Lipschitz-Killing curvatures of excursion sets for two dimensional random fields

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Joint works with...

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What is the question?

- Let $X : \mathbb{R}^2 \mapsto \mathbb{R}$ be a **stationary isotropic random field**

For example:

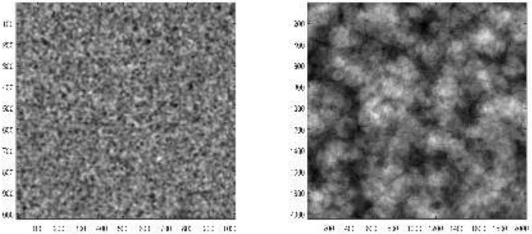

Figure: Gaussian field with covariance function $e^{-\kappa^2 \|x\|^2}$, $\kappa = 100/2^{10}$ (left), Shot noise field with random disks of radius $R = 50$ or 100 (with $\mathbb{P} = 1/2$) (right).

What is the question?

- $X : \boldsymbol{R}^2 \mapsto \boldsymbol{R}$ is a **stationary isotropic random field**
- X is observed on a **rectangle** T trough its *excursion sets at level $u \in \boldsymbol{R}$*

$$E_X(u) := X^{-1}([u, \infty)) = \{t \in \boldsymbol{R}^2,\ X(t) \geq u\}$$

we observe: $\boxed{T \cap E_X(u_0)}$ for a fixed level u_0: *sparse information*.

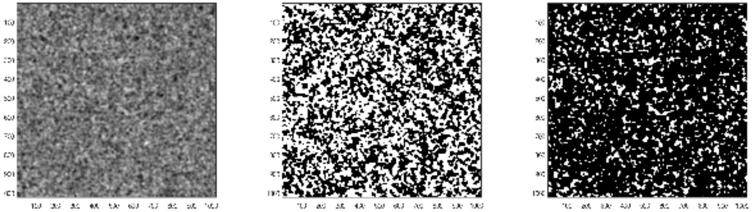

Figure: Gaussian field with covariance function $e^{-\kappa^2 |x|^2}$, $\kappa = 100/2^{10}$ (left) and two excursion sets for $u = 0$ (center) and $u = 1$ (right).

What is the question?

- $X : \mathbb{R}^2 \mapsto \mathbb{R}$ is a **stationary isotropic random field**
- X is observed on a **rectangle** T trough its *excursion sets at level $u \in \mathbb{R}$*

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Problems

1. **Inference problem:** is it possible to recover parameters of X?
2. **Testing:** Is X Gaussian or not? Is X symmetric or not?

Tool: Geometry of the excursion sets $T \cap E_X(u)$.

Contents

Lipschitz-Killing (LK) curvatures for excursion sets

If $d = 2$, for a "**nice**" Borel set A one can define 3 LK curvatures;

- Euler characteristic ($\sharp$(connected component) - $\sharp$(holes)) of A, related to the *connectivity*,
- Perimeter of A, related to the *regularity*,
- Area of A, related to the *occupation density*.

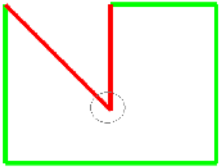
Applications: Cosmology, 2D x-ray images (detection of osteoporosis, mammograms),...

Lipschitz-Killing curvatures for excursion sets

Question: How to **properly** define these quantities for $T \cap E_X(u)$?

Tool: Curvature measures for Positive Reach (PR) sets[1]

Intuitively, "*A is a **positive reach set** if one can roll a ball of positive radius along the exterior boundary of A keeping in touch with A.*"



[1] Federer H., *Curvature measures*, Trans. Amer. Math. Soc. 93 (1959), 418-491.

Lipschitz-Killing curvatures for excursion sets

Definition of Curvature measures

Let A be a positive reach set. Define for any Borel set $U \subset \mathbb{R}^2$

$$\underbrace{\Phi_0(A,U)}_{\text{Euler characteristic}} = \frac{\mathrm{TC}(\partial A, U)}{2\pi}, \quad \underbrace{\Phi_1(A,U)}_{\frac{1}{2}\text{Perimeter}} = \frac{|\partial A \cap U|_1}{2} \text{ and } \underbrace{\Phi_2(A,U)}_{\text{Area}} = |A \cap U|,$$

where

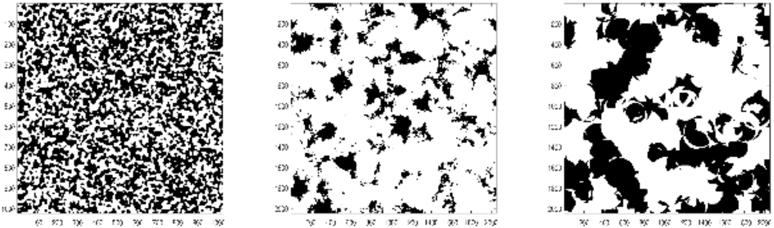
- $\mathrm{TC}(\partial A, U)$ is the integral over U of the curvature along the positively oriented curve ∂A
- $|\cdot|_1$ the 1-dim Hausdorff measure; $|\cdot|$ the 2-dim Lebesgue measure.

Remark: The measures $\Phi_i(A,\cdot)$ are additive : locally finite Union of sets with Positive Reach (UPR)

$A = T \cap E_X(u)$ is in the UPR class *a.s.* if, *e.g.*

- X is of class C^2 *a.s.*
- $E_X(u)$ is locally given by a finite union of disks.

Lipschitz-Killing curvatures for excursion sets



✓ Student random field; $E_X(u) \in$ UPR *a.s.*

✓ Shot noise field, $B = 1$ *a.s.* $E_X(u) \notin$ PR *a.s.* but $E_X(u) \in$ UPR *a.s.*

✗ Shot noise field, $B \pm 1$ *a.s.* $E_X(u) \notin$ PR *a.s.* and $E_X(u) \notin$ UPR *a.s.* (see Biermé et Desolgneux, 2016).

Matlab functions

`bwarea`, `bwperim` and `bweuler`

Lipschitz-Killing curvatures for excursion sets

Let X be a stationary isotropic random field defined on $\mathbb{R}^2$ and let T be a bounded rectangle in $\mathbb{R}^2$ with non empty interior.

Quantities of interest: *If $T \cap E_X(u)$ is a UPR set, define, for $i \in \{0, 1, 2\}$,*

Normalized LK curvatures

$$C_i^{/T}(X,u) := \frac{\Phi_i(T \cap E_X(u), T)}{|T|}$$ *(empirically accessible)*

Assuming the limits exist,

LK densities

$$C_i^*(X,u) := \lim_{T \nearrow \mathbb{R}^2} \mathbb{E}[C_i^{/T}(X,u)]$$ *(involves parameters of the field).*

Question: How can we compute $C_i^*(X,u) := \lim_{T \nearrow \mathbb{R}^2} \mathbb{E}[C_i^{/T}(X,u)]$?

Gaussian Kinematic formula

Question: How can we compute $C_i^*(X,u) := \lim_{T \nearrow R^2} \mathbb{E}[C_i^{/T}(X,u)]$?

- **Area**: $C_2^*(X,u) = \mathbb{E}[C_2^{/T}(X,u)] = \mathbb{P}(X(0) \geq u)$.
- **Gaussian type fields**: $X = F(\mathrm{G})$ where $\mathbb{V}(G_i'(0)) = \lambda I_2$, $\lambda > 0$.

Gaussian Kinematic formula

$$\mathbb{P}(\mathrm{G}(0) \in Tube(F,\rho)) = C_2^*(X,u) + \rho\frac{2\sqrt{2}}{\sqrt{\lambda\pi}}C_1^*(X,u) + \rho^2\frac{\pi}{\lambda}C_0^*(X,u) + O(\rho^3),$$

where

$$Tube(F,\rho) := \{x \in \boldsymbol{R}^k \text{ such that } \operatorname{dist}(x, F^{-1}([u,\infty))) \leq \rho\},$$

as $\rho \to 0^+$.

1. Lipschitz-Killing curvatures for excursion sets (LK)
2. Inference using LK curvatures
3. Removing assumption "the field is standard"
4. Test to compare two images of excursion sets
5. LK curvatures for perturbed model

Two bias problems

1. Numerical challenge of the control of the bias in the limit of an infinitely fine resolution, *i.e.* **pixelization error**.

2. Statistical challenge of the control of the bias due to the intersection of the excursion set with an **observation window**.

Two bias problems

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Numerical challenge of pixelization error in square tiling

Diffculties of estimating the perimeter of the smooth level set from a pixelated image.

Let

$$Z_{i,j}^{(m)}(u) := \mathbb{1}_{\{X_{i,j} \geq u\}}, \text{ for } i,j \in \{1, \ldots, m\},$$

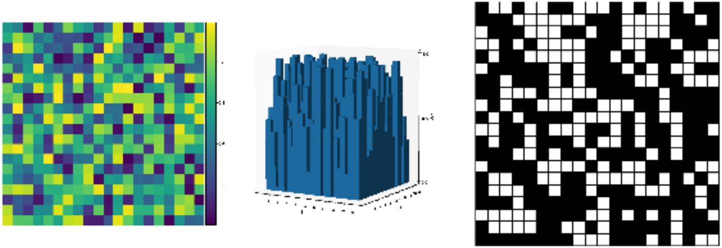

Figure: Left and center panels: Image of size (20×20) realization of a Uniform white noise model. Right panel: Obtained binary image for $u = 0.5$.

Pixelization error in a square tiling

We consider, for each edge w, the maximal and minimal values on the two sides of w. Then, the perimeter is given by

$$\mathcal{P}_m(u) := \sum_{w \in \text{ set of edges}} \left(f_-^{(u)}(w) - f_-^{(u)}(w)\right), \quad \text{where}$$

$$f_+^{(u)}(w) = \max(Z_{l,k-1}(u), Z_{l,k}(u)) \text{ and } f^{(u)}(w) = \min(Z_{l,k-1}(u), Z_{l,k}(u))$$

for w the common edge between cells (see Biermé et Desolgneux, 2021).

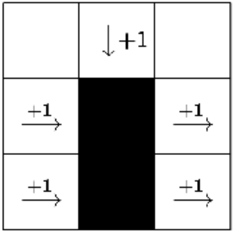

Figure: Computation of the perimeter of a binary image with $m = 3$. Here $\mathcal{P}_3 = 5$.

Numerical challenge of pixelization error in square tiling

Numerical study of the **dimensional constant** for the bias in the hyper-cubic tiling

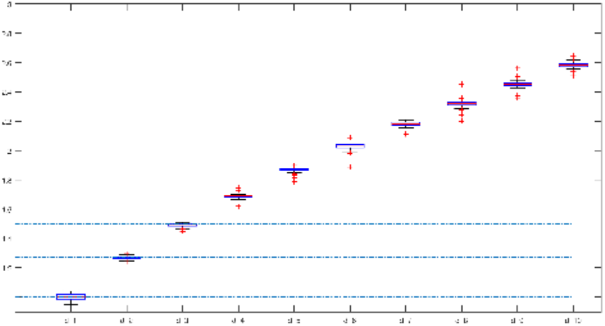

Figure: Centered and unit variance Normal field with $r(x) = e^{-\kappa^2 \|x\|^2}$, for $\kappa = 100/2^{10}$ for $d = 1$ to $d = 10$. We display the boxplots on $M = 10000$ samples of the **ratio between estimated perimeter and theoretical one** for 50 different values of the threshold u. Theoretical known dimensional constants $c_1 = 1$, $c_2 = \frac{4}{\pi}$ and $c_3 = \frac{3}{2}$ are displayed in horizontal dashed blue lines.

Kinematic formula and inference

Observations: we observe $T \cap E_X(u)$ for T a rectangle in $\mathbb{R}^2$.

Naive approach: $\widehat{C_i^*(X,u)} := C_i^{/T}(X,u) = \frac{\Phi_i(T \cap E_X(u), T)}{|T|}$

$$\Phi_i(T \cap E_X(u), T) = \underbrace{\Phi_i(E_X(u), \mathring{T})}_{\text{contains all information on } C_i^*} + \underbrace{\Phi_i(T \cap E_X(u), \partial T)}_{\text{Observation bias}}.$$

Proposition (Biermé, DB, Duval, Estrade, 2019)

If X is a centered, unit variance, stationary, isotropic such that either

- *X is a "smooth Gaussian type field"*
- *or X is a "shot noise field with bounded disks",*

then

$$\mathbb{E}[C_0^{/T}(X,u)] = C_0^*(X,u) + \frac{1}{\pi} C_1^*(X,u) \frac{|\partial T|_1}{|T|} + C_2^*(X,u) \frac{1}{|T|},$$

$$\mathbb{E}[C_1^{/T}(X,u)] = C_1^*(X,u) + \frac{1}{2} C_2^*(X,u) \frac{|\partial T|_1}{|T|},$$

$$\mathbb{E}[C_2^{/T}(X,u)] = C_2^*(X,u).$$

Kinematic formula and inference

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Kinematic formula and inference

Proposition (Biermé, DB, Duval Estrade, 2019)

The following quantities are unbiased estimators of $C^*_{0,1,2}(X, u)$,

$$\widehat{C}_{0,T}(X,u) = C_0^{/T}(X,u) - \frac{|\partial T|_1}{\pi|T|} C_1^{/T}(X,u) + \left(\frac{1}{2\pi}\left(\frac{|\partial T|_1}{|T|}\right)^2 - \frac{1}{|T|}\right) C_2^{/T}(X,u),$$

$$\widehat{C}_{1,T}(X,u) = C_1^{/T}(X,u) - \frac{|\partial T|_1}{2|T|} C_2^{/T}(X,u),$$

$$\widehat{C}_{2,T}(X,u) = C_2^{/T}(X,u) \quad \textit{(no edge correction)}.$$

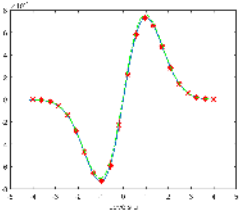
- As $T \nearrow \mathbb{R}^2$, CLT for $C_i^{/T}(X,u)$ known in particular cases (Gaussian, chi-square for $d=1$), asymptotic variances not (always) explicit (see, *e.g.*, DB, Estrade, León 2017 here). (Case $i = 2$ AREA - see below.)
- Difficult to get "general CLT results"
- Optimal choice of u ?

Inference: Gaussian field X

Let $\lambda > 0$ be the second spectral moment of X.

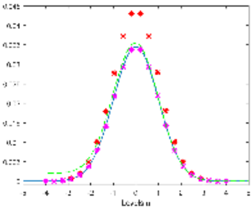
$u \mapsto$ Euler characteristic

$(2\pi)^{-3/2}\,\lambda\, u\, e^{-u^2/2}$



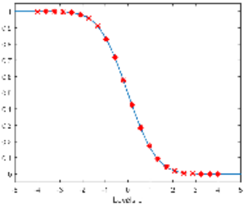
$u \mapsto \frac{1}{2}$perimeter

$\frac{1}{4}\,\lambda^{1/2}\, e^{-u^2/2}$



$u \mapsto$ area

$\int_u^{\infty} \frac{e^{-v^2/2}}{\sqrt{2\pi}}\, dv$



Theoretical $u \mapsto C_0^*(X,u)$, $C_1^*(X,u)$ and $C_2^*(X,u)$.

Unbiased (observation window) $\widehat{C}_{0,T}(X,u)$, $\widehat{C}_{1,T}(X,u)$ and $\widehat{C}_{2,T}(X,u)$.

Unbiased (pixelization error) $\frac{\pi}{4}\,\widehat{C}_{1,T}(X,u)$.

Naive estimates: $u \mapsto C_0^{I,T}(X,u)$ and $C_1^{I,T}(X,u)$.

Inference for parameters of chi-squared and t random fields : see here

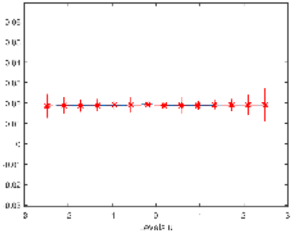
Inference: Gaussian field X

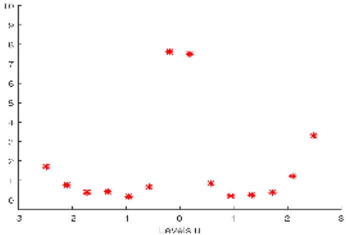
Estimation of λ: based on $\hat{C}_{0,T}(X,u)$, u fixed (`bweuler`).

Proposition (Biermé, DB, Duval Estrade, 2019)

Let $\hat{\lambda}_T(u) := \frac{(2\pi)^{3/2} e^{u^2/2}}{u} \widehat{C}_{0,T}(X,u)$. *Then,*

$$\sqrt{|T|}\left(\widehat{\lambda}_T(u) - \lambda\right) \xrightarrow[T \nearrow R^2]{d} \mathcal{N}(0, \Sigma(u)), \quad \textit{for some} \quad \Sigma(u) < +\infty,$$





Gaussian field with covariance function $e^{-\kappa^2 \|x\|^2}$.
LEFT Estimate $\hat{\lambda}_T(u)$ with associated confidence intervals for $M = 100$ sample simulations
RIGHT empirically estimated variance $\widehat{\Sigma}(u)$

Inference: a shot-noise field

Considered shot-noise model

$$S_\Phi(t) = \sum_{(x_i, b_i, r_i) \in \Phi} b_i \mathbf{1}_{r_i D}(t - x_i) \, , \quad \text{for } t \in \boldsymbol{R}^2,$$

where Φ is a stationary Poisson point process on $\boldsymbol{R}^2 \times \boldsymbol{R}^* \times \boldsymbol{R}^-$ with intensity measure $\nu \, Leb_{R^2} \otimes dF_B \otimes dF_R$.

Example:

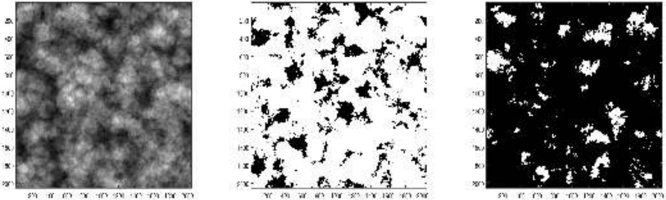
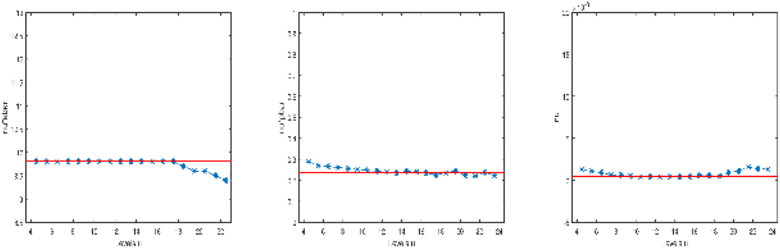


Figure: Shot-noise field with $B = 1$, *a.s.*, $\nu = 5 \times 10^{-4}$ with random disks of radius $R = 50$ or $R = 100$ (each with probability 0.5) (left) and two excursion sets for $u = 7.5$ (center) and $u = 14.5$ (right).

Inference: a shot-noise field

- Euler characteristic $= \nu\, e^{-\nu\bar{a}}\, \frac{(\nu\bar{a})^{\lfloor u \rfloor}}{\lfloor u \rfloor!} \left(1 - \nu \frac{\bar{p}^2}{4\pi} + \lfloor u \rfloor \frac{\bar{p}^2}{4\pi\bar{a}}\right)$
- $\frac{1}{2}$ perimeter $= \frac{1}{2} e^{\;\nu\bar{a}}\, \frac{(\nu\bar{a})^{\lfloor u \rfloor}}{\lfloor u \rfloor!}\, \nu\bar{p}$
- Area $= e^{\;\nu\bar{a}} \sum_{k>u} \frac{(\nu\bar{a})^k}{k!}$

where $\overline{p} = 2\pi\, \mathbb{E}[R]$ and $\overline{a} = \pi\, \mathbb{E}[R^2]$.



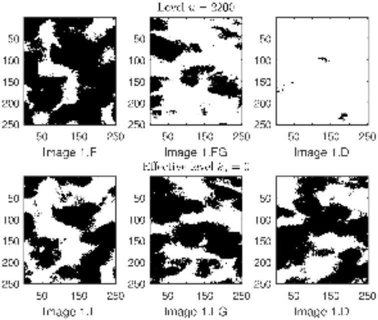
Shot-noise field with $B = 1$, a.s., with random disks of radius $R = 50$ or $R = 100$;
Left From $\hat{C}_{2,T} \Rightarrow \widehat{\nu\, \bar{a}}(u)$; **Center** From $\hat{C}_{1,T} \Rightarrow \widehat{\nu\, \bar{p}}(u)$; **Right** From $\hat{C}_{0,T} \Rightarrow \hat{\nu}(u)$

Removing assumption "the field is standard"

- The mean/variance (μ, σ^2) of X provide information on the LK curvatures

  $\hookrightarrow$ μ: black and white zones in comparable proportions
  $\hookrightarrow$ σ^2: range of levels of non-degenerate excursion sets

- From $E_X(u)$: impossible to estimate (μ, σ^2) (*sparse information*)
- Image comparison: what if the underlying fields have distant (μ, σ^2)?



Gaussian case: *effective level* and *spectral moment*

($\mathcal{A}0$) X is Gaussian stationary, isotropic, $\mathbb{E}[X(0)] = \mu$, $\mathbb{V}(X(0)) = \sigma^2 > 0$ and $\mathbb{V}(X'(0)) = \lambda I_2$ for $\lambda > 0$ (second spectral moment).
$t \mapsto X(t)$ are almost surely of class C^3.

Definition

Define the effective (observation) level:

$$s_u := \frac{u - \mu}{\sigma}$$

and the effective spectral moment:

$$a := \frac{\lambda}{\sigma^2}.$$

Notice that if $(\mu, \sigma^2) = (0, 1)$, then $(s_u, a) = (u, \lambda)$.

Gaussian case: *effective level* and *spectral moment*

Proposition (DB, Duval 2020)

Under $(\mathcal{A}0)$, *denote* $\psi(x) = \mathbb{P}(\mathcal{N}(0,1) \geq x)$, *it holds*

$$\mathbb{E}[C_0^{/T}(X,u)] = \frac{\psi(s_u)}{|T|} + \frac{\sqrt{a}}{2\pi} e^{-\frac{1}{2}s_u^2} \frac{|\partial T|_1}{2|T|} + \frac{a}{(2\pi)^{3/2}} e^{-\frac{1}{2}s_u^2} s_u,$$

$$\mathbb{E}[C_1^{/T}(X,u)] = \psi(s_u) \frac{|\partial T|_1}{2|T|} + \frac{\sqrt{a}}{4} e^{-\frac{1}{2}s_u^2}$$

$$\mathbb{E}[C_2^{/T}(X,u)] = \psi(s_u).$$

$\Rightarrow$ We get asymptotically normal estimators of s_u and a

$$\widehat{s}_{u,T} := \psi^{-1}(\widehat{C}_{2,T}(X,u)) \quad \widehat{a}_{u,T} := \frac{\widehat{C}_{0,T_1}(X,u)(2\pi)^{3/2}}{\widehat{s}_{u,T_2} \exp\{-\frac{1}{2}(\widehat{s}_{u,T_2})^2\}},$$

where $(|T_1| = |T_2|,\ T_1 \cup T_2 \subset T,\ T_1 \cap T_2 = \emptyset)$ with unknown limit variance.

Effective level estimation procedure for the limit variance

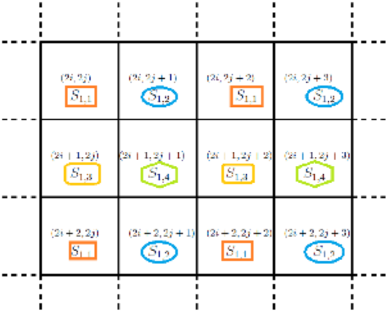
- Boils down to estimate the limite variance of $\widehat{C}_{2,T}(X,u)$
- Sub-windows estimation procedure: consistent under strong assumptions (see Plante et al. (2010), Bulinski et al. (2012))

Decompose T in M_N^2 distinct sub-rectangles $V^{(N,(i,j))}$, $1 \leq i,j \leq M_N$,

$$\widehat{\Sigma}^2_{\widehat{C}_2,(u,v)} = \frac{1}{M_N^2-1}\sum_{i,j=1}^{M_N} \xi_N^{(i,j)}(u)\xi_N^{(i,j)}(v)$$

$$-\left(\frac{1}{M_N^2-1}\sum_{i,j=1}^{M_N} \xi_N^{(i,j)}(u)\right)\left(\frac{1}{M_N^2-1}\sum_{i,j=1}^{M_N} \xi_N^{(i,j)}(v)\right)$$

where $\xi_N^{(i,j)}(u) := \widehat{C}_2^{/V^{(N,(i,j))}}(X,u)$.



Effective level estimation procedure for the limit variance

($\mathcal{A}1$) Correlation function $t \mapsto \rho(t)$ is decreasing and $|\rho(t)| \leq (1 + \|t\|)^{-\gamma}$, $\gamma > 2$.

Theorem (DB, Duval, 2020)

Let X a Gaussian random field satisfying ($\mathcal{A}0$) and ($\mathcal{A}1$). Then, it holds that

$$\widehat{\Sigma}^2_{C_2^*,(u,v)} \xrightarrow[N\to\infty]{\mathbb{P}} \Sigma^2_{C_2^*,(u,v)},\ \forall\,(u,v).$$

Sketch of proof:

i) Show that $\mathbb{V}(\widehat{\Sigma}^2_{C_2^*,(u,v)}) \to 0$ as $T \uparrow \boldsymbol{R}^2$,

ii) Key points 1) estimators are identically distributed,

2) $\mathrm{dist}(V^{(N,(i,j))}, V^{(N,(i',j'))}) \to \infty$, as $N \to \infty$.

which permit to establish the desired result.

iii) Using the following result:

Effective level estimation procedure for the limit variance

Proposition (DB, Duval, 2020)

Let X a Gaussian random field satisfying ($\mathcal{A}0$) and ($\mathcal{A}1$).
Set

$$G = (X - \mu)/\sigma.$$

Let T and T' be such that $|T| = |T'|$ and $\mathrm{dist}(T, T') \to \infty$.

Then, it holds that,

$$\begin{aligned}\mathbb{E}[\mathcal{L}_2(G, s_{u_1}, T)\mathcal{L}_2(G, s_{u_2}, T)\mathcal{L}_2(G, s_{u_3}, T')\mathcal{L}_2(G, s_{u_4}, T')] \\ = \psi(s_{u_1})\psi(s_{u_2})\psi(s_{u_3})\psi(s_{u_4})|T|^4 + o(|T|^3),\end{aligned}$$

where

$$\mathcal{L}_2(G, s_u, T) := |T|\,\widehat{C}_2^{|T}(X, u) \quad \text{and} \quad \psi(s_u) = C_2^*(X, u).$$

Effective level estimation procedure for the limit variance

Elements to prove this auxiliary result

1 **Itô-Wiener chaos decomposition for $\mathcal{L}_2$** (Nourdin and Peccati, 2012)

$$\mathcal{L}_2(G, s_u, T) = \sum_{q=0}^{+\infty} \frac{\beta_q(s_u)}{q!} \int_T H_q(G(t))\, dt,$$

where H_q is the q-th Hermite polynomial, the chaotic coefficients: $\beta_0(s_u) = \psi(s_u)$ and $\beta_q(s_u) = \varphi(s_u) H_{q-1}(s_u)$, such that

$$\|\beta_0\|_\infty \leq 1 \quad \text{and} \quad \|\beta_q\|_\infty \leq c_\beta \frac{\sqrt{(q-1)!}}{q^{\frac{1}{12}}}, \quad q \geq 1,$$

(see, e.g., Szegö (1959)).

2 **Diagram formula** (Taqqu (1977)) to compute/control

$$\mathbb{E}[H_{k_1}(G(t_1))H_{k_2}(G(t_2))H_{k_3}(G(t_3))H_{k_4}(G(t_4))]$$

Key point: $\mathrm{dist}(T, T') \to \infty$.

Effective level estimation procedure for the limit variance

Elements to prove this auxiliary result

1 **Itô-Wiener chaos decomposition for $\mathcal{L}_2$** (Nourdin and Peccati, 2012)

$$\mathcal{L}_2(G, s_u, T) = \sum_{q=0}^{+\infty} \frac{\beta_q(s_u)}{q!} \int_T H_q(G(t))\, dt,$$

where H_q is the q-th Hermite polynomial, the chaotic coefficients: $\beta_0(s_u) = \psi(s_u)$ and $\beta_q(s_u) = \varphi(s_u) H_{q-1}(s_u)$, such that

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2 **Diagram formula** (Taqqu (1977)) to compute/control

$$\mathbb{E}[H_{k_1}(G(t_1)) H_{k_2}(G(t_2)) H_{k_3}(G(t_3)) H_{k_4}(G(t_4))]$$

Key point: $\text{dist}(T, T') \to \infty$.

Comments

- To proof is lengthly and relies on technical computations.
- Conjecture: Should remain valid to get the consistency for the empirical variance of the *Euler characteristic*.

Possible uses

- Test " H_0 : X is Gaussian field." (strength: does not rely on the estimation of the covariance function) details here
- Asymptotic interval for $\mu := \mathbb{E}[X(0)]$
- Test to compare two images of excursion sets

1. Lipschitz-Killing curvatures for excursion sets (LK)
2. Inference using LK curvatures
3. Removing assumption "the field is standard"
4. Test to compare two images of excursion sets
5. LK curvatures for perturbed model

Mammograms

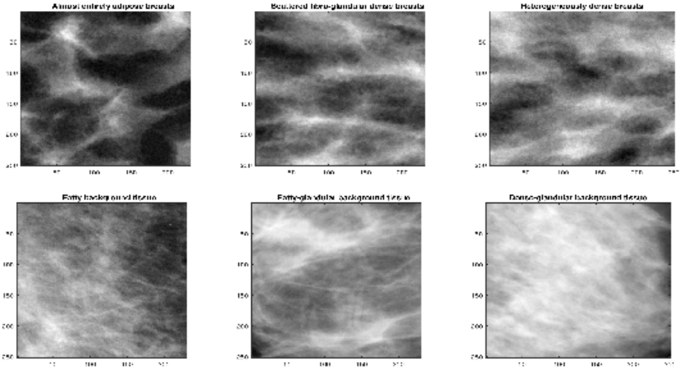

Figure: Synthetic (first row) and real digital (second row) mammograms studies. Group (F) (left), Group (FG) (center) and Group (D) (right). Image size: 251×251. See details here.

(F) Almost entirely adipose breasts;

(FG) Scattered fibro-glandular dense breasts;

(D) Heterogeneously dense breasts.

Can we compare the excursion sets of 2 images?

We observe $E_Y(u_Y)$ and $E_Z(u_Z)$, where Y and Z are Gaussian fields satisfying $(\mathcal{A}0)$ and $(\mathcal{A}1)$

with possibly different mean, variance, spectral moment or correlation function.

Is it allowed to compare their LK?

$$H_0 \,:\, s_{u_Y}(Y) = s_{u_Z}(Z) \qquad H_1 \,:\, s_{u_Y}(Y) \neq s_{u_Z}(Z).$$

Let $q_{1-\frac{\alpha}{2}}$ such that $\mathbb{P}(|N(0,1)| \leq q_{1-\frac{\alpha}{2}}) = 1 - \frac{\alpha}{2}$. The test

$$\phi_{T(N)} = \mathbf{1}_{\left\{\sqrt{\frac{1}{\hat{\Sigma}_{Y,Z}}}\left|\hat{s}_{u_Y,T} - \hat{s}_{u_Z,T}\right| > q_{1-\frac{\alpha}{2}}\right\}}.$$

has asymptotic level α and is consistent.

Importance of the *effective level*

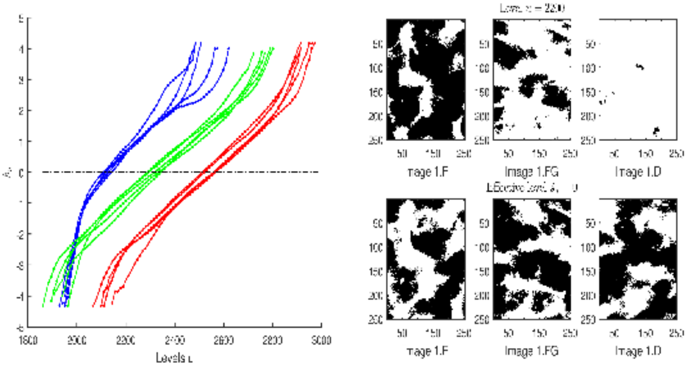

Figure: Synthetic digital mammograms. **Left:** Estimated $\widehat{s}_u$ (group (F) in blue, (FG) green, (D) red). **Right:** Excursion sets for a fixed level $u = 2200$ (first row) and for the three adaptive levels $\widetilde{u}$, such that $|\widehat{s}_{\widetilde{u}}| < 10^{-2}$ (second row).

Testing

We now test

$H_0 : s_{\widetilde{u}_Y}(Y) = s_{\widetilde{u}_Z}(Z)$ versus $H_1 : s_{\widetilde{u}_Y}(Y) \neq s_{\widetilde{u}_Z}(Z)$,

for $Y, Z \in \{1.F, 1.FG, 1.D\}$, where $\widetilde{u}_Y$ and $\widetilde{u}_Z$ are the adaptive levels such that

$$|\widehat{s}_{\widetilde{u}}| < 10^{-2}, \textit{ i.e.}, \text{ the associated } \widehat{C}_{0,T}(\widetilde{u}) \approx 0.$$

1.F versus 1.FG	1.F versus 1.D	1.FG versus 1.D
0.9858	0.9511	0.9642

Table: $p-$values for the synthetic digital mammograms study.

Testing

We now consider 1000 different *not−adaptive* values of u in a grid $\mathcal{G}$;

$$\forall\, u \in \mathcal{G}, \quad H_0 \;:\; s_u(Y) = s_u(Z) \qquad \text{versus} \qquad H_1 \;:\; s_u(Y) \neq s_u(Z),$$

for Y, Z images of this synthetic mammograms data-set.

Intra-classes analysis **Inter-classes analysis** **Inter-classes analysis**

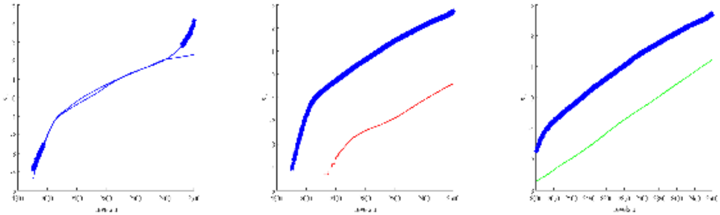

Figure: Synthetic digital mammograms study. Estimation of $\widehat{s}_u$ for 1000 different values of $u \in \mathcal{G}$ and couples of images: 2.F and 3.F (first panel); 1.F versus 5.D (second panel); 1.F and 3.FG (third panel). In bold marked points we represent the cases where the test rejects H_0 for a significant level $\alpha = 0.2$. Group (F) is displayed using blue curves, (FG) green curves and (D) red ones.

LK curvatures of $E(u)$ of perturbed Gaussian model

Definition (Perturbed Gaussian field)

Let X be a random variable such that $\mathbb{E}[X] = 0$ and $\mathbb{E}[|X|^3] < +\infty$.
Let g be a Gaussian random field defined on R^2 with C^3 trajectories.

We assume that g is

- stationary, isotropic with $\mathbb{E}[g(0)] = 0$, $\mathrm{Var}\, g(0) = \sigma_g^2$
- its covariance function $r(t) = Cov(g(0), g(t))$ satisfies

$$|r(t)| = O(\|t\|^{-\alpha}), \quad \text{for some } \alpha > 2 \quad \text{as} \quad \|t\| \to \infty.$$

with X independent of g.

Let $\epsilon > 0$. We consider the following perturbed field

$$f(t) = g(t) + \epsilon X, \; t \in R^2.$$

LK curvatures of $E(u)$ of perturbed Gaussian model

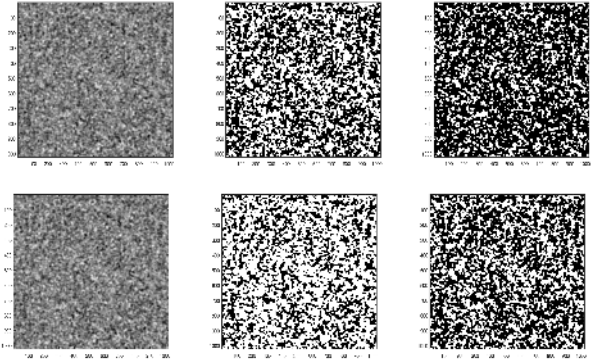

Figure: **Gaussian random field and its perturbed counter-part** with covariance $r(s) = \sigma_g^2 e^{-\kappa^2 \|s\|^2}$, for $\sigma_g = 2$, $\kappa = 100/2^{10}$ in a domain of size $2^{10} \times 2^{10}$ pixels, with $\epsilon = 1$ and $X \sim t(\nu = 5)$. **First row:** A realization of Gaussian random field g (left) and the two associated excursion sets for $u = 0$ (center) and $u = 1$ (right). **Second row**: The associated realization of a perturbed Gaussian random field f (left) and two excursion sets for $u = 0$ (center) and $u = 1$ (right).

LK curvatures of $E(u)$ of perturbed Gaussian model

Proposition (DB, Estrade, Rossi 2020)

Then, for small $\epsilon > 0$, *it holds that*

$$\mathbb{E}[C_0^{/T}(f,u)] = C_0^*(g,u)\left(1 + \frac{\epsilon^2 \mathbb{E}[X^2]}{2\sigma_g^2}\left(H_2\left(\frac{u}{\sigma_g}\right) - 2\right)\right)$$
$$+ \frac{1}{\pi} C_1^*(g,u)\left(1 + \frac{\epsilon^2\,\mathbb{E}[X^2]}{2\sigma_g^2} H_2\left(\frac{u}{\sigma_g}\right)\right)\frac{|\partial T|_1}{|T|}$$
$$+ \left(C_2^*(g,u) + \epsilon^2\,\mathbb{E}[X^2]\frac{\pi}{\lambda} C_0^*(g,u)\right)\frac{1}{|T|} + O\left(\epsilon^3\left(1 + \frac{|\partial T|_1}{2|T|} + \frac{1}{|T|}\right)\right),$$

$$\mathbb{E}[C_1^{/T}(f,u)] = C_1^*(g,u) + C_2^*(g,u)\frac{|\partial T|_1}{2|T|}$$
$$+ \epsilon^2\,\mathbb{E}[X^2]\left(\frac{C_1^*(g,u)}{2\sigma_g^2} H_2\left(\frac{u}{\sigma_g}\right) + C_0^*(g,u)\frac{\pi}{\lambda}\frac{|\partial T|_1}{2|T|}\right) + O\left(\epsilon^3\left(1 + \frac{|\partial T|_1}{2|T|}\right)\right),$$

$$\mathbb{E}[C_2^{/T}(f,u)] = C_2^*(g,u) + \epsilon^2\,\mathbb{E}[X^2]\frac{\pi}{\lambda} C_0^*(g,u) + O(\epsilon^3),$$

where $H_2(y) = y^2 - 1$, *for* $y \in R$ *and* λ *is the second spectral moment of* g.

LK curvatures of $E(u)$ of perturbed Gaussian model

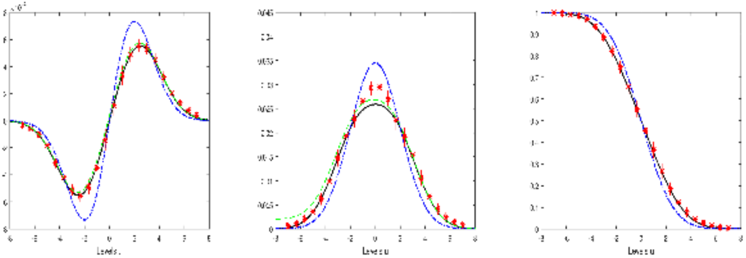

Figure: **Perturbed Gaussian random field** with covariance $r(s) = \sigma_g^2 e^{-\kappa^2 \|s\|^2}$, for $\sigma_g = 2$, $\kappa = 100/2^{10}$ in a domain of size $2^{10} \times 2^{10}$ pixels, with $X \sim t(\nu = 5)$ and $\epsilon = 1$.

Theoretical $u \mapsto C_j^*(f, u)$ are drawn in black plain lines.
Theoretical $u \mapsto C_j^*(g, u)$ in blue dashed lines.
Theoretical $u \mapsto C_0^{LT}(f, u)$ and $C_1^{LT}(f, u)$ in green dotted lines (left and center panels).
Averaged values on $M = 100$ sample simulations of $\widehat{C}_{j,T}(f, u)$ as a function of the level u by using red stars and empirical intervals.

Quantitative asymptotics for $C_2^{/T}(f,u)$

We are interested in the asymptotic distribution as $T \nearrow R^2$ of

$$
\begin{aligned}
Y_T^{\epsilon}(u) &= |T|^{1/2}\left(C_2^{/T}(f,u) - \mathbb{E}[C_2^{/T}(f,u)]\right)\\
&= |T|^{1/2}\left(C_2^{/T}(f,u) - \Psi\left(\frac{u-\epsilon X}{\sigma_g}\right)\right) + |T|^{1/2}\left(\Psi\left(\frac{u-\epsilon X}{\sigma_g}\right) - \mathbb{E}\left[\Psi\left(\frac{u-\epsilon X}{\sigma_g}\right)\right]\right)\\
&=: Z_T^{\epsilon}(u) + R_T^{\epsilon}(u).
\end{aligned}
$$

Theorem (DB, Estarde, Rossi 2020)

1. *For any fixed small $\epsilon > 0$ and $T \nearrow R^2$, it holds that*

$$d_W(Z_T^{\epsilon}(u), \Theta_{\epsilon}(u)) = O\left((\log|T|)^{-1/12}\right),$$

where the constant involved in the O-notation depends neither on ϵ nor on u and d_W is the Wasserstein distance between random variables.

2. *For $\epsilon \to 0$ and $T \nearrow R^2$. Let $T^{(N)} = NT$ and ϵ_N such that $\lim_{N\to\infty} N\epsilon_N = 0$. Then it holds that,*

$$d_W(Y_{T^{(N)}}^{\epsilon_N}(u), \mathcal{N}(0, v(u))) \underset{N\to\infty}{\longrightarrow} 0$$

What about the r.v. $\Theta_{\epsilon}(u)$ and the asymptotic variance $v(u)$?

What about the r.v. $\Theta_\epsilon(u)$ and the asymptotic variance $v(u)$?

$v(u)$ From the Gaussian case, we can get

$$v(u) = \frac{1}{2\pi} \int_{R^2} \int_0^{\rho(t)} \frac{1}{\sqrt{1-x^2}} \exp\left\{ -\frac{u^2}{\sigma_g^2(1+x)} \right\} \mathrm{d}x\, \mathrm{d}t$$

with $\rho(t) := corr(g(0), g(t)) = r(t)/\sigma_g^2$.

$\Theta_\epsilon(u)$ is a r.v.

- ↪ whose conditional distribution given $\{X = x\}$ is centered Gaussian with variance $v(u - \epsilon x)$.
- ↪ its probability density function h_ϵ can be expanded for small $\epsilon > 0$, as

$$h_\epsilon(y) = f_{\mathbf{BEP}}^{\delta=0}(y) + \frac{\epsilon^2\, \mathbb{E}[X^2]}{2} \left[\frac{3}{4} \frac{v'(u)^2}{v(u)^2} \left(f_{\mathbf{BEP}}^{\delta=0}(y) - 2 f_{\mathbf{BEP}}^{\delta=2}(y) + f_{\mathbf{BEP}}^{\delta=4}(y) \right) + \frac{1}{2} \frac{v''(u)}{v(u)} \left(- f_{\mathbf{BEP}}^{\delta=0}(y) + f_{\mathbf{BEP}}^{\delta=2}(y) \right) \right] + O(\epsilon^3),$$

where $f_{\mathbf{BEP}}^{\delta}(y)$ are *Bimodal Exponential Power* density functions.

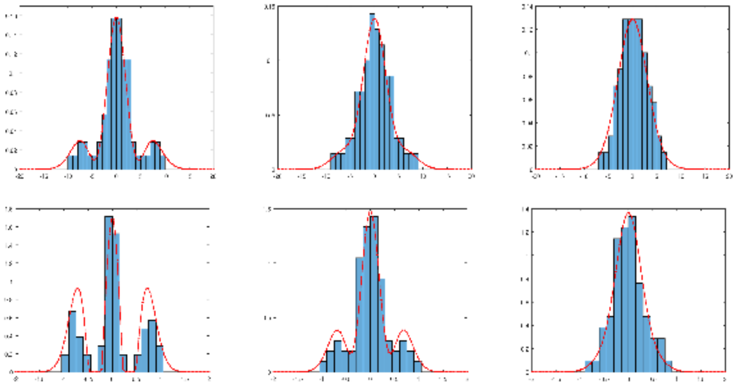

Figure: **Histogram for the study of density h_ϵ of Z_T^ϵ when X is t-distributed,** for $u = 1.5$ (first row) and $u = 3$ (second row), based on 300 Monte-Carlo independent simulations. In particular we chose $\epsilon = 0.5$ (first column), $\epsilon = 0.3$ (second column) and $\epsilon = 0.1$ (third column). Resulting theoretical h_ϵ density is drawn by using red plain line.

Conclusion and Discussion

- Literature and contribution

↪ We propose a unified framework and unbiased estimators

Limitation: Difficult to get (joint) “CLT” results

↪ We only observe $E_X(u)$

Testing and inference usually require the knowledge of X (estimation of the covariance function / of marginal distribution)

↪ Removed the assumption of centering and unit variance

- Perspectives:

↪ Control of variance of the pixelization error to pass Central limit Theorems (with C. Duval)

↪ LK for Gaussian mixtures with a view towards inference for spatial extremes (with A. Estrade and T. Opitz)

↪ Synthetic *morphological indicators* using these geometric features.

We presented some results from :

Abaach, Biermé, DB *Testing marginal symmetry of digital noise images through the perimeter of excursion sets*, Preprint, **2021**.

DB, Duval, *Statistics for Gaussian Random Fields with Unknown Location and Scale using Lipschitz-Killing Curvatures*, Scandinavian Journal of Statistics, **2020**.

DB, Estrade, Rossi *On the excursion area of perturbed Gaussian fields*, ESAIM: PS, **2020**.

Biermé, DB, Duval, Estrade, *Lipschitz-Killing curvatures of excursion sets for two dimensional random fields*, Electronic Journal of Statistics, **2019**.

DB, Estrade, León, *A test of Gaussianity based on the Euler characteristic of excursion sets*, Electronic Journal of Statistics, **2017**.

Thank you very much for your attention!

Central limit theorem for $C_0^{/T_i}(X, u_i)$, $d \geq 1$

Let T_1 and T_2 be two cubes in $\boldsymbol{R}^d$ s.t. $|T_1| = |T_2|$ and $dist(T_1, T_2) > 0$ and let u_1 and u_2 belong to $\boldsymbol{R}$ ($u_1 \neq u_2$ or $u_1 = u_2$).

Theorem (DB, Estrade & León, 2017)

Let

$$Z_i^{(N)} = |T_i^{(N)}|^{1/2}(C_0^{/T_i^{(N)}}(X, u_i) - \mathbb{E}[C_0^{/T_i^{(N)}}(X, u_i)]), \quad \textit{for } i = 1, 2.$$

Then, under the same hypothesis as above,

$$\left(Z_1^{(N)}, Z_2^{(N)}\right) \xrightarrow[N\to\infty]{distrib} \mathcal{N}\left(0, \begin{pmatrix} V(u_1) & 0 \\ 0 & V(u_2) \end{pmatrix}\right)$$

Note that $dist(T_1^{(N)}, T_2^{(N)}) \underset{N\to\infty}{\longrightarrow} \infty$.

Also a joint CLT holds for **a large domain** $T^{(N)}$ and **various levels** (see next slide).

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Central limit theorem for $C_0^{/T}(X, u_i)$, $d \geq 1$

Theorem (DB, Estrade & León, 2017)

Let T be a cube in $\mathbb{R}^d$ and let u_1 and u_2 belong to $\mathbb{R}$. For any integer $N > 0$, we introduce

$$\zeta_i^{(N)} = |T^{(N)}|^{1/2}(C_0^{/T^{(N)}}(X, u_i) - \mathbb{E}[C_0^{/T^{(N)}}(X, u_i)]), \quad \text{for } i = 1, 2.$$

Then

$$\left(\zeta_1^{(N)}, \zeta_2^{(N)}\right) \xrightarrow[N \to \infty]{distrib} \mathcal{N}\left(0, \begin{pmatrix} V(u_1) & V(u_1, u_2) \\ V(u_1, u_2) & V(u_2) \end{pmatrix}\right)$$

where $V(u_1, u_2)$ is given by

$$V(u_1, u_2) = \int_{\mathbb{R}^d} (G(u_1, u_2, t)\, D(t)^{-1/2} - C(u_1)C(u_2))\, dt + (2\pi\lambda)^{-d/2} g(\max(u_1, u_2))$$

with

$$G(u_1, u_2, t) = \mathbb{E}[1_{[u_1, \infty)}(X(0))\, 1_{[u_2, \infty)}(X(t))\, \det(X''(0))\, \det(X''(t)) \,|\, X'(0) = X'(t) = 0].$$

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Asymptotic variance of Euler characteristic, $d \geq 1$

In order to make $T \nearrow \mathbb{R}^d$, we introduce

$$T^{(N)} = \{Nt \,:\, t \in T\} \text{ with } T \text{ a fixed cube in } \mathbb{R}^d.$$

Theorem (DB, Estrade & León, 2017)

Let X be Gaussian, stationary, isotropic, of class $C^3(\mathbb{R}^d)$ and with "fast decay of the covariance",

$$\lim_{N\to+\infty} \mathrm{Var}\left(\frac{\Phi_0(T^{(N)} \cap E_X(u), T^{(N)})}{|T^{(N)}|^{1/2}}\right) = V(u) \in (0, +\infty)$$

with $V(u) = \int_{\mathbb{R}^d} (G(u,t)\, D(t) - C(u,\lambda)^2)\, dt + (2\pi\lambda)^{-d/2}\, g(u)$ *and*

$C(u) = (2\pi)^{-(d-1)/2}\, \lambda^{d/2}\, H_{d-1}(u)\, e^{-u^2/2}$,
$g(u) = \mathbb{E}[\mathbf{1}_{[u,\infty)}(X(0))\, |\det(X''(0))|]$,
$G(u,t) = \mathbb{E}[\mathbf{1}_{[u,\infty)^2}(X(0), X(t))\, \det(X''(0)\, X''(t)) \,/\, X'(0) = X'(t) = 0]$,
$D(t) = p_{X'(0),X'(t)}(0,0) = (2\pi)^{-d} \det(\lambda^2 I_d - r''(t)^2)^{-1/2}$.

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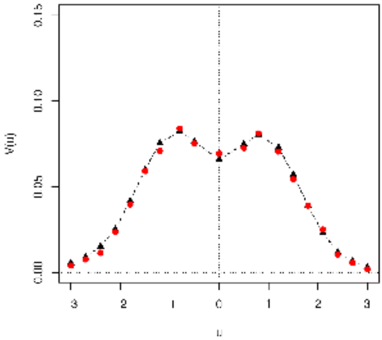
Asymptotic variance of Euler characteristic

In the case $d = 1$, we have an *explicit formula for* $V(u)$



black triangles:	numerical evaluation of $V(u)$
red dots:	empirical variance $\mathrm{Var}\left(\frac{\Phi_0(T \cap E_X(u), T)}{\lvert T \rvert^{1/2}}\right)$
	based on 300 Monte-Carlo simulations
	X Gaussian with $r(t) = e^{-t^2}$, $\lvert T \rvert = 200$

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Testing $H0$ $\{X$ is Gaussian$\}$

i Based on two levels $u_1 \neq u_2$ and using

$$R_{u_1,u_2} := \frac{C_0^*(X,u_2)}{C_0^*(X,u_1)} \overset{H0}{=} \frac{u_2}{u_1}\, e^{\frac{1}{2}(u_1^2-u_2^2)} : \qquad \text{KNOWN}.$$

ii Let $0 < u_1 < u_2$ and T_1 and T_2 two rectangles in $\mathbb{R}^2$, $\mathrm{dist}(T_1,T_2) > 0$ and $|T_1| = |T_2| > 0$, define $T_i^{(N)} = \{Nt : t \in T_i\}$, for $i = 1,2$.

Consider the test statistic

$$\hat{R}_{u_1,u_2,N} := \frac{\hat{C}_{0,T_2^{(N)}}(X,u_2)}{\hat{C}_{0,T_1^{(N)}}(X,u_1)}.$$

iii Then, under $H0$ it holds that

$$\sqrt{|T_1^{(N)}|}\left(\hat{R}_{u_1,u_2,N} - R_{u_1,u_2}\right) \xrightarrow[N\to\infty]{d} \mathcal{N}(0,\Sigma(u_1,u_2)),$$

where $\Sigma(u_1,u_2) < \infty$.

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Testing $H0$ $\{X \text{ is Gaussian}\}$

i Based on two levels $u_1 \neq u_2$ and using

$$R_{u_1,u_2} := \frac{C_0^*(X,u_2)}{C_0^*(X,u_1)} \stackrel{H0}{=} \frac{u_2}{u_1} e^{\frac{1}{2}(u_1^2-u_2^2)} : \qquad \text{KNOWN.}$$

ii Let $0 < u_1 < u_2$ and T_1 and T_2 two rectangles in $\mathbb{R}^2$, $\text{dist}(T_1,T_2) > 0$ and $|T_1| = |T_2| > 0$, define $T_i^{(N)} = \{Nt : t \in T_i\}$, for $i = 1, 2$.

Consider the test statistic

$$\hat{R}_{u_1,u_2,N} := \frac{\hat{C}_{0,T_2^{(N)}}(X,u_2)}{\hat{C}_{0,T_1^{(N)}}(X,u_1)}.$$

iii Then, under $H0$ it holds that

$$\sqrt{|T_1^{(N)}|}\left(\hat{R}_{u_1,u_2,N} - R_{u_1,u_2}\right) \xrightarrow[N\to\infty]{d} \mathcal{N}(0,\Sigma(u_1,u_2)),$$

where $\Sigma(u_1,u_2) < \infty$.

Build a test with asymptotic level α: $1_{\left\{\sqrt{\frac{1}{\hat{V}(\hat{R}_{u_1,u_2,N})}}(\hat{R}_{u_1,u_2,N}-R_{u_1,u_2})\geq q_{1-\alpha}\right\}}$

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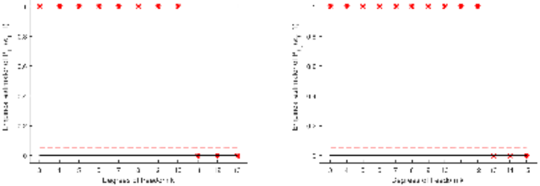
Testing $H0$ $\{X \text{ is Gaussian}\}$

$H1(k) : \exists\, k \geq 3,\ X$ is Student(k)

$$R_{u_1,u_2} \overset{H1}{=} \frac{u_2}{u_1}\left(1 - \frac{(u_2^2 - u_1^2)}{k - 2 + u_2^2}\right)^{\frac{k-1}{2}}.$$



Student random field with unit variance and different degrees of freedom.
$k \rightarrow$Power of the test with $u_1 = 1$ and $u_2 = 2$ (left) or $u_2 = 3$ (right).

For k too large or $u_2 \sim u_1$: the test fails, indeed

$$R_{u_1,u_2}(H1) = R_{u_1,u_2}(H0)\left(1 + O\left(\frac{1}{k}\left(\frac{u_2}{u_1} - 1\right)\right)\right).$$

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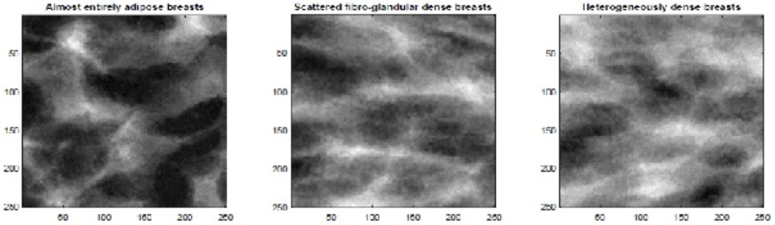
Testing $H0$ $\{X$ is Gaussian$\}$

- Consistency is hard to establish theoretically
- Different alternative: power of Gaussian field
- Real data example (2D digital mammograms)[§]



[§] Collaboration with Z. Li, GE Healthcare France, department *Mammography*.

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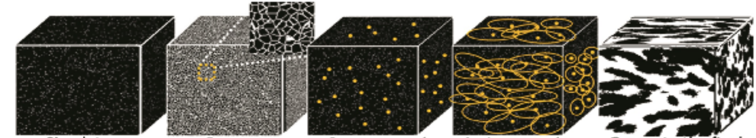
Real data example (2D digital mammograms)

We consider a recent 3D solid breast texture model inspired by the morphology of medium and small scale fibro-glandular and adipose tissue observed in clinical breast computed tomography (bCT) images[§].



We consider 15 simulated 2D digital images

(F) Almost entirely adipose breasts;

(FG) Scattered fibro-glandular dense breasts;

(D) Heterogeneously dense breasts.

[§]see Li, Desolneux, Muller and Carton (2016).

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Real data example (2D digital mammograms)

Group	Level	Image				
F	α	1.F	2.F	3.F	4.F	5.F
	0.2	84	76	**248**	**683**	**651**
	0.1	41	41	**136**	**613**	**565**
	0.05	27	8	**57**	**491**	**467**
FG	α	1.FG	2.FG	3.FG	4.FG	5.FG
	0.2	65	119	58	43	**900**
	0.1	19	71	28	12	**858**
	0.05	10	35	15	6	**797**
D	α	1.D	2.D	3.D	4.D	5.D
	0.2	**389**	**230**	**347**	**575**	**468**
	0.1	**267**	**164**	**210**	**411**	**312**
	0.05	**190**	**126**	**142**	**288**	**242**

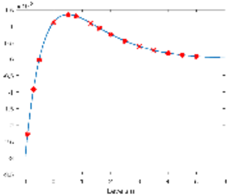
Table: Number of p−values associated to the 1000 different values of $u_2 \in [-3, 3]$ that are smaller than the significant α-levels.
In bold text the numbers larger than $\alpha \times 1000$ for which H0 is rejected.

Remark : $\nu_D < \nu_F < \nu_{FG}$

Inference: Chi square with k degrees of freedom

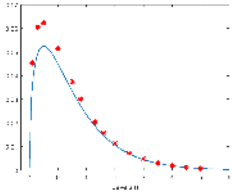
$u \mapsto$ Euler characteristic

$\frac{\lambda(k-u\sqrt{2k})^{(k-2)/2}}{\pi 2^{k/2}\Gamma(k/2)}(u\sqrt{2k}+1)\exp^{-\frac{k+u\sqrt{2k}}{2}}$



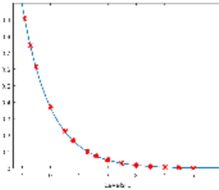
$u \mapsto \frac{1}{2}$ perimeter

$\frac{\sqrt{\pi}\lambda(k+u\sqrt{2k})^{(k-1)/2}}{2^{(k-1)/2}\Gamma(k/2)}\exp^{-\frac{k+u\sqrt{2k}}{2}}$



$u \mapsto$ Area

$\mathbb{P}(\chi^2_k \geq k + u\sqrt{2k})$



Chi square field with $k = 2$ degrees of freedom.

Unbiased $\widehat{C}_{0,T}(X,u)$, $\widehat{C}_{1,T}(X,u)$ and $\widehat{C}_{2,T}(X,u)$

Theoretical $u \mapsto C_0^*(X,u)$, $C_1^*(X,u)$ and $C_2^*(X,u)$

- From $(\hat{C}_{2,T}, \hat{C}_{0,T}) \Rightarrow (\hat{k}_T, \hat{\lambda}_T)$

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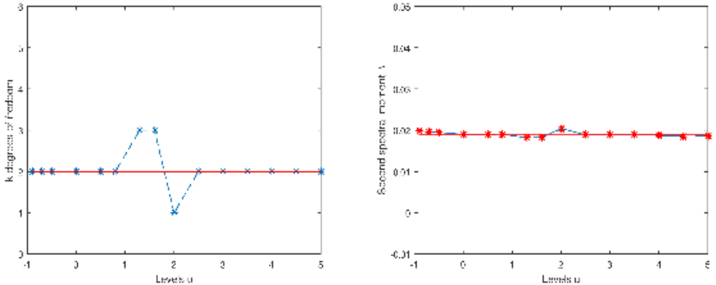

Chi square field with $k = 2$; **LEFT** From $\hat{C}_{2,T} \Rightarrow \widehat{K}(u)$, **RIGHT** From $\hat{C}_{0,T}$ and by using $\widehat{K}(u) \Rightarrow \hat{\lambda}_{T,\widehat{K}(u)}(u)$.

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Heuristic definition of Euler characteristic for compact sets

- $EC(A) =$ nber of disjoint intervals in $A \subset \boldsymbol{R}$
- $EC(A) =$ nber of connected components $-$ nber of holes in $A \subset \boldsymbol{R}^2$

If $A = \{t \in \boldsymbol{T} : X(t) \geq u\}$, with $\boldsymbol{T}$ a rectangle in $\boldsymbol{R}^d$ and $u \in \boldsymbol{R}$, there exists a rather tractable formula (theory of Morse functions):

$$EC(\{t \in T : X(t) \geq u\}) = \sum_{k=0}^{d-1} \sum_{\text{face } J \in \partial_k T} \cdots + EC(X, \mathring{T}, u)$$

Actually, $\sum_{k=0}^{d-1} \sum_{\text{face } J \in \partial_k T} \cdots = o(|T|)$ as $|T| \to \infty$

$\Rightarrow$ "Modified" Euler characteristic $EC(X, \boldsymbol{T}, u)$

Heuristic definition of Euler characteristic for compact sets

$$
\begin{aligned}
EC(X, T, u) &= \sum_{\ell=0}^{d} (-1)^{\ell} \mu_{\ell}(T, u) \\
\mu_{\ell}(T, u) &= \#\{t \in T : X(t) \geq u, X'(t) = 0, \text{ind } (X''(t)) = d - \ell\},
\end{aligned}
$$

with ind stands for the number of negative eigenvalues.

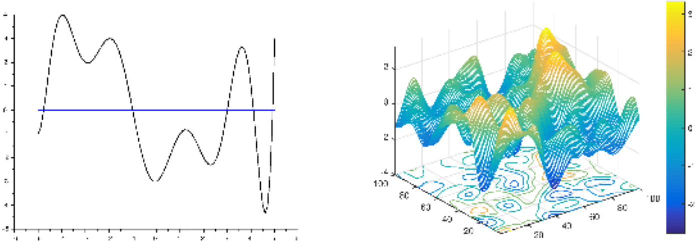

Figure: ($d = 1$), $EC(X, T, u) = \#\{\text{max of } X \text{ above } u \text{ in } \mathring{T}\} - \#\{\text{min of } X \text{ above } u \text{ in } \mathring{T}\}$
($d = 2$), $EC(X, T, u) = \#\{\text{local extrema of } X \text{ above } u \text{ in } \mathring{T}\} - \#\{\text{local saddle points of } X \text{ above } u \text{ in } \mathring{T}\}$