# Bayesian multi-objective optimization with constraints

## Application to the design of a commercial aircraft environment control system

### Atelier du GdR MASCOT NUM, 13 mai 2016, IHP, Paris

Taking into acount stochastics in optimization problems

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## 1 - The multi-objective optimization problem

Multi-objective optimization problem with constraints:

$$\begin{cases} \text{Minimize} & f(x) \\ \text{Subject to} & x \in \mathbb{X} \quad \text{and} \quad c(x) \leq 0 \end{cases}$$

with

- $\mathbb{X} \subset \mathbb{R}^d$ → search space
- $f = (f_j)_{1 \leq j \leq p}$ → vector of objective functions to be minimized
- $c = (c_i)_{1 \leq i \leq q}$ → vector of constraint functions

- Sequentially approximate the set of feasible and non-dominated solutions:

$$\Gamma = \{x \in \mathbb{X} : c(x) \leq 0 \text{ and } \nexists x' \in \mathbb{X} \text{ such that } c(x') \leq 0 \text{ and } f(x') \prec f(x)\}$$

Assumptions:

- Both $f$ and $c$ are smooth, nonlinear, expensive-to-evaluate functions (black-box)
- The available simulation budget is very limited ($\approx 10d$)

# 1 - Positioning and contributions

Positioning:

- We adopt a Bayesian approach to this optimization problem.
- We focus on highly constrained problems. Tipically it is assumed that no feasible observation is available at the start of the optimization procedure.

Main contributions:

- Proposal of an Expected Improvement (EI) criterion for the constrained multi-objective optimization problem.
- Study on the use of Sequential Monte Carlo techniques for the optimization of improvement based sampling criteria.
- Implementation of the associated algorithm, which we call BMOO.

# 1 - Publications

Associated publications:

- P. Feliot, J. Bect, and E. Vazquez. *A bayesian approach to constrained single- and multi-objective optimization*. Journal of Global Optimization, 2016.
- P. Feliot, Y. Le Guennec, J. Bect, and E. Vazquez. *Design of a commercial aircraft environment control system using Bayesian optimization techniques*. ENGOPT, 2016 (to appear).

01 Introduction

## 02 The BMOO algorithm

- Expected Hypervolume Improvement (EHVI) criterion
- Extending the EHVI to handle constraints
- Criterion calculation and optimization
- Structure of the algorithm
- Illustration

03 Environment control system

04 Optimization of the system

05 Conclusions

## 2 - Expected Hypervolume Improvement criterion

- We first consider the multi-objective optimization problem without constraints.

Expected Hypervolume Improvement (EHVI) sampling criterion:

- Given evaluation results $(f(X_1), \ldots, f(X_n))$, with $f(X_i) = (f_1(X_i), \ldots, f_p(X_i))$, $1 \leq i \leq n$, define:

$$\begin{cases} \mathbb{B} & = & \left\{y \in \mathbb{R}^p;\ y \leq y^{\text{upp}}\right\}, y^{\text{upp}} \in \mathbb{R}^p \\ H & = & \{y \in \mathbb{B}; \exists x \in \mathbb{X}, f(x) \prec y\} \\ H_n & = & \{y \in \mathbb{B}; \exists i \leq n,\ f(X_i) \prec y\} \end{cases}$$

- Loss function:

$$\varepsilon_n(\underline{X}, f) = |H \setminus H_n| \ ,$$

- Hypervolume improvement:

$$\begin{aligned} I_n\left(X_{n-1}\right) & = \varepsilon_n(\underline{X}, f) - \varepsilon_{n+1}(\underline{X}, f) \\ & = |H_{n+1}| - |H_n| \end{aligned}$$

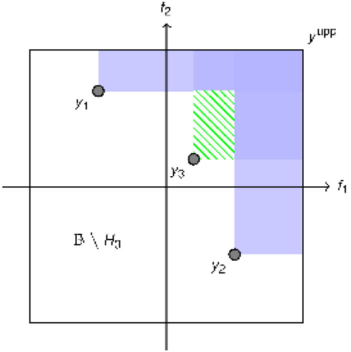



Figure: Hypervolume improvement yielded by the observation of $y_3$

## 2 - Expected Hypervolume Improvement criterion

Expected Hypervolume Improvement (EHVI) sampling criterion:

- Assume a vector-valued Gaussian random process model $\xi = (\xi_1, \ldots, \xi_p)$ of $f = (f_1, \ldots, f_p)$
- The EHVI[a] comes from taking the expectation of the improvement with respect to the posterior probability of $\xi$:

$$\begin{aligned} \mathrm{EI}_n(x) &= \mathbb{E}_n\left(|H_{n+1}| - |H_n|\right) \\ &= \mathbb{E}_n\left(\int_{\mathbb{B}\setminus H_n} \mathbb{1}_{\xi(x)\prec y}\,\mathrm{d}y\right) \\ &= \int_{\mathbb{B}\setminus H_n} \mathbb{P}_n\left(\xi(x) \prec y\right)\mathrm{d}y \end{aligned}$$

- The next sample is chosen as $X_{n+1} = \underset{x\in\mathbb{X}}{\operatorname{argmax}}\ \mathrm{EI}_n(x)$, which requires solving an auxiliary optimization problem at each iteration.

[a] References: Emmerich [2005], Emmerich et al. [2006]

## 2 - Constraint handling

- We now consider the multi-objective problem with constraints.

Constraint handling in the literature:[b]

- Emmerich [2005] and Couckuyt et al. [2014] propose to multiply the EHVI by $\mathbb{P}_n(\xi_c(x) \leq 0)$
  - This corresponds to $I_n = (|H_{n+1}| - |H_n|) \mathbb{1}_{c(x)<0}$ under the independence assumption.
  - The progress on constraint resolution is not measured.
- Gramacy et al. [2015] recently proposed to use the *Augmented Lagrangian* approach.
- Our proposal is to extend the Pareto domination rule to handle the constraints.

[b] References: See e.g. Conn et al. [1991], Sasena [2002], Parr et al. [2012], Gelbart [2015]

## 2 - Extended domination rule

Extended domination rule definition:[c]

- Denote $\mathbb{Y}_o = \mathbb{R}^p$, the objective space and $\mathbb{Y}_c = \mathbb{R}^q$ the constraint space, and let $\mathbb{Y} = \mathbb{Y}_o \times \mathbb{Y}_c$.
- We shall say that $y_1 \in \mathbb{Y}$ dominates $y_2 \in \mathbb{Y}$, which will be written as $y_1 \lhd y_2$, if $\psi(y_1) \prec \psi(y_2)$, where $\prec$ is the usual Pareto domination rule.

$$\begin{aligned} \psi : \quad \mathbb{Y}_o \times \mathbb{Y}_c \quad &\rightarrow \quad \overline{\mathbb{R}}^p \times \mathbb{R}^q \\ (y_o, y_c) \quad &\mapsto \quad \begin{cases} (y_o, 0) & \text{if } y_c \leq 0, \\ (+\infty, \max(y_c, 0)) & \text{otherwise.} \end{cases} \end{aligned}$$

- Properties of the extended domination rule:
  - Feasible solutions are compared wrt their objectives values.
  - Non-feasible solutions are compared wrt their component wise constraint violations.
  - Feasible solutions always dominate non-feasible ones.

[c] References: Ray et al. [2001], Fonseca and Fleming [1998], Oyama et al. [2007]

# 2 - Expected extended hypervolume improvement

- Similarly to the EHVI derivation, we can define the following sets.

$$\begin{cases} \mathbb{B} & = & \mathbb{B}_o \times \mathbb{B}_c, \text{ where } \mathbb{B}_o \subset \mathbb{Y}_o \text{ and } \mathbb{B}_c \subset \mathbb{Y}_c \text{ are bounded hyper-rectangles} \\ H & = & \{y \in \mathbb{B};\ \exists x \in \mathbb{X}\ (f(x), c(x)) \lhd y\} \\ H_n & = & \{y \in \mathbb{B};\ \exists i \leq n\ (f(X_i), c(X_i)) \lhd y\} \end{cases}$$

- Expected hypervolume improvement criterion with constraints:

$$\begin{aligned} \mathrm{EI}_n(x) &= \mathbb{E}_n\left(|H_{n+1}| - |H_n|\right) \\ &= \mathbb{E}_n\left(\int_{\mathbb{B}\setminus H_n} \mathbb{1}_{\xi(x) \lhd y}\, \mathrm{d}y\right) \\ &= \int_{\mathbb{B}\setminus H_n} \mathbb{P}_n\left(\xi(x) \lhd y\right) \mathrm{d}y, \end{aligned}$$

- where $\xi = (\xi_o, \xi_c)$, with $\xi_o = (\xi_{o,1}, \ldots, \xi_{o,p})$ and $\xi_c = (\xi_{c,1}, \ldots, \xi_{c,q})$.

# 2 - Criterion calculation and optimization

Criterion calculation:

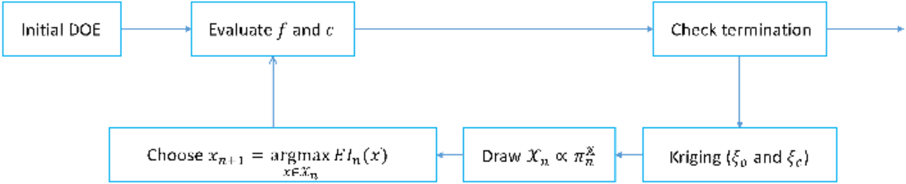
- The criterion takes the form of a multi-dimensional integral for which no closed form solution exists.
- The BMOO algorithm implements a Monte Carlo approximation of the integral using sequential Monte Carlo techniques.

Criterion optimization:

- The criterion optimization is performed using sequential Monte Carlo techniques as well.
- At time $n$, the next sample is chosen out of a population $\mathcal{X}_n$ of particles distributed according to a density of interest $\pi_n^{\mathbb{X}}$.

## 2 - Structure of the algorithm

Structure of the algorithm:



## 2 - Illustration on a toy example

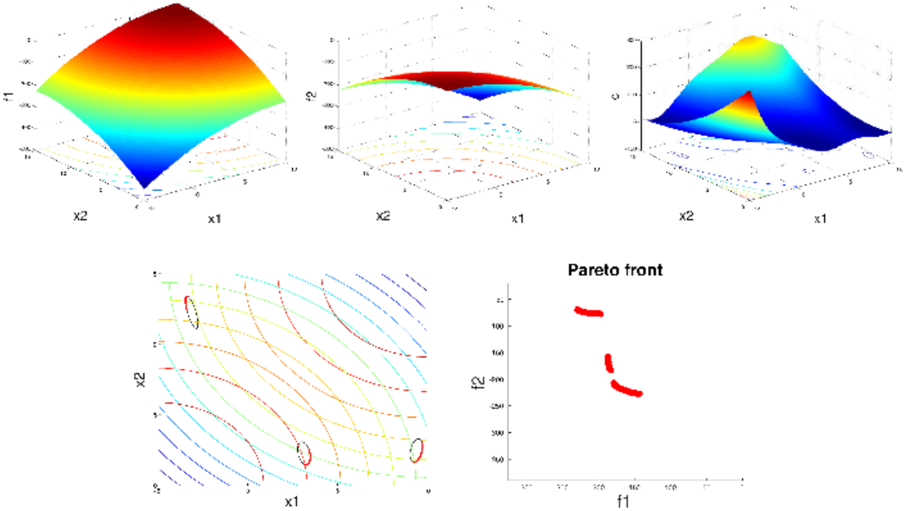


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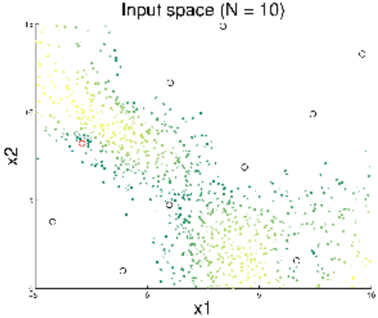
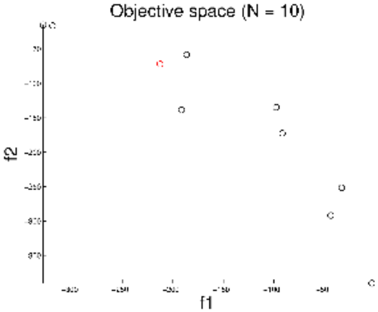
Illustration with $\pi_n^{\mathbb{X}}(x) \propto \mathbb{P}_n(\xi(x) \in (\mathbb{B} \setminus H_n))$ and $\pi_n^{\mathbb{Y}}(y) \propto L_2^{opt}(y)$



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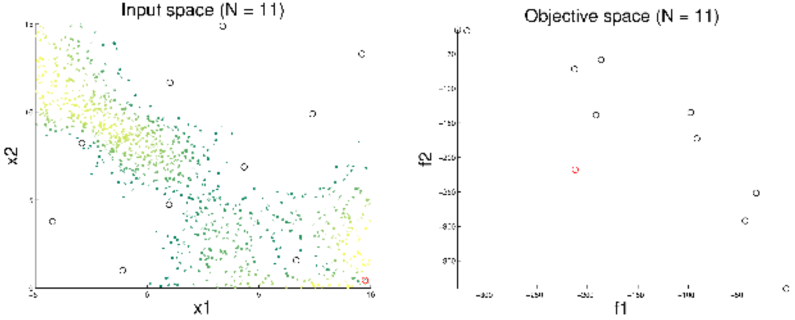

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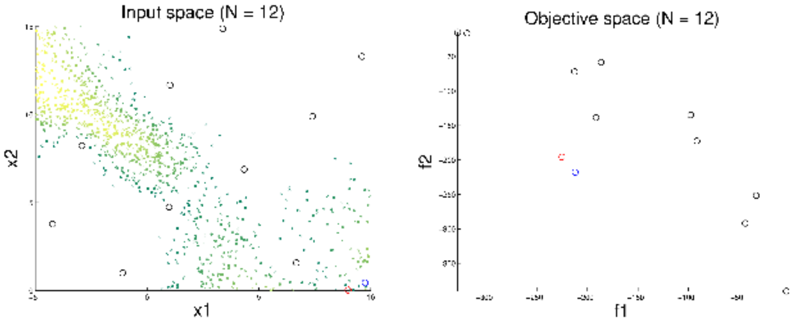

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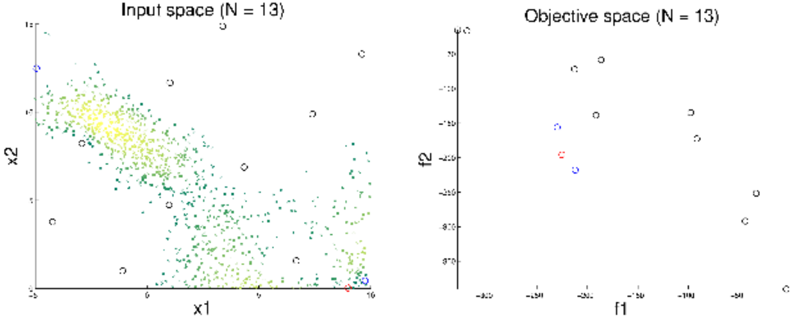

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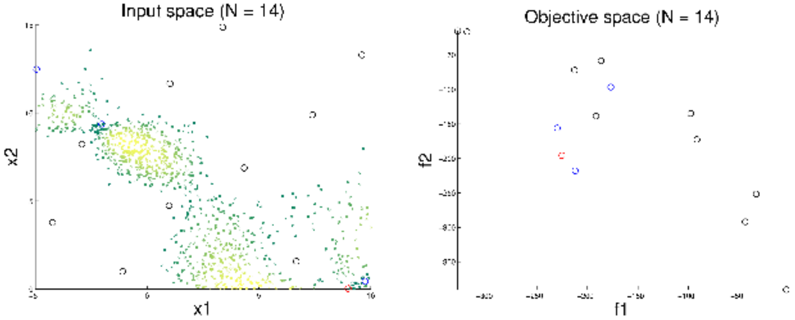

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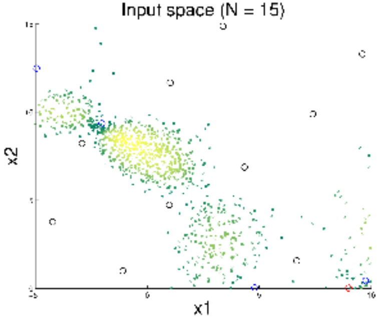


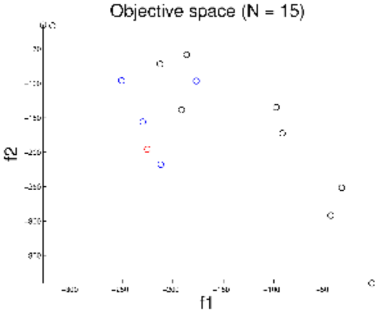

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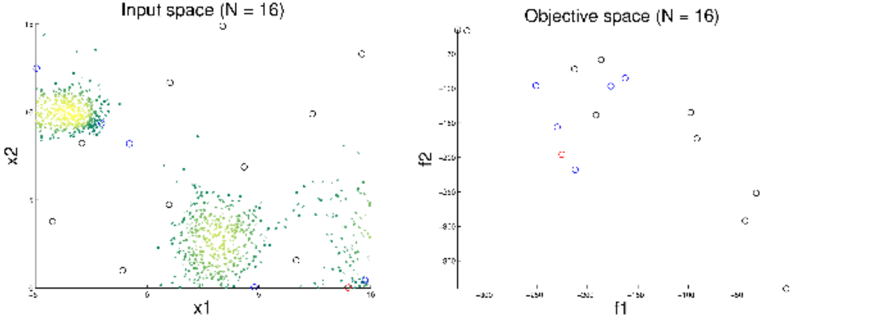

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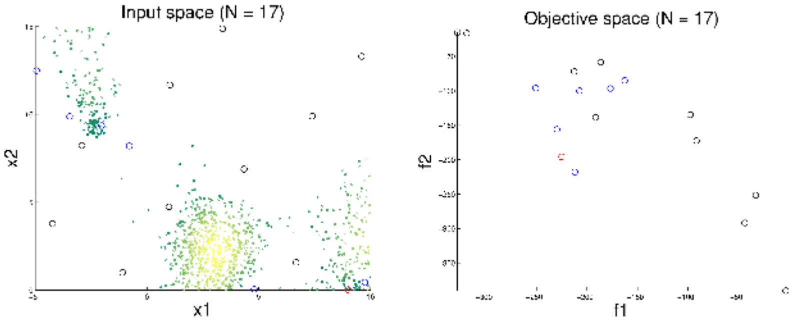

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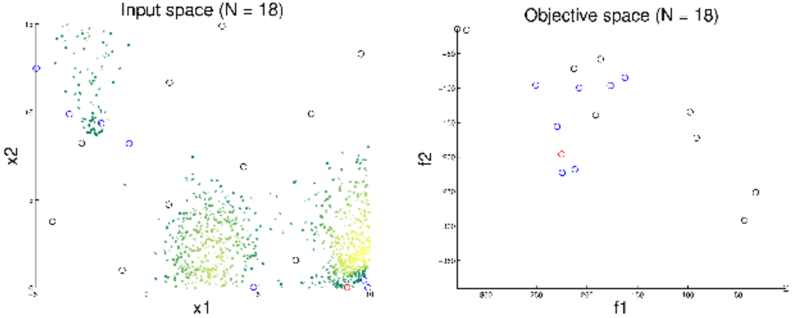

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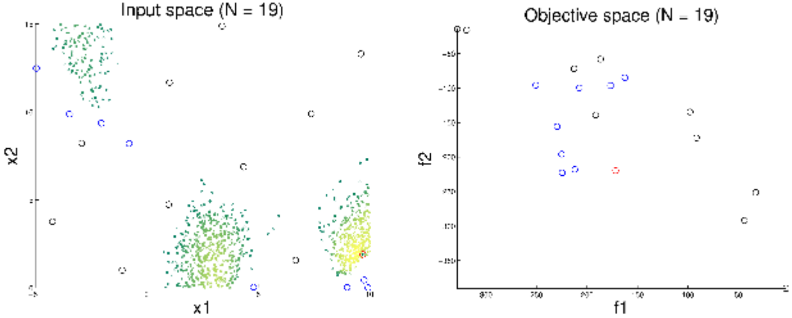

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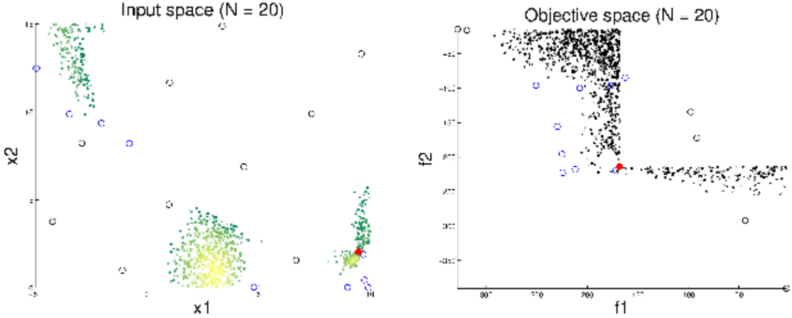

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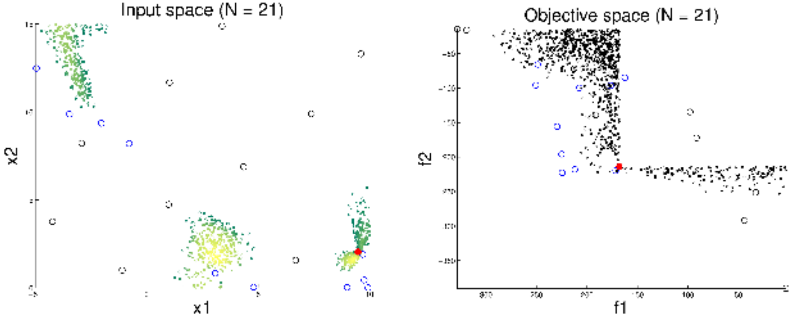

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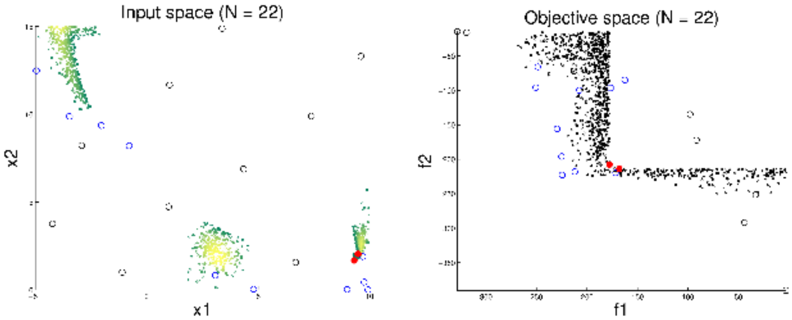

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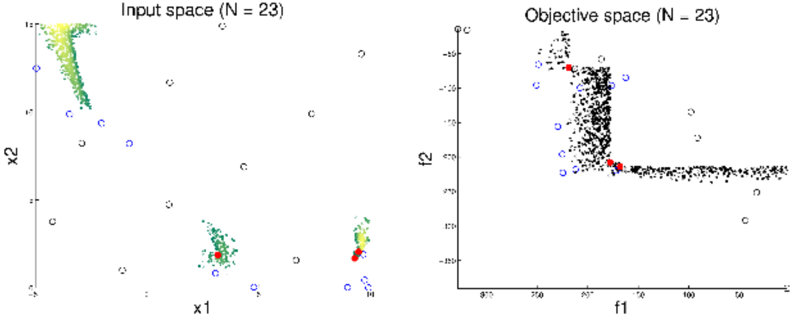

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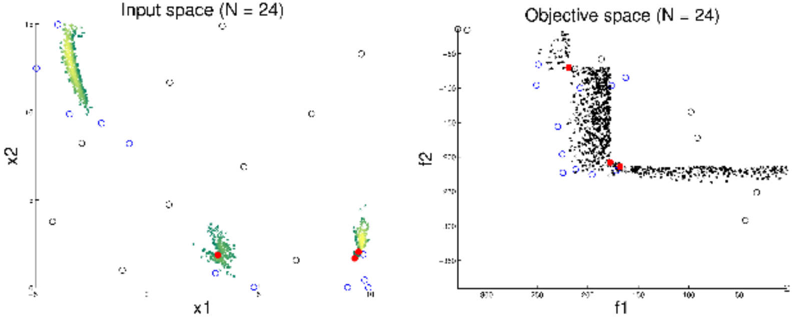

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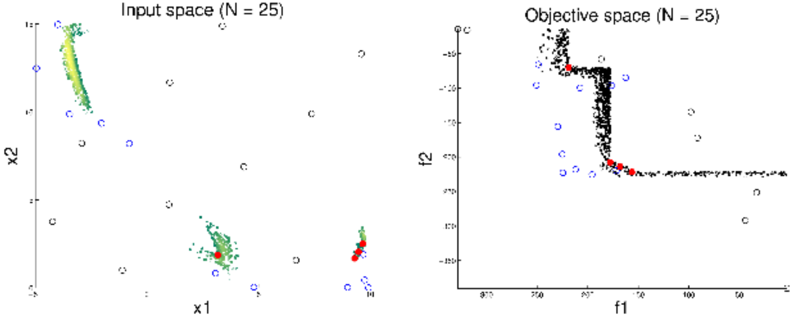

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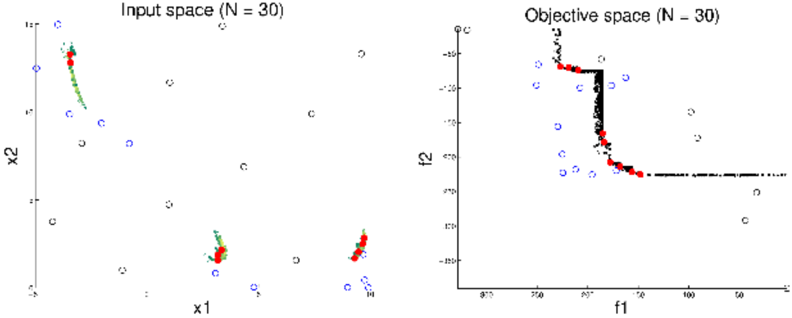

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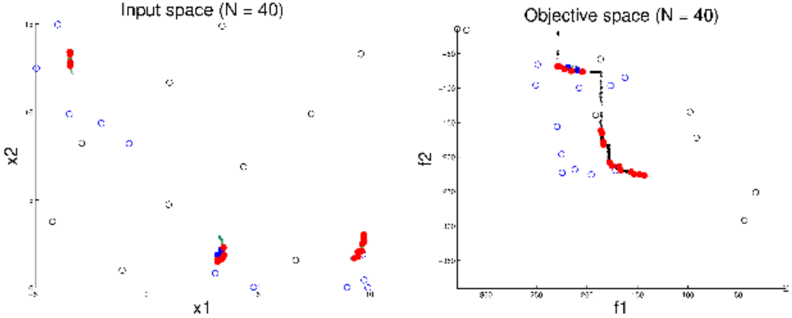

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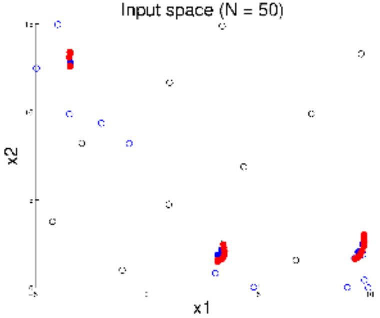
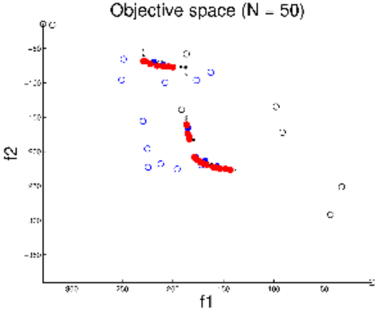
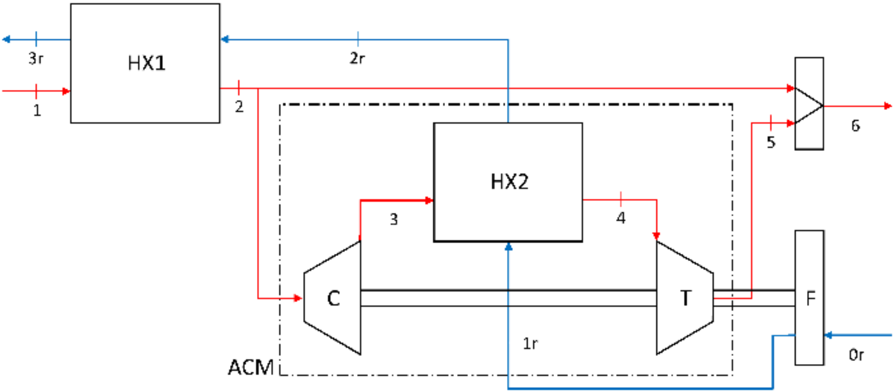


# 3 - Presentation of the problem

Architecture of the ECS:



## 3 - Presentation of the problem

Sizing scenario:

- Aircraft-on-ground, full of passengers, with 50°C outside.

$$\mathcal{P}_{HT} = \mathcal{P}_{out} + \mathcal{P}_{eq} + N_{pax}\mathcal{P}_{pax} + N_{crew}\mathcal{P}_{crew}$$

- Keep the cabin temperature at $T_c = 24^\circ$C.

$$\mathcal{P}_{HT} \leq \dot{m}c_p\left(T_c - T_5\right)$$

- Keep the cabin pressure close to the atmospheric pressure.

$$P_{min} \leq P_5 \leq P_{max}$$

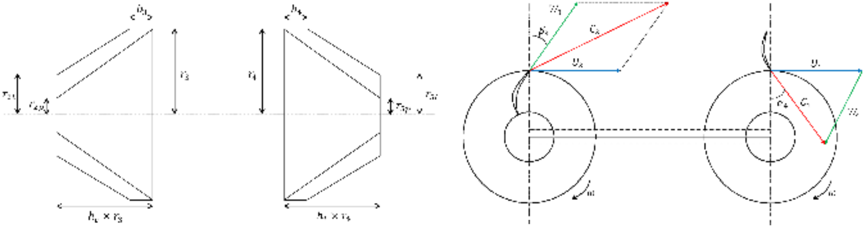
- The air injected into the cabin must lie between $T_{min} = 15^\circ$C and $T_{max} = 25^\circ$C.

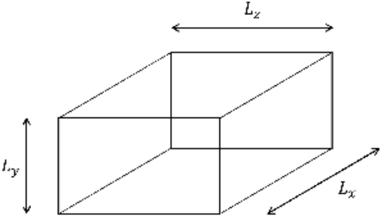
Objectives:

- Minimize the mass of the system.
- Minimize its entropy generation rate.

# 3 - One dimensional model of the system

Parametrization:





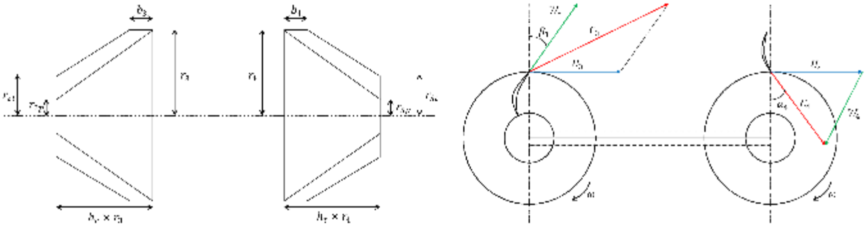

- 18 variables: $\dot{m}$, $\dot{m}_r$, $r_3$, $r_{2p}$, $r_{2t}$, $b_3$, $\beta_3$, $r_4$, $r_{5p}$, $r_{5t}$, $b_4$, $\alpha_4$, $L_{x1}$, $L_{y1}$, $L_{z1}$, $L_{x2}$, $L_{y2}$, $L_{z2}$.

# 3 - One dimensional model of the system

- The system is ruled by a non-linear system of 13 equations with 13 unknowns: $T_{t2}, T_{t3}, T_{t4}, T_{t5}, T_{t2r}, T_{t3r}, P_{t2}, P_{t3}, P_{t4}, P_{t5}, P_{t2r}, P_{t3r}$ and $\omega$.

$$\begin{cases} \dot{m}c_p(T_{t1} - T_{t2}) & = & \dot{m}_r c_p(T_{t3r} - T_{t2r}) \\ \dot{m}c_p(T_{t3} - T_{t4}) & = & \dot{m}_r c_p(T_{t2r} - T_{t1r}) \\ c_p(T_{t1} - T_{t2}) & = & \epsilon_1 c_p(T_{t1} - T_{t2r}) \\ c_p(T_{t3} - T_{t4}) & = & \epsilon_2 c_p(T_{t3} - T_{t1r}) \\ P_{t2} - P_{t1} & = & \Delta P_{HX} \\ P_{t4} - P_{t3} & = & \Delta P_{HX} \\ P_{t3} & = & P_{t2}\left(1 + \eta_C \frac{T_{t3} - T_{t2}}{T_{t2}}\right)^{\frac{\gamma}{\gamma-1}} \\ P_{t5} & = & P_{t4}\left(1 + \frac{1}{\eta_T}\frac{T_{t5} - T_{t4}}{T_{t4}}\right)^{\frac{\gamma}{\gamma-1}} \\ \dot{W}_C & = & \dot{m}\left(r_3^2\omega^2 - \frac{\dot{m}\tan(\beta_3)}{2\pi\rho b_3}\omega\right) \\ \dot{W}_T & = & -\frac{\dot{m}^2\tan(\alpha_4)}{2\pi\rho b_4}\omega \\ \dot{W}_C + \dot{W}_T + \frac{1}{\eta_F}\frac{\dot{m}_r^3}{2\rho^2 A_r^2} & = & 0 \\ \dot{W}_C & = & \eta_C \dot{m}c_p(T_{t3} - T_{t2}) \\ \dot{W}_T & = & \frac{1}{\eta_T}\dot{m}c_p(T_{t5} - T_{t4}) \end{cases}$$

# 3 - Formulation of an optimization problem



## Design constraints:

$$
\begin{array}{llllll}
d_1 & : & \dot{m} & \leq & \dot{m}_r\,, & \\
d_2 & : & b_3 & \leq & h_c r_3\,, & \\
d_3 & : & b_4 & \leq & h_t r_4\,, & \\
d_{4-5} & : & r_{2p} & \leq & r_{2t} & \leq \quad r_3\,, \\
d_{6-7} & : & r_{5p} & \leq & r_{5t} & \leq \quad r_4\,, \\
d_8 & : & \Delta & \geq & 0\,, & \\
d_9 & : & \dfrac{\tan(\beta_3)}{b_3} & \geq & -\dfrac{\tan(\alpha_4)}{b_4}\,. &
\end{array}
$$

# 3 - Formulation of an optimization problem

Simulation constraints:

$$
\begin{array}{lcccccc}
c_{1-2} & : & T_{min} & \leq & T_5 & \leq & T_{max}, \\
c_{3-4} & : & P_{min} & \leq & P_5 & \leq & P_{max}, \\
c_{5-6} & : & 0.5 & \leq & \epsilon_1 & \leq & 0.9, \\
c_{7-8} & : & 0.5 & \leq & \epsilon_2 & \leq & 0.9, \\
c_9 & : & C_2 & \leq & 0.95\sqrt{\gamma R T_2}, & & \\
c_{10} & : & C_3 & \leq & 0.95\sqrt{\gamma R T_3}, & & \\
c_{11} & : & C_4 & \leq & 0.95\sqrt{\gamma R T_4}, & & \\
c_{12} & : & C_5 & \leq & 0.95\sqrt{\gamma R T_5}, & & \\
c_{13} & : & r_3\omega & \leq & \sqrt{\gamma R T_3}, & & \\
c_{14} & : & r_4\omega & \leq & \sqrt{\gamma R T_4}, & & \\
c_{15} & : & \mathcal{P}_{\mathcal{H}\mathcal{T}} & \leq & \dot{m} c_p \left(T_c - T_5\right). & &
\end{array}
$$

# 3 - Formulation of an optimization problem

Summary:

- 18 variables: $\dot{m}$, $\dot{m}_r$, $r_3$, $r_{2p}$, $r_{2t}$, $b_3$, $\beta_3$, $r_4$, $r_{5p}$, $r_{5t}$, $b_4$, $\alpha_4$, $L_{x1}$, $L_{y1}$, $L_{z1}$, $L_{x2}$, $L_{y2}$, $L_{z2}$.
- Bound constraints on the variables.
- 9 design constraints: $d_{1-9}$.
- 15 simulation constraints: $c_{1-15}$.
- 2 objectives: $\mathcal{M}$ and $\dot{S}$.

Remarks:

- The design space is not hypercubic.
- Possibility of simulation failures (hidden constraints).

## 4 - Simulation failures management

Hidden constraints management:

- There are two cases where the model fails to produce a result:
  - The system can not be inverted.
  - The air flowrate becomes supersonic.
- This is taken into account in BMOO by multiplying the Expected improvement by a probability of observability[d].
  - A nearest-neighbour classifier is built on the observed/non-observed data to this end.
- The sampling density $\pi_n^{\mathbb{X}}$ is also multiplied by the probability of observability.

[c] Reference: Gramacy and Lee [2011]

# 4 - Handling non-hypercubic design spaces

Initial DoE:

- Denote $\mathbb{D}$ the domain delimited by the bound constraints and define:

$$\mathbb{X} = \mathbb{D} \setminus \{x \in \mathbb{D}; d(x) > 0\}$$

- If $|\mathbb{X}|/|\mathbb{D}|$ is not too small, a pseudo-maximin design on $\mathbb{X}$ can be achieved using the following process.
    - Sample a large population uniformly on $\mathbb{D}$.
    - Discard particles falling out of $\mathbb{X}$.
    - Prune the remaining particles to augment the maximin distance until the required number of particles remain.

During the optimization:

- The sampling density $\pi_n^{\mathbb{X}}$ is multiplied by $\mathbb{1}_{\mathbb{X}}$.

# 4 - Optimization results

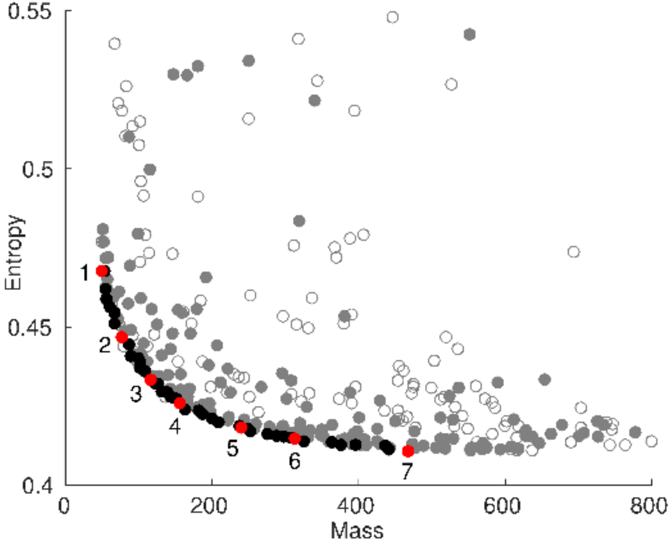
A few statistics:

- Initial DoE:
  - $N_{init} = 90$ samples.
  - 44 simulation failures.
  - No feasible observation

.

- Optimization process:
  - $N_{max} = 500$ samples.
  - 92 additional simulation failures.
  - First feasible point found after 25 iterations.

# 4 - Optimization results



# 4 - Optimization results

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\dot{m}$ | 2.95 | 2.92 | 2.94 | 2.94 | 2.94 | 2.95 | 2.94 |
| $\dot{m}_r$ | **7.74** | **6.86** | **5.63** | **5.06** | **4.64** | **4.40** | **4.27** |
| $r_{2o}$ | 0.07 | 0.05 | 0.05 | 0.03 | 0.03 | 0.07 | 0.04 |
| $r_{2t}$ | 0.10 | 0.08 | 0.08 | 0.08 | 0.06 | 0.09 | 0.10 |
| $r_3$ | 0.10 | 0.11 | 0.10 | 0.10 | 0.12 | 0.12 | 0.13 |
| $b_3$ | 0.01 | 0.01 | 0.05 | 0.05 | 0.04 | 0.02 | 0.03 |
| $\beta_3$ | 0.36 | 0.74 | 0.97 | -0.16 | 0.61 | 0.94 | 0.48 |
| $r_{5o}$ | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 |
| $r_{5t}$ | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| $r_4$ | 0.10 | 0.10 | 0.11 | 0.12 | 0.11 | 0.10 | 0.11 |
| $b_4$ | 0.02 | 0.02 | 0.04 | 0.02 | 0.04 | 0.03 | 0.03 |
| $\alpha_4$ | 1.04 | 0.50 | 0.89 | 1.01 | 0.44 | 0.79 | 0.30 |
| $L_{x1}$ | 0.67 | 0.65 | 0.68 | 0.68 | 0.63 | 0.69 | 0.70 |
| $L_{y1}$ | 0.65 | 0.68 | 0.61 | 0.67 | 0.67 | 0.66 | 0.65 |
| $L_{z1}$ | **0.03** | **0.04** | **0.07** | **0.12** | **0.17** | **0.20** | **0.32** |
| $L_{x2}$ | 0.66 | 0.69 | 0.66 | 0.66 | 0.70 | 0.68 | 0.69 |
| $L_{y2}$ | 0.69 | 0.53 | 0.68 | 0.65 | 0.65 | 0.68 | 0.65 |
| $L_{z2}$ | **0.03** | **0.06** | **0.09** | **0.10** | **0.17** | **0.25** | **0.36** |

# 5 - Conclusions

- BMOO: A Bayesian optimization algorithm for single- and multi-objective optimization.
  - Designed to address highly constrained problems.
  - Based on a generalized EHVI criterion defined using an extended domination rule.
  - SMC techniques for the criterion calculation and optimization.
  - Hidden constraints handling capability.
  - Non-hypercubic design spaces handling capability.
- The algorithm is applied to the design of a commercial aircraft environment control system with promising results.
- Directions for future work on this application case.
  - Taking into account uncertainties on key parameters of the simulation.
  - Sensitivity analysis on non-hypercubic design spaces.
  - Sensitivity analysis in constrained multi-objective optimization.

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# 06 Additional Material

# 7 - Adaptive procedure to set $\mathbb{B}_c$ and $\mathbb{B}_o$

Adaptive procedure to set $\mathbb{B}_c$ and $\mathbb{B}_o$:

- Assume that $n$ evaluation results $\xi(X_i)$, $1 \leq i \leq n$, are available. Then, we define the corners of $\mathbb{B}_o$ by

$$\begin{cases} y_{o,i,n}^{\text{low}} &= \min\left(\min_{i\leq n} \xi_{o,i}(X_i),\ \min_{x\in\mathcal{X}_n} \hat{\xi}_{o,i,n}(x) - \lambda_o \sigma_{o,i,n}(x)\right), \\ y_{o,i,n}^{\text{upp}} &= \max\left(\max_{i\leq n} \xi_{o,i}(X_i),\ \max_{x\in\mathcal{X}_n} \hat{\xi}_{o,i,n}(x) + \lambda_o \sigma_{o,i,n}(x)\right), \end{cases}$$

for $1 \leq i \leq p$, and the corners of $\mathbb{B}_c$ by

$$\begin{cases} y_{c,j,n}^{\text{low}} &= \min\left(0,\ \min_{i\leq n} \xi_{c,j}(X_i),\ \min_{x\in\mathcal{X}_n} \hat{\xi}_{c,j,n}(x) - \lambda_c \sigma_{c,j,n}(x)\right), \\ y_{c,j,n}^{\text{upp}} &= \max\left(0,\ \max_{i\leq n} \xi_{c,j}(X_i),\ \max_{x\in\mathcal{X}_n} \hat{\xi}_{c,j,n}(x) + \lambda_c \sigma_{c,j,n}(x)\right), \end{cases}$$

for $1 \leq j \leq q$, where $\lambda_o$ and $\lambda_c$ are positive numbers.

# 7 - $L_2^{opt}$ density for EHVI calculation

$L_2^{opt}$ density for EHVI calculation:

- Let $(x_k)_{1 \leq k \leq m_{\mathbb{X}}}$ be a population of candidates on which we want to compute the EHVI value, using a population of particles $(y_i)_{1 \leq i \leq m_{\mathbb{Y}}}$ distributed according to some density $\pi$.

$$\hat{I}_k^{\pi} = \frac{1}{m_{\mathbb{Y}}} \sum_{i=1}^{m_{\mathbb{Y}}} \frac{\mathbb{P}_n(\xi(x_k) \prec y_i)}{\pi(y_i)}$$

- We want to achieve a good approximation for all particles $(x_k)_{1 \leq k \leq m_{\mathbb{X}}}$. Using the $L_2$ norm, we get the following:

$$\begin{aligned} \mathbb{E}\left(\left\|\hat{I}^{\pi} - I\right\|_2^2\right) &= \mathbb{E}\left(\sum_{k=1}^{m_{\mathbb{X}}} (\hat{I}_k^{\pi} - I_k)^2\right) \\ &= \frac{1}{m_{\mathbb{Y}}} \sum_{k=1}^{m_{\mathbb{X}}} \left(\int \frac{\mathbb{P}_n(\xi(x_k) \prec y)^2}{\pi(y)^2} \pi(y) \mathrm{d}y - I_k^2\right) \end{aligned}$$

$$\rightarrow L_2^{opt}(y) \propto \sqrt{\sum_{k=1}^{m_{\mathbb{X}}} \mathbb{P}_n(\xi(x_k) \prec y)^2}$$

# 7 - A density for heavily constrained problems

A density for problems with a lot of constraints:

- Suppose $q = d$, $\mathbb{X} = [-1/2, 1/2]^q$ and $c_j : x = (x_1 \dots, x_q) \mapsto |x_j| - \frac{\varepsilon}{2}$, $\varepsilon \in (0; 1]$.
- Thus, the feasible domain is $C = [-\varepsilon/2, \varepsilon/2]^q$ and the volume of the subset of $\mathbb{X}$ where exactly $k$ constraints are satisfied is

$$V_k \approx \binom{q}{k} \varepsilon^k (1 - \varepsilon)^{q-k}.$$

- Assume moreover that the Gaussian process models are almost perfect, i.e.,

$$\mathbb{P}_n\left(\xi_{c,j}(x) \le 0\right) \approx \begin{cases} 1, & \text{if } c_j(x) \le 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

- Further assume $n = 1$ with $X_1 = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ so that the probability of improvement $\mathbb{P}_n\left(\xi(x) \in G_1\right)$ is close to one everywhere on $\mathbb{X}$.
- The expected number of particles satisfying exactly $k$ constraints is $m\,V_k$
- If $q$ is large, the particles thus tend to concentrate in regions where $k \approx q\varepsilon$.
- To compensate for the decrease of $V_k$, we suggest using the following modified sampling density:

$$\pi_n^{\mathbb{X}} \propto \mathbb{E}_n\left(K(x)!\, \mathbb{1}_{\xi(x) \in G_n}\right),$$

where $K(x)$ is the number of constraints satisfied by $\xi$ at $x$.

# 7 - Intermediate subsets

Parametric construction of intermediate subsets:

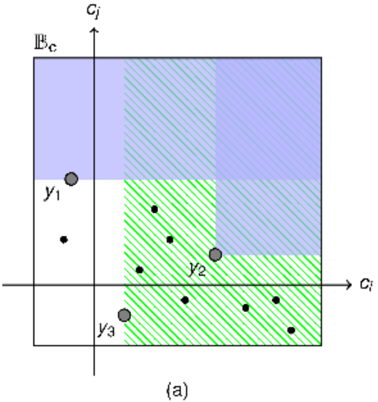


Figure: Procedure to construct intermediate subsets before the observation of a feasible solution.

# 7 - Intermediate subsets

Parametric construction of intermediate subsets:

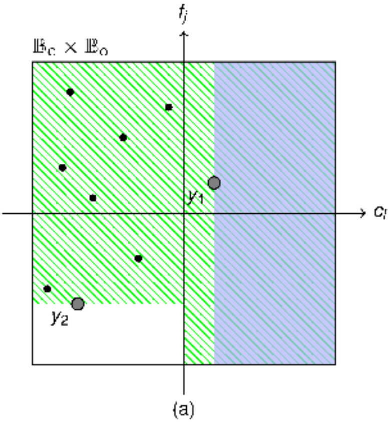


Figure: Procedure to construct intermediate subsets after the observation of a feasible solution.