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Roe Solver and Entropy Corrector for Hyperbolic Systems with Uncertain Coefficients

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Stochastic spectral methods and so-called Chaos expansions provide effective tools for uncertainty quantification (UQ) and propagation in numerical models. The determination of the stochastic solution can be achieved by means of non intrusive (sampling based) methods or a stochastic Galerkin projection procedure to derive a spectral problem for the stochastic modes. In this work, we consider the application of the Galerkin projection [3] to the resolution of hyperbolic systems of equations involving uncertain coefficients.

The stochastic Galerkin projection method has been applied successfully to different types of models (*e.g.* elliptic, parabolic, ordinary differential equations, ...). Its application to hyperbolic systems remains challenging essentially because the solutions can develop discontinuities in finite time (shock waves, contact discontinuities, ...). Indeed, whenever the propagation speed of the singularities is uncertain, the solution is also discontinuous in the stochastic space. This characteristic calls for appropriate discretization techniques and we rely on finite volume in space and piecewise continuous representations [1, 4, 6] at the stochastic level to prevent the emergence of Gibbs phenomena and aliasing errors associated with spectral representations based on smooth stochastic functionals.

Another difficulty raised by the Galerkin projection of stochastic hyperbolic systems is the need for appropriate numerical schemes for the discretization of the Galerkin fluxes. This point is crucial to obtain a stable numerical method while minimizing the numerical diffusion. We proposed in [5] to apply to the Galerkin system the first order Roe scheme derived in the deterministic context. This is motivated by the mathematical analysis of the Galerkin system which is shown to be hyperbolic provided that some conditions are satisfied, or at least close to an hyperbolic system in the most general situation and sufficiently fine stochastic discretization. However, the Roe scheme, which amounts to an upwinding with regard to the different wave components of the solution, requires the eigen decomposition of the Galerkin Jacobian matrix of the (large) Galerkin flux vector. To avoid this expensive decomposition of the Jacobian, we proposed instead an original technique which requires only an approximate knowledge of the Jacobian eigenvalues [5].

Finally, while effectively stabilizing the numerical method, the Roe scheme is known to provide unphysical (non entropic) solutions at sonic points. In this situation, a corrector is needed to construct numerical fluxes consistent with the physics. Again, different entropy correctors have been proposed in the literature, and we present an adaptation of the deterministic corrector of Dubois and Mehlman [2] to the stochastic Galerkin system. As for the Roe scheme, the computational efficiency of the corrector is a great concern as the dimension of the Galerkin system can be large. To improve the computational efficiency, we propose and test different indicators to decide a priori where the correction is needed, and provide inexpensive estimates of the eigenvectors of the Galerkin Jacobian matrix to avoid the need of its actual decomposition.

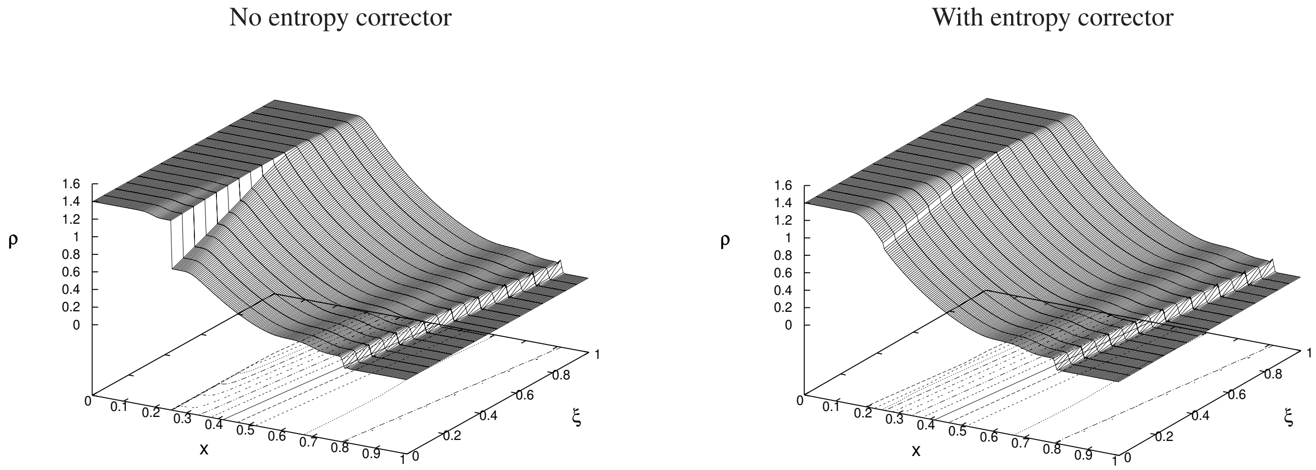

Figure 1: One-dimensional Sod shock tube problem, where the flow of an ideal gas is governed by the Euler equations. We consider an uncertainty on the initial Mach number which is parametrized using a uniform variable ξ in $[0,1]$. The initial conditions are such that sonic points appear for $\xi \in [0,0.6]$. The Figure shows a reconstruction of the stochastic density at $t = 1$ as a function of (x,ξ) obtained by the resolution of the derived Galerkin system with the Roe solver without (left) and with (right) the entropy corrector.

The resulting Roe scheme and entropy corrector are tested on nonlinear hyperbolic problems (Burgers and Euler equations) with uncertain initial conditions and physical properties. We rely on a fully intrusive approach for the computation of the flux vectors, upwinding matrices and correction. We demonstrate the numerical efficiency of the method and analyze its computational complexity. Simulation results are validated and contrasted with reference solutions computed by means of Monte Carlo sampling. An illustration of the effectiveness of the proposed method to remove non-entropic shocks is provided by Figure 1.

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Présentation courte de l'étudiante et contexte général de la thèse

TRYOEN Julie, née le 05/11/1984, titulaire d'un MASTER Analyse Numérique & EDP de l'UPMC (Paris VI), est actuellement en deuxième année de thèse à l'Université Paris Est, CERMICS, Ecole des Ponts, France et au LIMSI-CNRS, France, où elle effectue un travail de recherche sur les schémas numériques pour la propagation d'incertitudes paramétriques dans les systèmes hyperboliques.