Meta-model of a large credit risk portfolio in the Gaussian copula model

based on joint work with

F. Bourgey[1] E. Gobet[1] C. Rey[1]

[1]Ecole Polytechnique, Centre de Mathémathiques Appliquées,

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Stress Test
RISK Management and Financial Steering

The loss distribution

- The loss distribution of a large credit portfolio can be modeled as

$$\mathcal{L} = \sum_{k=1}^{K} l_k Y_k.$$

- K : number of obligors ($K \geq 10^5$).
- l_k : loss given default for the kth obligor.
- $Y_k = 1$ if the kth obligor defaults or 0 otherwise.
- $p_k = \mathbb{P}(Y_k = 1)$: default probability for the kth obligor.

Problem :

- Need to estimate $\mathbb{P}(\mathcal{L} > x)$ for credit risk measures (VaR, CVaR,...).
- Any Monte-Carlo (MC) simulation scheme is hugely time-consuming.

The one-factor Gaussian copula model [Li, 2000]

- Defaults are modeled by : $\quad Y_k = 1_{X_k \geq c_k}$.
- c_k : default boundary for the kth obligor.
- Dependence between obligors achieved through the correlated stochastic factors

$$(X_1, \ldots, X_K) \text{ where } (X_k)_{k=0\ldots K} \stackrel{d}{=} \mathcal{N}(0,1).$$

- More precisely,

$$X_k = \rho_k Z + \sqrt{1-\rho_k^2}\varepsilon_k.$$

- $Z \stackrel{d}{=} \mathcal{N}(0,1)$ systemic risk factor (economy).
- $(\epsilon_k)_{k=0\ldots K} \stackrel{d}{=} \mathcal{N}(0,1)$ idiosyncratic risks i.i.d and independent from Z.
- ρ_k : correlation parameter for the kth obligor.

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The one-factor Gaussian copula model [Li, 2000]

- Denote Φ the normal c.d.f. We choose $c_k = -\Phi^{-1}(p_k)$ so that

$$\mathbb{P}(Y_k = 1) = \mathbb{P}(X_k \geq c_k) = \mathbb{P}\left(\rho_k Z + \sqrt{1-\rho_k^2}\epsilon_k \geq c_k\right) = \Phi(-c_k) = p_k.$$

- The default event writes (we exclude the case $\rho_k = 0$) :

$$\{X_k \geq c_k\} = \{a_k \varepsilon_k + b_k \leq \frac{\rho_k}{|\rho_k|} Z\} \text{ where } a_k = \frac{-\sqrt{1-\rho_k^2}}{|\rho_k|},\ b_k = \frac{-\Phi^{-1}(p_k)}{|\rho_k|}.$$

- The loss rewrites :

$$\mathcal{L} = \sum_{k=1}^{K} l_k \mathbb{1}_{a_k \epsilon_k + b_k \leq \frac{\rho_k}{|\rho_k|} Z}.$$

- Note that both r.v.s are normally distributed :

$$\frac{\rho_k}{|\rho_k|} Z \overset{d}{=} \mathcal{N}(0,1), \qquad a_k \varepsilon_k + b_k \overset{d}{=} \mathcal{N}\left(b_k, a_k^2\right).$$

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Naive Monte-Carlo algorithm

Algorithm 1: Naive algorithm to sample from $\mathcal{L}$

Input: N (nonnegative integer), $(a_1, \dots, a_K)$ (vector of size K), $(b_1, \dots, b_K)$ (vector of size K), $(\rho_1, \dots, \rho_K)$ (vector of size K), $(l_1, \dots, l_K)$ (vector of size K)

Output: $\mathcal{L}^{(1:N)} = \left(\mathcal{L}^{(1)}, \dots, \mathcal{L}^{(N)}\right)$

$\mathcal{L}^{(1:N)} = \left(\mathcal{L}^{(1)}, \dots, \mathcal{L}^{(N)}\right) \leftarrow (0, \dots, 0)$ (vector of size N)

for $n = 1 \dots N$ **do**

- Generate $Z^{(n)} \sim \mathcal{N}(0,1)$ (scalar)
- Generate $(\epsilon_K)^{(n)} = \left(\epsilon_{K,1}^{(n)}, \dots, \epsilon_{K,K}^{(n)}\right) \sim \mathcal{N}(0, \mathrm{Id}_K)$ (vector of size K independent of $Z^{(n)}$)
- $\mathcal{L}^{(n)} \leftarrow 0$ (scalar)
- **for** $k = 1 \dots K$ **do**
  - $\mathcal{L}^{(n)} \leftarrow \mathcal{L}^{(n)} + l_k \mathbb{1}_{a_k \epsilon_{K,k}^{(n)} + b_k \leq \frac{\rho_k}{|\rho_k|} Z^{(n)}}$

Total cost: $\mathcal{O}(NK)$ operations. Very costly for large portfolios.

Some references

- CDO pricing. Explicit approximations for conditional quantities $\mathbb{E}\left[(\mathcal{L}-x)_+ \mid Z\right]$ (and then integrate over Z). Simpler since sum of independent variables. [Vasicek, 1991, Jiao et al., 2008].
- Use of characteristic functions for conditional loss and saddle-point techniques to provide approximations of the tranche function $\mathbb{E}\left[(\mathcal{L}-x)_+\right]$, see [Yang et al. (2006)].
- Importance sampling conditionally on Z; optimal tuning depends on x and suffers from large K [Glasserman and Li, 2005].

- Different route here as we wish to model the full distribution $\mathcal{L}$.
- We want to take advantage of the fact that K is large.

Probabilists' Hermite polynomials

- He_i : Probabilists' Hermite polynomial of degree $i \in \mathbb{N}$.
- Orthogonal for the standard normal measure :

$$\forall i, j \in \mathbb{N}, \int_{\mathbb{R}} \mathrm{He}_i(x)\mathrm{He}_j(x)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\mathrm{d}x = \begin{cases} i! & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- Three-term recurrence relation :

$$\mathrm{He}_0(x) = 1, \quad \mathrm{He}_1(x) = x, \quad \mathrm{He}_{i+2}(x) = x\mathrm{He}_{i+1}(x) - (i+1)\mathrm{He}_i(x).$$

The Wiener chaos decomposition

- Let $Z \overset{d}{=} \mathcal{N}(0,1)$ and a measurable function $\varphi : \mathbb{R} \mapsto \mathbb{R}$, s.t. $\mathbb{E}\left[\varphi^2(Z)\right] < +\infty$. The r.v. $\varphi(Z)$ can be decomposed into L_2 as

$$\varphi(Z) = \sum_{i=0}^{\infty} \alpha_i \mathrm{He}_i(Z), \quad \alpha_i = \mathbb{E}\left[\varphi(Z)\mathrm{He}_i(Z)\right]/i!.$$

Theorem

The indicator function has an explicit Wiener chaos decomposition: for any $c \in \mathbb{R}$,

$$\mathbb{1}_{c \le Z} = \sum_{i=0}^{\infty} \alpha_i(c)\mathrm{He}_i(Z), \quad \alpha_0(c) = \Phi(-c), \quad \alpha_i(c) = \frac{e^{-\frac{c^2}{2}}\mathrm{He}_{i-1}(c)}{i!\sqrt{2\pi}}.$$

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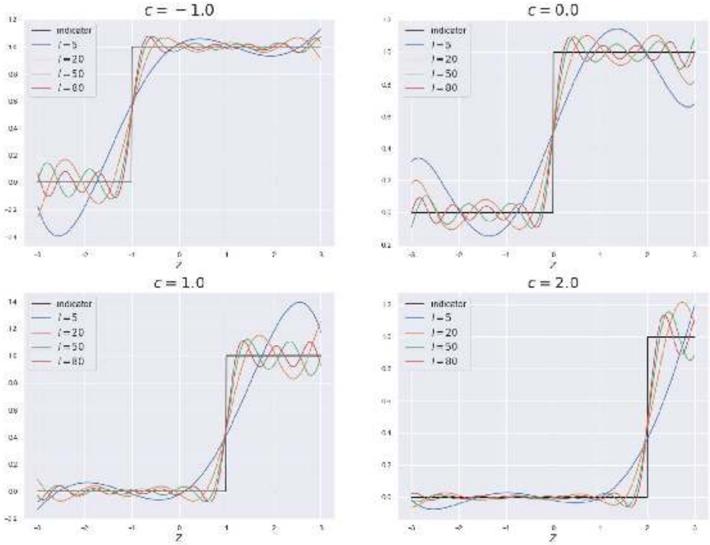
Chaos decomposition truncated at order $I \in \{5, 20, 50, 80\}$ and $c \in \{-1, 0, 1, 2\}$



Chaos decomposition of the loss $\mathcal{L}$

- Recall $\mathcal{L} = \sum_{k=1}^{K} l_k \mathbb{1}_{a_k \epsilon_k + b_k \leq \frac{\rho_k}{|\rho_k|} Z}$.
- Using the identity $\mathrm{He}_i\left(\frac{\rho_k}{|\rho_k|} Z\right) = \frac{\rho_k^i}{|\rho_k|^i} \mathrm{He}_i(Z)$, the loss rewrites :

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where : $\epsilon_{K,i} = \sum_{k=1}^{K} l_k \alpha_i \left(a_k \varepsilon_k + b_k\right) \frac{\rho_k^i}{|\rho_k|^i}$.

- The truncation up to $I \in \mathbb{N}$ gives the I-chaos decomposition :

$$\mathcal{L}_I = \sum_{i=0}^{I} \epsilon_{K,i} \mathrm{He}_i(Z).$$

Theorem

There exists $C > 0$ s.t. $\mathbb{E}\left[\left|\mathcal{L} - \mathcal{L}_I\right|^2\right] \leq C \frac{\left(\sum_{k=1}^{K} l_k\right)^2}{\sqrt{I}}$.

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Analysis of the $\epsilon_{K,i}$

Idea : for large K, approximate the vector $\epsilon_K = (\epsilon_{K,i})_{i=0\dots I}$ by a Gaussian vector for a fixed truncation parameter I.

More precisely, let $X \overset{d}{=} \mathcal{N}(0,1)$ and define for every $a, b \in \mathbb{R}$,

- $\mu_i(a,b) = \mathbb{E}\left[\alpha_i(aX+b)\right]$.
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Theorem

Suppose $\|s_K^{-1}\| \sup_{1\le k\le K} l_k^2 \xrightarrow[K\to+\infty]{} 0$ *with* $\|s_K^{-1}\| = \sup_{x\in\mathbb{R}^{I+1}, x\neq 0} \frac{|s_K^{-1}x|}{|x|}$.
Then, the following Central Limit Theorem (CLT) holds :

$$s_K^{-1/2}(\epsilon_K - m_K) \xrightarrow[K\to\infty]{d} \mathcal{N}(0, \mathrm{Id}_{I+1}).$$

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Some remarks so far

- Ignore the dependence of $\sigma_{i,j}(a_k, b_k)$ w.r.t. k:

$$(\sigma_{i,j}(a_k, b_k))_{i,j} = \mathcal{S},$$

  then $s_K = \left(\sum_{k=1}^{K} l_k^2\right) S$ and the theoretical (Lindeberg) condition in the CLT becomes

$$\frac{\sup_{1\leq k\leq K} l_k^2}{\sum_{k=1}^{K} l_k^2} \xrightarrow[K\to+\infty]{} 0.$$

- The larger the number of obligors K, the better the normal approximation.
- Recurrence relations hold for both μ_i and $\sigma_{i,j}$ (see next slide).
- The characteristics m_K, s_K can be computed off-line.
- Sampling from the meta-model thus reduces to sampling a I-dimensional Gaussian vector and the systemic risk factor Z.

Recurrence relation for $\mu_i(a,b)$ and $\sigma_{i,j}(a,b)$

Let $n \in \mathbb{N}^*$, $x \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$ positive definite and $X \overset{d}{=} \mathcal{N}(0,\Sigma)$, set

$$\Phi_\Sigma(x) := \mathbb{P}\left(X \le x\right) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{e^{-\frac{1}{2}t^\top \Sigma^{-1} t}}{(2\pi)^{\frac{n}{2}} \left(\det(\Sigma)\right)^{\frac{1}{2}}} \mathrm{d}t_1 \dots \mathrm{d}t_n.$$

In the case $n = 1$ and $\Sigma = 1$, we simply write Φ for Φ_Σ.

$$\begin{cases} \mu_0(a,b) = \Phi\left(-\frac{b}{\sqrt{1+a^2}}\right), \qquad \mu_1(a,b) = \frac{e^{-\frac{b^2}{2\left(1+a^2\right)}}}{\sqrt{2\pi}\sqrt{1+a^2}}, \\ \mu_{i+2}(a,b) = \frac{b}{(i+2)\left(1+a^2\right)}\mu_{i+1}(a,b) - \frac{i}{(i+2)(i+1)\left(1+a^2\right)}\mu_i(a,b), \end{cases}$$

$$\begin{cases} \sigma_{0,0}(a,b) = \Phi_\Sigma\left((-b,-b)^\top\right) - \mu_0(a,b)^2, \quad \Sigma = \begin{pmatrix} 1+a^2 & a^2 \\ a^2 & 1+a^2 \end{pmatrix}, \\ \sigma_{0,1}(a,b) = \mu_1(a,b)\left(\mu_0\left(\frac{a}{\sqrt{1+a^2}}, \frac{b}{1+a^2}\right) - \mu_0(a,b)\right), \\ \sigma_{0,i+2}(a,b) = \frac{b\sigma_{0,i+1}(a,b)}{(i+2)(1+a^2)} - \frac{i\sigma_{0,i}(a,b)}{(i+1)(i+2)(1+a^2)} - \frac{a^2}{(i+2)(1+a^2)}\mu_1(a,b)\mu_{i+1}\left(\frac{a}{\sqrt{1+a^2}}, \frac{b}{1+a^2}\right), \\ \sigma_{i+1,j+1}(a,b) = -\frac{(1+a^2)(j+2)\sigma_{i,j+2}(a,b)}{a^2(i+1)} + \frac{b\sigma_{i,j+1}(a,b)}{a^2(i+1)} - \frac{j\sigma_{i,j}(a,b)}{a^2(i+1)(j+1)} - \mu_{i+1}(a,b)\mu_{j+1}(a,b). \end{cases}$$

What to do in practice

- Replace $\mathcal{L}_I$ with its Gaussian approximation :

$$\mathcal{L}_I^{\mathrm{G}} = \sum_{i=0}^{I} \epsilon_{K,i}^{\mathrm{G}} \mathrm{He}_i(Z) \quad \text{where} \quad \epsilon_K^{\mathrm{G}} = \left(\epsilon_{K,i}^{\mathrm{G}}\right)_{i=0\dots I} \overset{d}{=} \mathcal{N}(m_K, s_K)$$

Algorithm 2: Gaussian-based approximation for the loss $\mathcal{L}$

Input: N (nonnegative integer), m_K (vector of size $I+1$), s_K (covariance matrix of size $(I+1, I+1)$)

Output: $\mathcal{L}_{I,N}^{\mathrm{G}} = \left(\left(\mathcal{L}_I^{\mathrm{G}}\right)^{(1)}, \dots, \left(\mathcal{L}_I^{\mathrm{G}}\right)^{(N)}\right)$

$\mathcal{L}_{I,N}^{\mathrm{G}} = \left(\left(\mathcal{L}_I^{\mathrm{G}}\right)^{(1)}, \dots, \left(\mathcal{L}_I^{\mathrm{G}}\right)^{(N)}\right) \leftarrow (0, \dots, 0)$ (vector of size N)

for $n = 1 \dots N$ **do**

  Generate $Z^{(n)} \sim \mathcal{N}(0,1)$ (scalar)

  Generate $\left(\epsilon_K^{\mathrm{G}}\right)^{(n)} \sim \mathcal{N}(m_K, s_K)$ (vector of size $I+1$ independent of $Z^{(n)}$)

  $\left(\mathcal{L}_I^{\mathrm{G}}\right)^{(n)} \leftarrow 0$ (scalar)

  for $i = 0 \dots I$ **do**

    $\left(\mathcal{L}_I^{\mathrm{G}}\right)^{(n)} \leftarrow \left(\mathcal{L}_I^{\mathrm{G}}\right)^{(n)} + \left(\epsilon_{K,i}^{\mathrm{G}}\right)^{(n)} \mathrm{He}_i\left(Z^{(n)}\right)$

Offline computational cost : $\mathcal{O}(KI) + \mathcal{O}(KI^2) + \mathcal{O}(I^3) = \mathcal{O}(KI^2)$

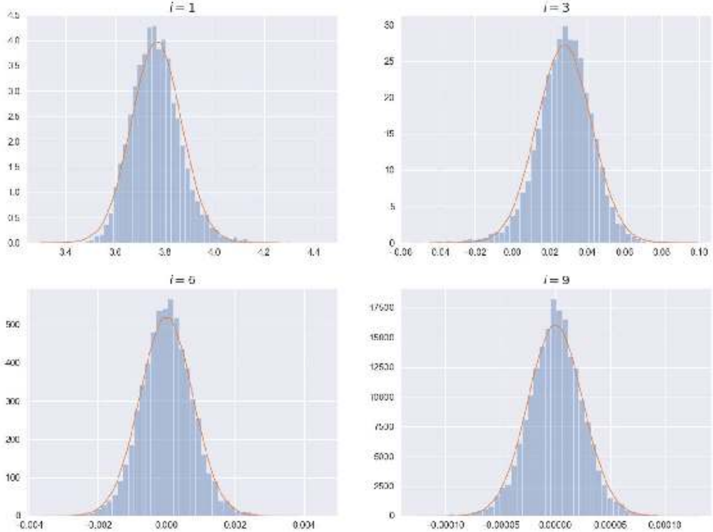
Online computational cost (N samples): $\mathcal{O}(NI^2)$

Total cost: $\mathcal{O}(KI^2) + \mathcal{O}(NI^2)$. Compare with the previous $\mathcal{O}(NK)$

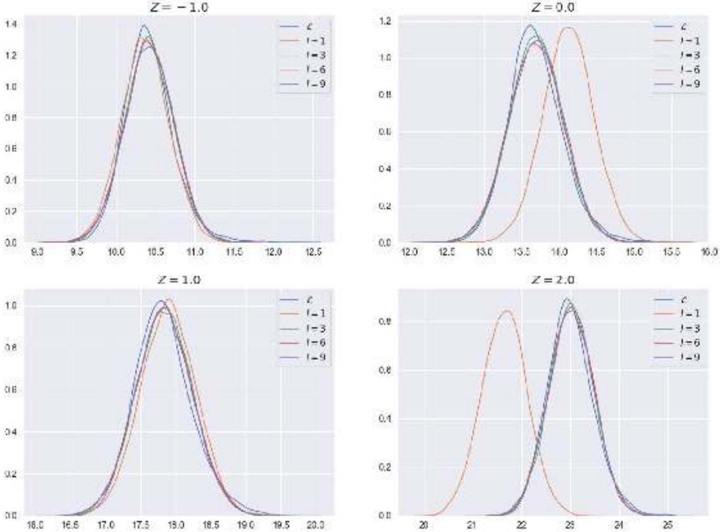
Portfolio A : A homogeneous portfolio

- We fix $K = 5 \times 10^5$.
- $\forall k \in \{1, \dots K\}$, $p_k = 0.01$, $\rho_k = 0.1$ and $l_k = 1/\sqrt{k}$.
- Simplified CLT condition holds since $\frac{\sup_{1\leq k\leq K} l_k^2}{\sum_{k=1}^K l_k^2} = \frac{1}{\sum_{k=1}^K 1/k} \to 0$ as $K \to \infty$.

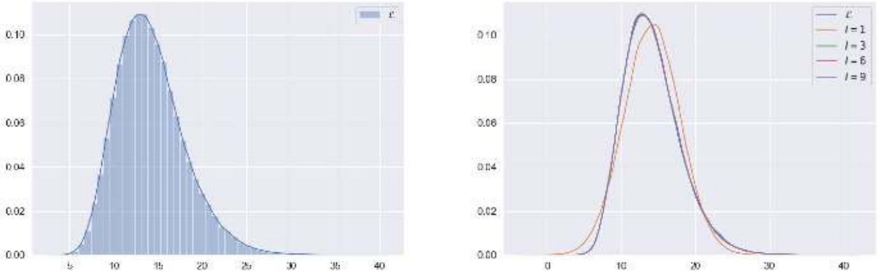
Histograms of $c_{K,i}$, density of $\mathcal{N}(\mathbb{E}[c_{K,i}], \mathbf{Var}(c_{K,i}))$ for $i \in \{1,3,6,9\}$ with $N = 10^4$ samples



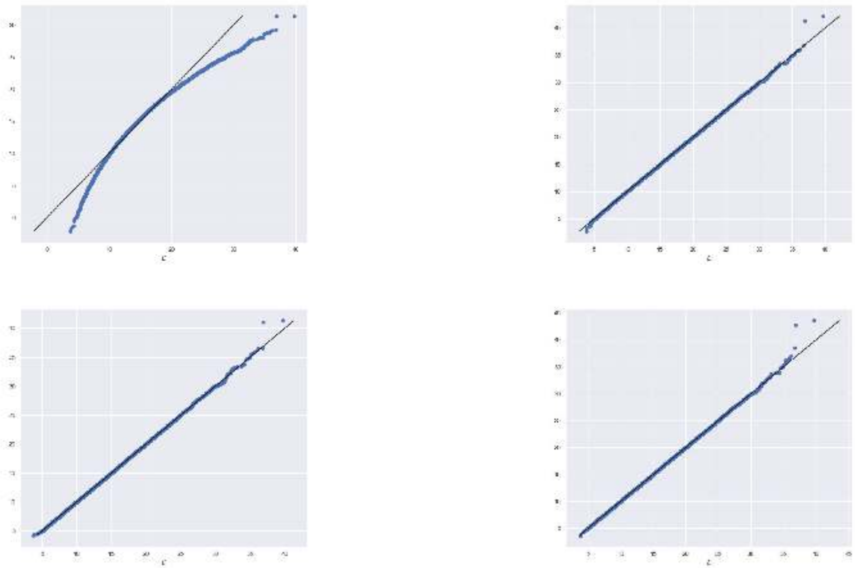
K.d.e of $\mathcal{L}_I^{\mathrm{C}}$ for $I \in \{1, 3, 6, 9\}$ and $\mathcal{L}$ conditionally on $Z \in \{-1, 0, 1, 2\}$ with $N = 10^4$ samples



K.d.e of $\mathcal{L}_l^G$ for $l \in \{1, 3, 6, 9\}$ and $\mathcal{L}$ with $N = 10^5$ samples



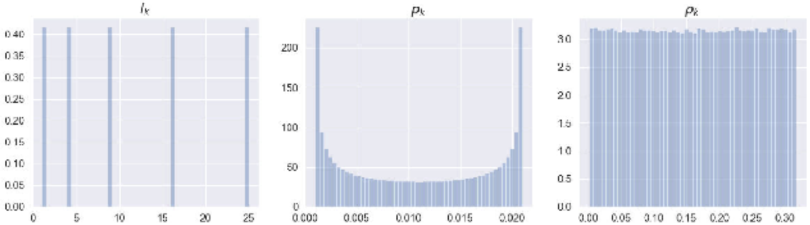
Q-Q plot of $\mathcal{L}_I^G$ for $I \in \{1, 3, 6, 9\}$ w.r.t. $\mathcal{L}$ with $N = 10^5$ samples



$\Longrightarrow$ Quantiles up to $\alpha = 99.998\%$ are accurate.

Portfolio B [Glasserman and Li, 2005]

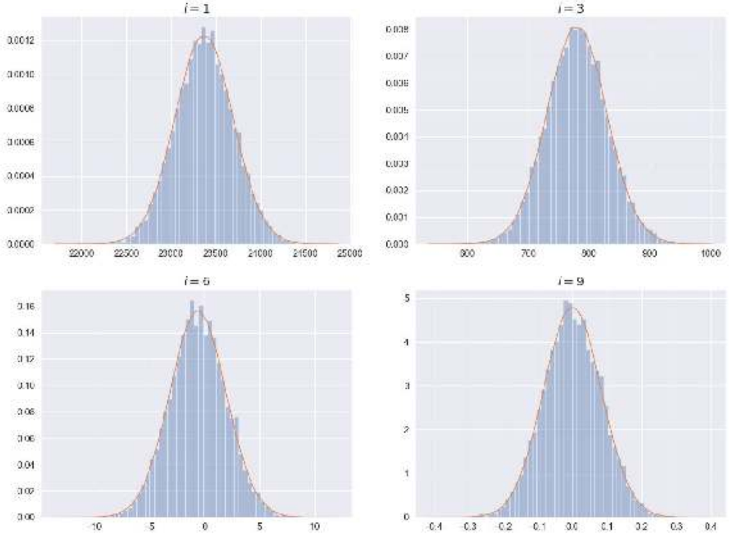
- Again take $K = 5 \times 10^5$.
- $\forall k \in \{1, \dots K\}, p_k = 0.01\left(1 + \sin\left(\frac{16\pi k}{K}\right)\right) + 0.001$
  $\rho_k \overset{d}{=} \mathcal{U}\left([0, 1/\sqrt{10}]\right) + 0.001,\ l_k = \left(\left\lceil \frac{5k}{K} \right\rceil\right)^2$



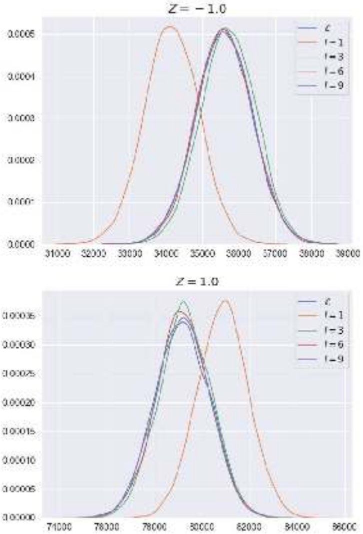
- p_k vary between 0.001% and 2.001% with a mean 1.0005%.
- l_k are $1, 4, 9, 16, 25$ with $K/5$ at each level.
- Simplified CLT condition holds since $\sup_{1 \le k \le K} l_k^2 = 25$ and

$$\sum_{k=1}^{K} \left(\left\lceil \frac{5k}{K} \right\rceil\right)^2 \ge \frac{25}{K^2} \sum_{k=1}^{K} k^2 = \frac{25}{K^2} \frac{K(K+1)(2K+1)}{6} \sim \frac{25}{3} K^2 \xrightarrow[K \to +\infty]{} +\infty.$$

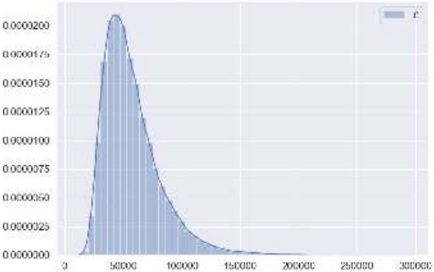
Histograms of $c_{K,i}$, density of $\mathcal{N}(\mathbb{E}[c_{K,i}], \mathbf{Var}(c_{K,i}))$ for $i \in \{1, 3, 6, 9\}$ with $N = 10^4$ samples

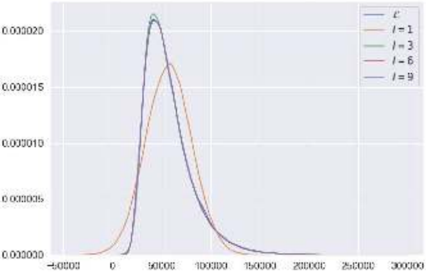


K.d.e of $\mathcal{L}_I^{\mathrm{G}}$ for $I \in \{1,3,6,9\}$ and $\mathcal{L}$ conditionally on $Z \in \{-1,0,1,2\}$ with $N = 10^4$ samples

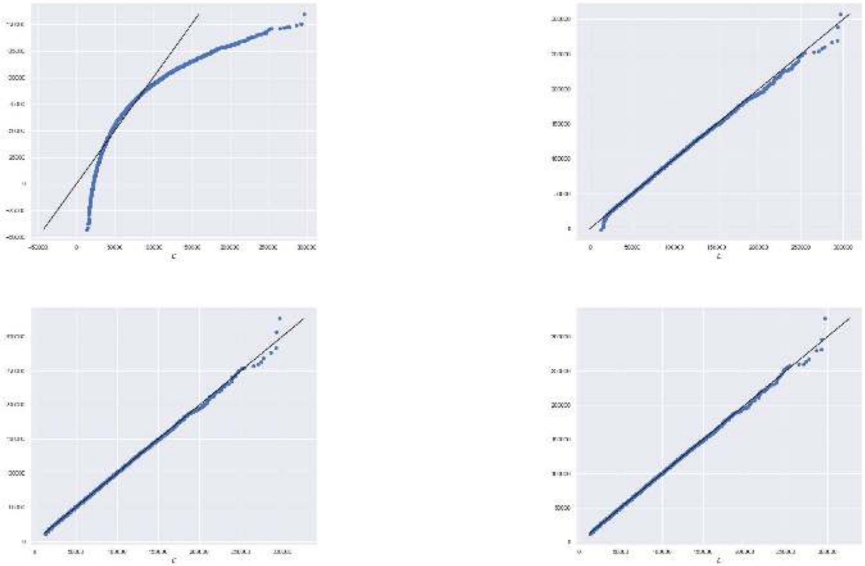


K.d.e of $\mathcal{L}_l^G$ for $l \in \{1, 3, 6, 9\}$ and $\mathcal{L}$ with $N = 10^5$ samples





Q-Q plot of $\mathcal{L}_I^G$ for $I \in \{1, 3, 6, 9\}$ w.r.t. $\mathcal{L}$ with $N = 10^5$ samples



$\Longrightarrow$ Quantiles up to $\alpha = 99.99\%$ are accurate.

Extension to a d-multi-factor-model

- Defaults are still modeled by : $Y_k = 1_{X_k \geq c_k}$ with $c_k = -\Phi^{-1}(p_k)$.
- d systemic factors (industry sector) : $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$.
- The stochastic factor X_k now takes the form :

$$X_k = \boldsymbol{\rho}_k \cdot \boldsymbol{Z} + \sqrt{1 - \|\boldsymbol{\rho}_k\|^2}\epsilon_k = \sum_{j=1}^{d} \rho_{kj} Z_j + \sqrt{1 - \|\boldsymbol{\rho}_k\|^2}\epsilon_k.$$

- Define $a_k = -\frac{\sqrt{1-\|\boldsymbol{\rho}_k\|^2}}{\|\boldsymbol{\rho}_k\|}$, $\quad b_k = -\frac{\Phi(p_k)}{\|\boldsymbol{\rho}_k\|}$, and set for every multi-indices $\boldsymbol{i} = (i_1, \ldots, i_d)\,, \boldsymbol{j} = (j_1, \ldots, j_d) \in \{0, \ldots I\}^d$,

$$\epsilon_{K,\boldsymbol{i}} = \sum_{k=1}^{K} l_k \left(\frac{\boldsymbol{\rho}_k}{\|\boldsymbol{\rho}_k\|}\right)^{\boldsymbol{i}} \left(\begin{array}{c} |\boldsymbol{i}| \\ \boldsymbol{i} \end{array}\right) \alpha_{|\boldsymbol{i}|}\left(a_k \epsilon_k + b_k\right),$$

$$\mathcal{L} = \sum_{i=0}^{\infty} \sum_{|\boldsymbol{i}|=i} \epsilon_{K,\boldsymbol{i}} \mathrm{He}_{i_1}(Z_1) \ldots \mathrm{He}_{i_d}(Z_d), \quad \mathcal{L}_I = \sum_{i=0}^{I} \sum_{|\boldsymbol{i}|=i} \epsilon_{K,\boldsymbol{i}} \mathrm{He}_{i_1}(Z_1) \ldots \mathrm{He}_{i_d}(Z_d). \tag{1}$$

- The L_2 estimate and CLT still hold in the d-multi-factor model.

Conclusion :

- Meta-modeling allows us to significantly reduce computational time.
- I small seems to already give an efficient approximation.
- The Gaussian characteristics can be computed off-line.
- We essentially sample I terms instead of K (large).
- Factorization of the economic factor Z w.r.t. the other variables.

On-going works :

- Global error analysis (PCE + Gaussian approximation)
- What if Gaussian approximation non valid?
- Other dependence modelling ($\neq$ Gaussian Copulas)

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