^2}} \leq \rho_s$$

$$E_{X_d; X_p} F_{r_3} T E_F F \mathbf{p}^T X_d; X_p; \mathbf{0} \mathbf{q} \leq \rho_s$$

Remark

Time-independent RBDO: easier to solve.

Table of contents

- 1 Presentation of the harmonic oscillator problem
- 2 Problem reformulation
- 3 From stationary to piecewise stationary**
- 4 Solving the reformulated problem with AK-ECO
- 5 Conclusion

Definition of a piece-wise stationary process

The period $0; T$ is decomposed into n_T intervals

$I_i = [t_i, t_{i+1}]$; $t_i = i T; i = 1; \dots; n_T$.

for fixed x_d, x_p the process $Y(x_d; x_p; \cdot; q)$ is defined as:

$$Y(x_d; x_p; t; q) = \sum_{i=1}^{n_T} Y_i(x_d; x_p; S_i; t; q) 1_{I_i}(t; q)$$

where $Y_i(x_d; x_p; S_i; t; q)$ is solution of the oscillator equation with external force $p(S_i; t; q)$

$S_1; \dots; S_{n_T}$ are i.i.d. discrete random variables such that $P(S_j = s^j) = p^j$ for $j = 1; \dots; 7$.

$p^j(\cdot; q)$ is a zero-mean stationary Gaussian process with spectral density depending on s^j .

Reformulated problem

Reliability-Based Design Optimization (RBDO)

min cost_d such that

$$E_{X_d; X_p; X_{r_1}} F_j \leq e^{-a_{T,pj}} \frac{\mu_{Y1,0}^2 \sigma_{Y1,0}^2}{\mu_{Y1,0}^2 \sigma_{Y1,0}^2} \leq \rho_s$$

$$E_{X_d; X_p; X_{r_2}} F_j \leq e^{-b_{T,pj}} \frac{\mu_{Y2,0}^2 \sigma_{Y2,0}^2}{\mu_{Y2,0}^2 \sigma_{Y2,0}^2} \leq \rho_s$$

$$E_{X_d; X_p} F_{r_3} T_j \leq \rho_s$$

Table of contents

- 1 Presentation of the harmonic oscillator problem
- 2 Problem reformulation
- 3 From stationary to piecewise stationary
- 4 Solving the reformulated problem with AK-ECO
- 5 Conclusion

Metamodeling strategy

First constraint:

$$\begin{aligned}
 & p_1 p_d q \quad E_{X_d; X_p; X_{r_1}} \quad F \quad \sum_{j=1}^7 e^{a_{T,pj}} p_{X_d; X_p; s^j} q^2 \quad \frac{a_{T,pj} p_{X_d; X_p; s^j} q_{X_{r_1}}}{m_{Y,1,0} p_{X_d; X_p; s^j} q} \\
 & E_{X_d; X_p; X_{r_1}} \quad F \quad \sum_{j=1}^7 e^{M} p_{X_d; X_p; X_{r_1}; s^j} q
 \end{aligned}$$

From a space-filling design of experiments (DoE) of the augmented space, calibration of a metamodel by Kriging:

$$\mathbb{M} p_{X_d; X_p; X_{r_1}; s^j} q \quad N \quad p_{X_d; X_p; X_{r_1}; s^j} q \quad p_{X_d; X_p; X_{r_1}; s^j} q^2$$

Idea of AK-ECO : succession of cycles composed of a local enrichment of the metamodels and an optimization resolution.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}(d^k)$.

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user + constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq d^{\text{min}}$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user + constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| < \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| < \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| < \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| < \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq d^{\text{min}}$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq d^{\text{min}}$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$ and $\tilde{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$.

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user + constraints evaluated with Monte Carlo replacing M by $\tilde{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$.

Stopping condition:

If $\|d^k - d^{k-1}\| \leq d^{\text{min}}$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$ and $\tilde{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$.

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$.

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \mathcal{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \mathcal{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \mathcal{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$ and $\tilde{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$.

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by \tilde{M}^{k-1} .

Stopping condition:

If $\|d^k - d^{k-1}\| < \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| < \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq d^{\text{min}}$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq d^{\text{min}}$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq d^{\text{min}}$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq d^{\text{min}}$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \mathcal{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \mathcal{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \tilde{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and $d^{\text{min}} = d^k$.

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \mathcal{N}$ DoE \mathcal{M}^k and $\tilde{M}^{k-1} \cup \tilde{N}$ \tilde{M}^k .

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by $\tilde{M}^k \cup \mathcal{N}$ d^k .

Stopping condition:

If $\|d^k - d^{k-1}\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$ and $\tilde{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$.

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by \tilde{M}^{k-1} .

Stopping condition:

If $\|d^{k-1} - d^k\| < \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| < \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Solving the RBDO problem: AK-ECO

Initialization: Initial design d^0 , DoE \mathcal{M}^0 and metamodel \tilde{M}^0 , cycle $k = 1$.

Optimization cycle k : d^{k-1} ; \mathcal{M}^{k-1} , DoE \mathcal{M}^{k-1}

Step 1: local enrichment of the metamodel:

Step 1.a: accuracy criterion

Step 1.b: selection of x_{enr} . Update DoE \mathcal{M}^{k-1} and recalibration of \tilde{M}^{k-1} .

DoE $\mathcal{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$ and $\tilde{M}^{k-1} \cup \tilde{\mathcal{M}}^{k-1}$.

Step 2: solve the reformulated RBDO starting from d^{k-1} .

optimization algorithm chosen by user
+ constraints evaluated with Monte Carlo replacing M by \tilde{M}^{k-1} .

Stopping condition:

If $\|d^{k-1} - d^k\| \leq \epsilon$ OR $|\text{cost}(d^k) - \text{cost}(d^{k-1})| \leq \epsilon$: AK-ECO ends and

Else: $k = k + 1$ and go back to step 1.

Comparison of AK-ECO with reference methods in RBDO

Results of different methods starting from μ^0 p 3; 35q

	MC	RIA[2]	PMA[2]	SORA[3]	Stieng[4]	AK-ECO
d^{\min}	(5.0, 35.78)	(5.0, 35.07)	(5.0, 35.04)	(5.0, 36.06)	(4.98, 37.82)	(5.0, 35.67)
cost $pd^{\min} q$	-14.22	-14.93	-14.96	-13.94	-11.89	-14.33
$p_1^{MC} d^{\min}$	0:6 10 ⁴	0:7 10 ⁴	0:8 10 ⁴	0:5 10 ⁴	0:1 10 ⁴	0:5 10 ⁴
$p_2^{MC} d^{\min}$	1:0 10 ⁴	1:0 10 ⁴	1:4 10 ⁴	0:7 10 ⁴	0:3 10 ⁴	1:0 10 ⁴
$p_3^{MC} d^{\min}$	0:1 10 ⁴	0:4 10 ⁴	0:4 10 ⁴	0:1 10 ⁴	0:02 10 ⁴	0:2 10 ⁴
n_{call}	3:36 10 ⁶	431256	29393	16072	32242	150

Table: Results of AK-ECO and the comparison methods

AK-ECO finds a reliable optimum with few calls to the expensive code.

[2] J. Tu, K.K. Choi and Y. H. Park. A New Study on Reliability-Based Design Optimization, Journal of Mechanical Design, 1999.

[3] X. Du and W. Chen, Sequential Optimization and Reliability Assessment Method for Efficient Probabilistic Design, Journal of Mechanical Design, 2004

[4] Lars Einar S. Stieng. Optimal design of offshore wind turbine support structures under uncertainty. PhD thesis, Norwegian University of Science and Technology, 2019.

Comparison of AK-ECO with reference methods in RBDO

Results of different methods starting from μ^0 p 3; 35q

	MC	RIA[2]	PMA[2]	SORA[3]	Stieng[4]	AK-ECO
d^{\min}	(5.0, 35.78)	(5.0, 35.07)	(5.0, 35.04)	(5.0, 36.06)	(4.98, 37.82)	(5.0, 35.67)
cost $pd^{\min} q$	-14.22	-14.93	-14.96	-13.94	-11.89	-14.33
$p_1^{MC} d^{\min}$	0:6 10 ⁴	0:7 10 ⁴	0:8 10 ⁴	0:5 10 ⁴	0:1 10 ⁴	0:5 10 ⁴
$p_2^{MC} d^{\min}$	1:0 10 ⁴	1:0 10 ⁴	1:4 10 ⁴	0:7 10 ⁴	0:3 10 ⁴	1:0 10 ⁴
$p_3^{MC} d^{\min}$	0:1 10 ⁴	0:4 10 ⁴	0:4 10 ⁴	0:1 10 ⁴	0:02 10 ⁴	0:2 10 ⁴
n_{call}	3:36 10 ⁶	431256	29393	16072	32242	150

Table: Results of AK-ECO and the comparison methods

✪ AK-ECO finds a reliable optimum with few calls to the expensive code.

[2] J. Tu, K.K. Choi and Y. H. Park. A New Study on Reliability-Based Design Optimization, Journal of Mechanical Design, 1999.

[3] X. Du and W. Chen, Sequential Optimization and Reliability Assessment Method for Efficient Probabilistic Design, Journal of Mechanical Design, 2004

[4] Lars Einar S. Stieng. Optimal design of offshore wind turbine support structures under uncertainty. PhD thesis, Norwegian University of Science and Technology, 2019.

Table of contents

- 1 Presentation of the harmonic oscillator problem
- 2 Problem reformulation
- 3 From stationary to piecewise stationary
- 4 Solving the reformulated problem with AK-ECO
- 5 Conclusion

Conclusion

Optimization problem with probabilistic constraints dependent on random variables and a piecewise-stationary stochastic Gaussian process t-RBDO.

Methodology in two parts:

- Reformulation of the constraints using limit theorems ;

- Resolution of the reformulated problem with a new adaptive kriging strategy AK-ECO .

Application of the methodology to the harmonic oscillator problem.
Not presented work:

- Quantification of the approximation errors made during the reformulation steps;

- Global enrichment of the metamodels before AK-ECO;

- RBDO-oriented GSA;

- Application of the methodology to the FOWT problem [5].

[5] A.Cousin, J. Garnier, M. Guiton, M. Munoz Zuniga. Optimization with probabilistic constraints of complex systems. Application to the design of an offshore wind turbine. ECCOMAS, 2021

References

M.R. Leadbetter, G. Lindgren and H. Rootzén.

Extremes and Related Properties of Random Sequences and Processes
Springer-Verlag, 1983.

B. Echard, N. Gayton and M. Lemaire.

AK-MCS: An active learning reliability method combining Kriging and Monte Carlo Simulation
Structural Safety, 33(2): 145-154, 2011.

V. Dubourg.

Adaptive surrogate models for reliability analysis and reliability-based design optimization
Phd Thesis, 2011.

M. Moustapha and B. Sudret.

Surrogate-assisted reliability-based design optimization: a survey and a new general framework
Structural and Multidisciplinary Optimization, 60(5): 2157-2176, 2019.

Table of contents

6 Appendix

General formulation

$\min_{d \in \Omega_d} \text{cost}(d)$ such that

$$P_{X_d, X_p, X_{r_I}, Y} \min_{r \in \mathcal{R}_s} \int_0^T \gamma_p(X_d, X_p; t) \mathbb{1}_{\{r \in X_{r_I}\}} \rho_s$$

$$P_{X_d, X_p, X_{r_{II}}, Y} \int_0^T f_p \gamma_p(X_d, X_p; t) dt \mathbb{1}_{\{r \in X_{r_{II}}\}} \rho_s.$$

Graphs of the PW stationary oscillator problem

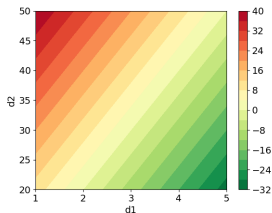


Figure: Level sets of the cost function

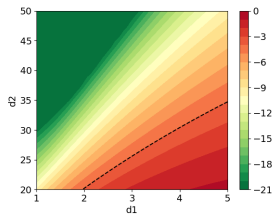


Figure: Level sets of the log of p_1

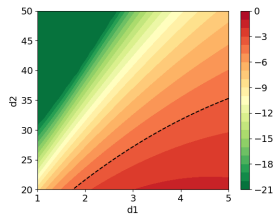


Figure: Level sets of the log of p_2

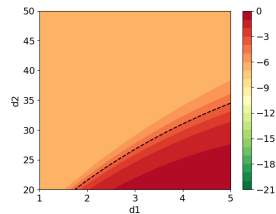


Figure: Level sets of the log of p_3

Reformulated constraints: probability version

Reformulated failure probabilities as expectations:

$$E_{X_d, X_p, X_{r_1}} F \prod_{j=1}^{n_s} e^{M_{1j} p_{X_d, X_p, X_{r_1}, s^j} q} = \rho_s \quad (1)$$

$$E_{X_d, X_p} F_{r_{11}} \prod_{j=1}^{n_s} T p^j M_{11} X_d, X_p, s^j = \rho_s \quad (2)$$

with $F p x q = 1 - \exp(-xq)$.

Reformulated failure probabilities as probabilities:

$$P_{X_d, X_p, X_{r_1}, X} X \prod_{j=1}^{n_s} \exp(-M_{1j} X_d, X_p, X_{r_1}, s^j) = 0 \quad \rho_s, \quad (3)$$

$$P_{X_d, X_p, X_{r_{11}}} X_{r_{11}} \prod_{j=1}^{n_s} T p^j M_{11} X_d, X_p, s^j = 0 \quad \rho_s \quad (4)$$

with X r.v with exponential distribution of parameter 1.

AK-ECO Step 1.a: accuracy criterion

$$p_{I,1}^{k-1} d^{k-1} = \frac{1}{N_{MC}} \prod_{i=1}^{N_{MC}} F_{j=1}^7 \exp \left(\mu_{I,1}^{k-1} x_{d,i}^j, x_{p,i}^j, x_{r_1,i}^j, s^j \right) \quad (5)$$

$$p_{I,1}^{k-1} p_{dq} = \frac{1}{n_{MC}} \prod_{i=1}^{n_{MC}} F_{j=1}^{n_s} \exp \left(\mu_{I,1}^{k-1} x_{d,i}^j, x_{p,i}^j, x_{r_1,i}^j, s^j \right) \quad (6)$$

$$p_{I,1}^{k-1} p_{dq} = \frac{1}{n_{MC}} \prod_{i=1}^{n_{MC}} F_{j=1}^{n_s} \exp \left(\mu_{I,1}^{k-1} x_{d,i}^j, x_{p,i}^j, x_{r_1,i}^j, s^j \right) \quad (7)$$

Accuracy criteria for the first constraint:

$$\frac{|p_{I,1}^{k-1} p_{dq}^{k-1} q - p_s|}{p_{I,1}^{k-1} p_{dq}^{k-1} q} \leq 1.$$

AK-ECO Step 1.b: enrichment criterion

$$p_1^k = d^k = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} F_{j=1}^7 e^{\mu^k \cdot p_{d^k-1, X_p^i, X_{r_1}^i, S^j}^q}$$

$$\max_{i \in \{1, \dots, N_{MC}\}, j \in \{1, \dots, 7\}} C_{aug} p_{d^k-1, X_p^i, X_{r_1}^i, S^j}^q \quad \text{with} \quad C_{aug} p_{X_d^i, X_p^i, X_{r_1}^i, S^j}^q$$

$$F_{j=1}^7 \approx e^{\mu^k \cdot p_{X_d^i, X_p^i, X_{r_1}^i, S^j}^q} = e^{\mu^k \cdot p_{X_d^i, X_p^i, X_{r_1}^i, S^j}^q} \cdot 2^{-k} \cdot p_{X_d^i, X_p^i, X_{r_1}^i, S^j}^q$$

$$F_{j=1}^7 \approx e^{\mu^k \cdot p_{X_d^i, X_p^i, X_{r_1}^i, S^j}^q} = e^{\mu^k \cdot p_{X_d^i, X_p^i, X_{r_1}^i, S^j}^q} \cdot 2^{-k} \cdot p_{X_d^i, X_p^i, X_{r_1}^i, S^j}^q$$

$$f_{X_d, X_p, X_{r_1}} p_{X_d^i, X_p^i, X_{r_1}^i}^q$$

$F(p, q) = \int \exp(p \cdot x) q$ and $f_{X_d, X_p, X_{r_1}}$ the pdf of $p_{X_d, X_p, X_{r_1}}^q$